Abstract

A three-valued function $f$ defined on the vertices of a graph $G = (V,E)$, $f : V \rightarrow \{-1,0,1\}$, is a minus dominating function if the sum of its function values over any closed neighborhood is at least one. That is, for every $v \in V$, $f(N[v]) \geq 1$, where $N[v]$ consists of $v$ and every vertex adjacent to $v$. The weight of a minus dominating function is $f(V) = \sum f(v)$, over all vertices $v \in V$. The minus domination number of a graph $G$, denoted $\gamma^-(G)$, equals the minimum weight of a minus dominating function of $G$. In this note, we establish a sharp lower bound on $\gamma^-(G)$ for regular graphs $G$.

1. Introduction

Let $G = (V,E)$ be a graph with vertex set $V$ and edge set $E$, and let $v$ be a vertex in $V$. If $v \in V$, the degree of $v$ in $G$ is written as $\deg v$. The graph $G$ is $r$-regular if $\deg v = r$ for all $v \in V$. In particular, if $r = 3$, then we call $G$ a cubic graph. The open neighborhood of $v$ is defined as the set of vertices adjacent to $v$, i.e., $N(v) = \{u \mid uv \in E\}$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$.

For any real valued function $g : V \rightarrow R$ and $S \subseteq V$, let $g(S) = \sum g(u)$ over all $u \in S$. A minus dominating function is defined in [2] as a function $g : V \rightarrow \{-1,0,1\}$ such that $g(N[v]) \geq 1$ for all $v \in V$. The minus domination number for a graph $G$ is $\gamma^-(G) = \min\{g(V) \mid g$ is a minus dominating function on $G\}$. The concept of minus domination in graphs is studied in [1–3].

Zelinka [4] established the following lower bound on $\gamma^-(G)$ for a cubic graph $G$.

Theorem A. For every cubic graph $G$ of order $n$, $\gamma^-(G) \geq n/4$. 
In this note we generalize the result of Theorem A to $r$-regular graphs.

**Theorem 1.** For every $r$-regular graph $G = (V, E)$ of order $n$,

$$\gamma^-(G) \geq \frac{n}{r + 1},$$

and this bound is sharp.

**Proof.** Let $f$ be a minus dominating function on $G$ satisfying $f(V) = \gamma^-(G)$. We consider the sum $N = \sum_{v \in V} \sum_{u \in N[v]} f(u)$, where the outer sum is over all $v \in V$ and the inner sum is over all $u \in N[v]$. This sum counts the value $f(u)$ exactly $\text{deg } u + 1$ times for each $u \in V$, so

$$N = \sum_{u \in V} (\text{deg } u + 1) f(u) = (r + 1) \sum_{u \in V} f(u) = (r + 1) f(V).$$

On the other hand,

$$N = \sum_{v \in V} \sum_{u \in N[v]} f(u) = \sum_{v \in V} f(N[v]) \geq \sum_{v \in V} 1 = n.$$

Consequently, $\gamma^-(G) = f(V) \geq n/(r + 1)$. That the lower bound is sharp is easily seen by considering a complete graph on $r + 1$ vertices and assigning the value 1 to one vertex and the value 0 to the remaining $r$ vertices to produce a minus dominating function of weight $n/(r + 1) = 1$. 

**References**


