An accurate closed-form approximate solution for the quintic Duffing oscillator equation

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ABSTRACT
An accurate closed-form solution for the quintic Duffing equation is obtained using a cubication method. In this method the restoring force is expanded in Chebyshev polynomials and the original nonlinear differential equation is approximated by a cubic Duffing equation in which the coefficients for the linear and cubic terms depend on the initial amplitude. The replacement of the original nonlinear equation by an approximate cubic Duffing equation allows us to obtain explicit approximate formulas for the frequency and the solution as a function of the complete elliptic integral of the first kind and the Jacobi elliptic function cn, respectively. Excellent agreement of the approximate frequencies and periodic solutions with the exact ones is demonstrated and discussed and the relative error for the approximate frequency is lower than 0.37%.

Keywords: Nonlinear oscillator; Approximate solutions; quintic Duffing equation; Chebyshev polynomials.

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1. Introduction

Nonlinear oscillations in engineering, physics, applied mathematics and in many real world applications has been a topic to intensive research for many years and several methods have been used to find approximate solutions to nonlinear oscillators [1, 2]. In general, given the nature of a nonlinear phenomenon, the approximate methods can only be applied within certain ranges of the physical parameters and to certain classes of problems. The purpose of this paper is to calculate analytical approximations to the periodic solutions to the quintic Duffing oscillator [3-5]. To do this, the Chebychev series expansion of the restoring force is used [6-10] and the original nonlinear differential equation is approximated by a cubic Duffing equation in which the coefficients for the linear and cubic terms depend on the initial amplitude and this last equation can be exactly solved. The replacement of the original nonlinear equation by an approximate Duffing equation allows us to obtain an approximate frequency-amplitude relation as a function of the complete elliptic integral of the first kind and the solution in terms of the Jacobi elliptic function \( cn \). A major advantage of the procedure is that allows us easily the calculation of approximate frequency and solution.

2. Solution procedure

A quintic Duffing oscillator is an example of a conservative autonomous oscillatory system, which can be described by the following second-order differential equation

\[
\frac{d^2x}{dt^2} + f(x) = 0 \tag{1}
\]

with initial conditions

\[
x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \tag{2}
\]

where \( f(x) = x + \varepsilon x^5 \) is an odd function and \( \varepsilon \) is a positive constant parameter. We denote the angular frequency of these oscillations by \( \omega \) and it is a function of the initial amplitude \( A \). The quintic Duffing equation is difficult to handle because of the presence of strong nonlinearity and it has no known closed-form solution [3, 4].
Eq. (1) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. In order to approximately solve this equation by means a cubication procedure based on the expansion of the nonlinear restoring force in terms of the Chebyshev polynomials as can be seen in the papers of Denman [6] and Jonckheere [7]. To do this we first introduce a reduced variable \( \varepsilon y = x / A \) in Eqs. (1) and (2) and we obtain

\[
\frac{d^2 y}{dt^2} + \frac{1}{A} f(Ay) = 0
\]  
(3)

\[y(0) = 1 \quad \text{and} \quad \frac{dy}{dt}(0) = 0\]  
(4)

Eq. (3) can be written as follows

\[
\frac{d^2 y}{dt^2} + g(y) = 0, \quad g(y) = y + \lambda y^5
\]  
(5)

where \( \lambda = \varepsilon A^4 \).

A cubication of the nonlinear function \( g(y) \) can be done by expanding \( g(y) \) in terms of Chebyshev polynomials of the first kind \( T_n(y) \) [7]. Doing this gives

\[
g(y) = \sum_{n=0}^{\infty} b_{2n+1} T_{2n+1}(y)
\]  
(6)

where

\[
b_{2n+1} = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1 - y^2}} g(y) T_{2n+1}(y) dy
\]  
(7)

and we have taken into account that \( g(y) \) is an odd function of \( y \). The first two polynomials are

\[
T_1(y) = y, \quad T_3(y) = 4y^3 - 3y
\]  
(8)
Note that the nonlinear function $g(y)$ (Eq. (5)) has not a cubic term, however this cubic term appears if this function is expanded in terms of the Chebyshev polynomials of the first kind. The cubication process consists in dropping all the terms proportional to $T_{2n+1}(y)$ for $n \geq 2$, so we approximate Eq. (6) retaining only the first two terms as follows [9, 10]

$$g(y) = \sum_{n=0}^{1} b_{2n+1} T_{2n+1}(y) = b_1 T_1(y) + b_3 T_3(y) = (b_1 - 3b_3)y + 4b_3 y^3 = \alpha y + \beta y^3$$  \hspace{1cm} (9)

where

$$b_1 = 1 + \frac{5}{8} \lambda, \quad b_3 = \frac{5}{16} \lambda$$  \hspace{1cm} (10)

$$\alpha = 1 - \frac{5}{16} \lambda, \quad \beta = \frac{5}{4} \lambda$$  \hspace{1cm} (11)

The nonlinear differential equation in (1) can be then approximated by the nonlinear differential equation

$$\frac{d^2 y}{dt^2} + \alpha(\lambda)y + \beta(\lambda)y^3 = 0$$  \hspace{1cm} (12)

This last equation is now of the form of a cubic Duffing equation which can be exactly solved. As we can see, the cubication procedure consists in approximating the original nonlinear differential equation (1) by Eq. (12)—which is the nonlinear differential equation for the cubic Duffing oscillator. Then the approximate frequency and solution for the initial equation will be the exact frequency and solution for the cubic Duffing equation. It is well known that the nonlinear oscillator given by the Eq. (12) has a solution, which can be written in terms of the Jacobian elliptic function $cn$ [2]
\[ y(t) = \text{cn}(t \sqrt{\alpha + \beta}; m), \quad m = \frac{\beta}{2(\alpha + \beta)} \]  

(13)

and its frequency is given in terms of the complete elliptic integral of the first kind

\[ \omega = \frac{\pi \sqrt{\alpha + \beta}}{2K(m)} \]  

(14)

defined as follows

\[ K(m) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \]  

(15)

From Eq. (11) we can write the following approximate equations for the frequency and solution for the quintic Duffing oscillator

\[ \omega(\lambda) = \frac{\pi \sqrt{1 + \frac{15}{16} \lambda}}{2K\left(\frac{10\lambda}{16+15\lambda}\right)} \]  

(16)

\[ y(t) = \text{cn}\left(t \sqrt{1 + \frac{15}{16} \lambda}; \frac{10\lambda}{16+15\lambda}\right) \]  

(17)

As we can see these expression are very simple and easily computed with the help of programs such as MATHEMATICA.

3. Comparison with the exact and other approximate solutions
We illustrate the accuracy of the approach by comparing the approximate solutions previously obtained with the exact frequency \( \omega_e(\lambda) \) and the exact solution \( y_e(t) \) of Eq. (1).

By integrating Eq. (1) and using the initial conditions in Eq. (2), we arrive at

\[
\omega_e(\lambda) = \frac{\pi}{2} \left[ \int_0^1 \frac{du}{\sqrt{1 - u^2 + \frac{1}{3} \lambda (1 - u^6)}} \right]^{-1} \quad \text{(18)}
\]

For small values of \( \lambda \) it is possible to take into account the following power series expansions

\[
\omega_e(\lambda) = 1 + \frac{5}{16} \lambda - \frac{215}{3072} \lambda^2 + \ldots = 1 + \frac{5}{16} \varepsilon A^4 - \frac{215}{3072} \varepsilon^2 A^8 + \ldots \quad \text{(19)}
\]

\[
\omega(\lambda) = 1 + \frac{5}{16} \lambda - \frac{161.25}{3072} \lambda^2 + \ldots = 1 + \frac{5}{16} \varepsilon A^4 - \frac{161.25}{3072} \varepsilon^2 A^8 + \ldots \quad \text{(20)}
\]

For very large values of \( \lambda \) it is possible to take into account the following power series expansions

\[
\omega_e(\lambda) = \frac{\pi}{12} \frac{\Gamma(2/3)}{\Gamma(7/6)} \sqrt{\lambda} + \ldots = 0.746834 \sqrt{\lambda} + \ldots = 0.746834 \sqrt{\varepsilon A^2} + \ldots \quad \text{(21)}
\]

\[
\omega(\lambda) = \frac{\sqrt{15\pi}}{8K(2/3)} \sqrt{\lambda} + \ldots = 0.749605 \sqrt{\lambda} + \ldots = 0.749605 \sqrt{\varepsilon A^2} + \ldots \quad \text{(22)}
\]

Furthermore, we have the following equations

\[
\lim_{\lambda \to 0} \frac{\omega(\lambda)}{\omega_e(\lambda)} = 1, \quad \lim_{\lambda \to \infty} \frac{\omega(\lambda)}{\omega_e(\lambda)} = \frac{3\sqrt{5\pi} \Gamma(7/6)}{4K(2/3)\Gamma(2/3)} = 1.00371 \quad \text{(23)}
\]
In Figure 1 we plotted the relative error for the approximate frequency $\omega(\lambda)$ (Eq. (16)). As we can see from this figure, the maximum relative error is 0.37%.

Lai et al [3] have approximately solved the quintic Duffing oscillator applying the Newton-harmonic balancing approach and they obtained that the maximum relative errors for the approximate frequency are 5.86%, 1.08% and 0.23% for the first-, second- and third-order approximation, respectively. As we can see, only the third-order approximate frequency is better than that obtained in this paper, but its maximum relative error (0.23%) is not very different to the relative error (0.37%) obtained using the cubication method considered here. However, as we can see in reference [3], to obtain the third-order approximate frequency using Newton-harmonic balancing approach is more difficult than to obtain Eq. (16). However, we can see that the method considered in this paper is very simple and easy to apply.

The exact periodic solutions $y_e(t)$ achieved by numerically integrating Eq. (5), and the proposed approximate periodic solution $y(t)$ in Eq. (17) for one complete cycle are plotted in Figures 2 and 4 for $\lambda = 0.1$ and for $\lambda \to \infty$, respectively, whereas in Figures 3 and 5 we plotted the difference $\Delta = y_e - y$ for $\lambda = 0.1$ and for $\lambda \to \infty$, respectively. In these figures parameter $h$ is defined as $h = \omega_c t / (2\pi)$. In the limit $\lambda \to \infty$ we use the expression

$$y_\infty(t) = \text{cn} \left( \frac{3\sqrt{5}\pi \Gamma(7/6)h}{\Gamma(2/3)} \frac{2}{3} \right)$$  \hspace{1cm} (24)

for the approximate frequency. These figures show that Eq. (17) provides a good approximation to the exact periodic solution.

4. Conclusions

A cubication method for the quintic Duffing oscillator based on the Chebyshev series expansion of the restoring force has been analyzed and discussed and an approximate frequency-amplitude relationship has been obtained. In this procedure, instead of
approximately solve the original nonlinear differential equation, the original nonlinear equation is replaced by a related ‘cubic’ Duffing equation that approximates the ‘quintic’ Duffing equation closely enough to provide useful solutions. The cubic Duffing equation is exactly solved in terms of the complete elliptic integral of the first kind and the Jacobi elliptic function \( cn \) and these exact solutions are the approximate solutions to the quintic Duffing equation. Using this approach, accurate explicit approximate expressions for the frequency and periodic solution are obtained and these approximate solutions are valid for small as well as large amplitude of oscillation. Excellent agreement between the approximate frequency and the exact one has been demonstrated. The discrepancy of this approximate frequency with respect to the exact one is lower than 0.37%. Finally, using the relationship between the complete elliptical integral of the first kind and the arithmetic-geometric mean and the harmonic balance method closed-form expressions for the approximate frequency and the solution are obtained in terms of elementary functions.

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References


Figure captions

Figure 1.- Relative error for the approximate frequency obtained using the cubication procedure.

Figure 2.- Comparison of the approximate solution (dashed line and triangles) with the numerical exact solution (continuous line and squares) for $\lambda = 0.1$.

Figure 3.- Difference between normalized exact and approximate solutions for $\lambda = 0.1$.

Figure 4.- Comparison of the approximate solution (dashed line and triangles) with the numerical exact solution (continuous line and squares) for $\lambda \to \infty$.

Figure 5.- Difference between normalized exact and approximate solutions for $\lambda \to \infty$. 
FIGURE 1
FIGURE 2
FIGURE 3
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FIGURE 5