MULTIGRID METHODS FOR
MULTILEVEL CIRCULANT MATRICES

STEFANO SERRA-CAPIZZANO† AND CRISTINA TABLINO-POSSIO‡

Abstract. We introduce a multigrid technique for the solution of multilevel circulant linear systems whose coefficient matrix has eigenvalues of the form $f(x_j^{(n)})$, where $f$ is continuous and independent of $n = (n_1, \ldots, n_d)$, and $x_j^{(n)} = 2\pi j/n = (2\pi j_1/n_1, \ldots, 2\pi j_d/n_d)$, $0 \leq j_r \leq n_r - 1$. The interest of the proposed technique pertains to the multilevel banded case, where the total cost is optimal, i.e., $O(N)$ arithmetic operations (ops). $N = \prod_{r=1}^d n_r$, instead of $O(N \log N)$ ops arising from the use of FFTs. In fact, multilevel banded circulants are used as preconditioners for elliptic and parabolic PDEs (with Dirichlet or periodic boundary conditions) and for some two-dimensional image restoration problems where the point spread function (PSF) is numerically banded, so that the overall cost is reduced from $O(k(\varepsilon, n)N \log N)$ to $O(k(\varepsilon, n)N)$, where $k(\varepsilon, n)$ is the number of PCG iterations to reach the solution within an accuracy of $\varepsilon$. Several numerical experiments concerning one-rank regularized circulant discretization of elliptic 2-differential operators over one-dimensional and two-dimensional square domains with mixed boundary conditions are performed and discussed.

Key words. circulant algebra, two-grid and multigrid iterations, multilevel matrices

AMS subject classifications. 65F10, 65F15, 15A12

DOI. 10.1137/S1064827501388509

1. Prelude. Let $f$ be a $d$-variate trigonometric polynomial defined over the hypercube $Q^d$, with $Q = (0, 2\pi]$ and $d \geq 1$, and having degree $c = (c_1, c_2, \ldots, c_d)$, $c_r \geq 0$, $r = 1, \ldots, d$, with regard to the variables $u = (u_1, u_2, \ldots, u_d)$. From the Fourier coefficients of $f$,

$$a_j = \frac{1}{(2\pi)^d} \int_{Q^d} f(u) \exp(-i(j, u)) \, du,$$

with $(j, u) = \sum_{r=1}^d j_r u_r$, one can build the sequence of Toeplitz matrices $\{T_n(f)\}$, $n = (n_1, \ldots, n_d)$, where $T_n(f) \in C^{N(n) \times N(n)}$ and $N(n) = \prod_{r=1}^d n_r$. It is clear that the Fourier coefficient $a_j$ equals zero if the condition $|j| \leq c$ is violated (i.e., if there exists an index $r$ such that the absolute value of $j_r$ exceeds $c_r$). The matrix $T_n(f)$ is said to be the Toeplitz matrix of order $n$ generated by $f$ (see [32]) and can be conveniently written in terms of Jordan blocks and of their powers as follows:

$$T_n(f) = \sum_{|j| \leq c} a_j J_n^{[j]} = \sum_{|j_1| \leq c_1} \cdots \sum_{|j_d| \leq c_d} a_{(j_1, \ldots, j_d)} J_{n_1}^{[j_1]} \otimes \cdots \otimes J_{n_d}^{[j_d]}.$$

In the above relation, $\otimes$ denotes the tensor product; $J_n^{[j]}$ denotes the Jordan matrix of order $m$ whose $(s, t)$ entry equals 1 if $s - t = j$, and equals zero otherwise; while $J_n^{[j]}$, where $j$ and $n$ are multi-indices, is the tensor product of all $J_n^{[j]}$ for $r = 1, \ldots, d$.

More explicitly, the $2m - 1$ matrices $\{J_n^{[l]}\}$, $l = 0, \pm 1, \ldots, \pm (m - 1)$, are the canonical
basis of the linear space of $m \times m$ (unilevel) Toeplitz matrices, and the tensor notation emphasizes the $d$-level Toeplitz structure of $T_n(f)$. Indeed, the set $\{J_m^{[1]}\}$ is the canonical basis of the linear space of the $N(n) \times N(n)$ $d$-level Toeplitz matrices.

Accordingly, the $d$-level circulant matrix of size $N(n)$, generated by the same polynomial $f$ (see, e.g., [32]), is defined as

$$S_n(f) = \sum_{|j| \leq r} a_j Z_m^j = \sum_{|j_1| \leq c_1} \cdots \sum_{|j_d| \leq c_d} a_{(j_1,\ldots,j_d)} Z_{n_1}^{j_1} \otimes \cdots \otimes Z_{n_d}^{j_d},$$

where the matrix

$$Z_m = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \end{bmatrix}$$

is the cyclic permutation Toeplitz matrix that generates the unilevel circulants and that can be viewed as a one-rank correction of the basic Jordan block $J_m^{[1]}$.

In addition, if $F_m$ denotes the $m$th Fourier matrix whose $(s,t)$ entry is given by the quantity $(e^{2\pi t/m})/\sqrt{m}$ for $0 \leq s, t \leq m-1$, then it is well known that $S_n(f) = F_n F_m^T$, where $F_n = F_{n_1} \otimes \cdots \otimes F_{n_d}$ is the $d$-level Fourier matrix and $D_j^{[n]} = \text{Diag}_{0 \leq j \leq n - e^T} f(x_j^{[n]})$ with $e = (1,\ldots,1)^T \in \mathbb{N}^d$. Here, the vector relation $0 \leq j \leq n - e^T$ and the expression $x_j^{[n]} = 2\pi j/n = (2\pi j/n_1,\ldots,2\pi j/n_d)$ are intended componentwise.

Under the assumption that $c = \text{degree}(f)$ is such that $c_r \leq [(n_r - 1)/2]$ for any $r = 1,\ldots,d$, the matrix $S_n(f)$ is the Strang or natural circulant preconditioner of the corresponding Toeplitz matrix $T_n(f)$ [8]. We observe that the above-mentioned assumption $c_r \leq [(n_r - 1)/2]$ is fulfilled at least definitely, since each $c_r$ is a fixed constant and $n_r$ is the size at level $r$: in actuality, in real applications it is natural to suppose that $n_r$ is large if we assume that $S_n(f)$ comes from a discretization process of an infinite-dimensional problem.

Now, let $C_n(f)$ be the $n$th Cesaro sum of $f$ given by

$$C_n(f) = \sum_{j=(0,\ldots,0)}^{n-e^T} \frac{F_j(f)}{N(n)},$$

with $(F_j(f))(u) = \sum_{|i| \leq j} a_i e^{i(u)}$ being the $j$th Fourier expansion of $f$. Then degree$(C_n(f)) = \text{degree}(f)$ and $C_n(f) = S_n(C_n(f))$ is the Chan [9] optimal preconditioner of $T_n(f)$.

Besides Toeplitz linear systems (see, for instance, [7]), these banded circulant structures have been used in the context of preconditioning of finite differences discretizations of PDEs over hyperrectangular domains [5, 17, 18, 19, 21]. In this case, by the consistency condition, it is known that $f$ is a nonnegative trigonometric polynomial which vanishes at the origin and can be chosen positive elsewhere (see, e.g., [27]). Therefore, the singular circulant matrix $S_n(f)$ is usually replaced by

$$\tilde{S}_n(f) = S_n(f) + \left( \min_{\|x\|_\infty = 1} f(x^{[n]}_j) \right) \frac{e e^T}{N(n)},$$
which is positive definite and can be used as preconditioner.

On the other hand, $C_n(f)$ is always positive definite since $C_n(f)$, with $f$ being a nonnegative polynomial, can vanish if and only if $f$ is identically zero (see [25]). However, the clustering properties related to the modified Strang preconditioner are better than those of the optimal one in the case of nonnegative generating functions with zeros (for a rigorous analysis of this phenomenon refer to [31, 10]).

Now, if we consider PCG-like methods for the solution of a linear system $A_n x_n = b_n$, then the cost per iteration is given by

(a) the solution of $M_n y_n = \gamma_n$ with the preconditioner $M_n$;
(b) a constant number of matrix-vector products with matrix $A_n$;
(c) computations of lower order complexity (vector-vector operations, etc.).

In the case where $A_n = T_n(f)$, the overall cost of (b) and (c) is of $O(N(n))$ arithmetic operations (ops) due to the bandedness of $T_n(f)$, while the cost of (a) is of $O(N(n) \log N(n))$ ops due to the use of FFTs. The multigrid-type method that we propose in this paper reduces the cost of (a) to $O(N(n))$ ops when $M_n$ is a multilevel banded circulant; moreover, since the proposed technique is an iterative method, we also have more numerical stability with respect to the use of FFTs.

Indeed, the proposed technique can be also extended to the case where

$$
\tilde{M}_n = M_n + \sum_{t=1}^{s} \vartheta_t f_{t_i} \left[ f_{t_i}^H \right],
$$

$M_n$ being a banded circulant and $\sum_{t=1}^{s} \vartheta_t f_{t_i} \left[ f_{t_i}^H \right]$ being a special s-rank correction with $f_{t_i}^q$ denoting the $q$th Fourier column of $F_n$. Of course, this extension is of interest since it allows us to treat the case of the modified Strang preconditioner given in (1.4).

The paper is organized as follows. In section 2 we recall definitions and convergence results concerning two-grid and multigrid iterations. In section 3 we define our multigrid technique for unilevel circulants. Section 4 is devoted to the proof of convergence of the two-grid iteration, while in subsections 4.2 and 4.3 we briefly discuss the optimal choice of some parameters and the convergence analysis regarding the multigrid method ($V$ cycle). Section 5 addresses the extension of these results to the multilevel case, and section 6 is devoted to the related proofs of convergence. In section 7 some variations on the theme of the discussed algorithms are given: we consider a modification for dealing with both the non-Hermitian case and the case of structured low-rank corrections (subsection 7.1); we propose to view the multigrid algorithm in the framework of the multi-iterative approach [23] (subsection 7.2); and we discuss a further enriched version of the multigrid technique (subsection 7.3). Finally, in section 8 several numerical experiments on polynomially ill-conditioned linear systems coming from discretized differential and integral problems are reported and discussed.

2. Premises. Let us consider the generic linear system $A_n x_n = b_n$, where $A_n \in \mathbb{C}^{N(n) \times N(n)}$, $N(n) = \prod_{r=1}^{d} n_r$, and $x_n, b_n \in \mathbb{C}^{N(n)}$. Let $p^k_n \in \mathbb{C}^{N(n) \times N(k)}$, $k_r < n_r$ for any $r = 1, \ldots, d$, be a given full-rank matrix, and let us consider a class of iterative methods of the form

$$
x_n^{(j+1)} = V_n x_n^{(j)} + \tilde{b}_n := V_n (x_n^{(j)} , \tilde{b}_n),
$$

where $V_n := I_n - R_n^{-1} A_n \in \mathbb{C}^{N(n) \times N(n)}$ and $\tilde{b}_n := R_n^{-1} b_n \in \mathbb{C}^{N(n)}$. 


A two-grid method (TGM) is defined by the following algorithm [16]:

\[
\text{TGM}(V_{n,\text{pre}}^{\nu_{\text{pre}}}, V_{n,\text{post}}^{\nu_{\text{post}}}, p_{n}^{k})(x_{n}^{(j)})
\]

0. \( \hat{x}_{n} = V_{n,\text{pre}}^{\nu_{\text{pre}}}(x_{n}^{(j)}, \hat{b}_{n,\text{pre}}) \)
1. \( d_{n} = A_{n}\hat{x}_{n} - b_{n} \)
2. \( d_{k} = (p_{n}^{k})^{H}d_{n} \)
3. \( A_{k} = (p_{n}^{k})^{H}A_{n}p_{n}^{k} \)
4. Solve \( A_{k}y_{k} = d_{k} \)
5. \( \hat{x}_{n} = \hat{x}_{n} - p_{n}^{k}y_{k} \)
6. \( x_{n}^{(j+1)} = V_{n,\text{post}}^{\nu_{\text{post}}}(\hat{x}_{n}, \tilde{b}_{n,\text{post}}) \)

Steps 1–5 define the “coarse grid correction” that depends on the projector operator \( p_{n}^{k} \), while step 0 and step 6 consist, respectively, in applying \( \nu_{\text{pre}} \) times and \( \nu_{\text{post}} \) times a “presmoothing iteration” and a “postsmoothing iteration” of the generic form given in (2.1). The global iteration matrix of \( TGM := TGM(V_{n,\text{pre}}^{\nu_{\text{pre}}}, V_{n,\text{post}}^{\nu_{\text{post}}}, p_{n}^{k}) \) is then given by

\[
TGM(V_{n,\text{pre}}^{\nu_{\text{pre}}}, V_{n,\text{post}}^{\nu_{\text{post}}}, p_{n}^{k}) = V_{n,\text{post}}^{\nu_{\text{post}}} \left[ I_{n} - p_{n}^{k} ((p_{n}^{k})^{H}A_{n}p_{n}^{k})^{-1} (p_{n}^{k})^{H}A_{n} \right] V_{n,\text{pre}}^{\nu_{\text{pre}}}
\]

In [12, 13], using specific analytical properties of the generating function \( f \), the authors defined a fast TGM in the case where the matrix \( A_{n} \) belongs to the Toeplitz and \( \tau \) class (i.e., the algebra associated to the most known sine transform \([4]\) and generated by the Toeplitz matrix \( T_{\alpha}(\cos(u)) \)). In the present paper, we propose a multigrid technique in the case where \( A_{n} \) is a multilevel banded circulant matrix.

First, we recall some convergence results [22] from the theory of the algebraic multigrid method, which are the main theoretical tools applied in the following sections.

By \( \| \cdot \|_{2} \) we denote the Euclidean norm on \( \mathbb{C}^{m} \) and the associated induced matrix norm over \( \mathbb{C}^{m \times m} \). If \( X \) is positive definite, \( \| \cdot \|_{X} = \| X^{1/2} \cdot \|_{2} \) denotes the Euclidean norm weighted by \( X \) on \( \mathbb{C}^{m} \) and the associated induced matrix norm. Finally, if \( X \) and \( Y \) are Hermitian matrices, then the notation \( X \preceq Y \) means that \( Y - X \) is nonnegative definite. In the following sections we use some functional norms as well on the spaces \( L^{\infty}(Q^{d}) \) with positive integer \( d \): more precisely, the usual \( L^{\infty} \) norm \( \| \cdot \|_{\infty} \) defined as \( \| f \|_{\infty} = \sup_{x \in Q^{d}} | f(x) | \) and the weighted \( L^{1} \) norm \( \| \cdot \|_{1} \) defined as \( \| f \|_{1} = \frac{1}{\pi^{d/2}} \int_{Q^{d}} | f(x) | \) (the Haar measure).

**Theorem 2.1** (see [22]). Let \( A_{n} \) be a positive definite matrix of size \( N(n) \) and let \( V_{n} \) be defined as in the TGM algorithm. Suppose that there exists \( \alpha_{\text{post}} > 0 \) independent of \( n \) such that

\[
\| V_{n,\text{post}} x_{n} \|_{A_{n}}^{2} \leq \| x_{n} \|_{A_{n}}^{2} - \alpha_{\text{post}} \| x_{n} \|_{A_{n}, D_{n}^{-1} A_{n}}^{2} \quad \forall x_{n} \in \mathbb{C}^{N(n)},
\]

where \( D_{n} \) is the main diagonal of \( A_{n} \), and that there exists \( \alpha_{\text{pre}} > 0 \) independent of \( n \) such that

\[
\| V_{n,\text{pre}} x_{n} \|_{A_{n}}^{2} \leq \| x_{n} \|_{A_{n}}^{2} - \alpha_{\text{pre}} \| V_{n,\text{pre}} x_{n} \|_{A_{n}, D_{n}^{-1} A_{n}}^{2} \quad \forall x_{n} \in \mathbb{C}^{N(n)}.
\]

Assume that there exists \( \gamma > 0 \) independent of \( n \) such that

\[
\min_{y_{k} \in \mathbb{C}^{N(n)}} \| x_{n} - p_{n}^{k} y_{k} \|_{D_{n}}^{2} \leq \gamma \| x_{n} \|_{A_{n}}^{2} \quad \forall x_{n} \in \mathbb{C}^{N(n)}.
\]
Then $\gamma \geq \alpha_{\text{post}}$ and

$$\|TGM(V_{n,\text{pre}}^\nu, V_{n,\text{post}}^\nu, P_n^k)\| A_n \leq \sqrt{1 - \frac{\alpha_{\text{post}}}{\gamma}} \frac{1}{1 + \frac{\alpha_{\text{pre}}}{\gamma}}$$

3. Multigrid method for unilevel circulant matrices. Let $A_n = S_n(f)$ be a unilevel circulant matrix generated by a univariate trigonometric polynomial $f$. In order to provide a general method to obtain a projector operator from an arbitrary banded circulant matrix $P_n$, for some bandwidth independent of $n$, we introduce the operator $T_n^k \in \mathbb{R}^{n \times k}, k = n/2$, where

$$(T_n^k)_{i,j} = \begin{cases} 1 & \text{for } i = 2j - 1, j = 1, \ldots, k, \\ 0 & \text{otherwise,} \end{cases}$$

and we define the projector operator as $P_n^k = P_n T_n^k \in \mathbb{C}^{n \times k}$.

In analogy with the $\tau$ case, proposed in [12, 13] and analyzed in more detail in [26], the operator $T_n^k$ represents a spectral link between the space of the frequencies of size $n$ and the corresponding space of frequencies of size $k$ according to the representation

$$[T_n^k]^T F_n = (\sqrt{2})^{-1}[1, 1] \otimes F_k,$$

where $F_m$ is the unilevel Fourier matrix of size $m$. This simple relation (see [28] for the details of the proof) is the key step in defining an algebraic multigrid method, since it allows us to obtain again a circulant matrix at the lower level.

Notice that relation (3.2) holds also if $F_n$ and $F_k$ are replaced by the Hartley transform matrices (see, e.g., [7] for the Hartley transform) of sizes $n$ and $k$, respectively. Therefore the same basic structure $T_n^k$ can be used for defining good prolongation operators for the multigrid solution of Hartley linear systems.

Moreover, it is important to preserve the "structure" also in a stronger sense: in applying the $V$ cycle multigrid procedure to $A_n := S_n(f)$ with nonnegative $f$, we have to require that the matrix $A_k$, obtained at the lower level, be circulant with a nonnegative generating function. Therefore, the choice of the circulant matrix $P_n$ is made by considering its eigenvalue function $\rho$, which sets the weights of the frequencies in the projector; in other words, the spectral behavior of $P_n$ selects the subspace in which the original problem is projected and solved. More precisely, if $f$ has a unique zero $x^0 \in (0, 2\pi]$, then we consider $\hat{x} = \pi + x^0$ (mod $2\pi$), and we set $P_n = S_n(p)$, where $p$ is trigonometric polynomial defined as

$$p(x) = \left(2 - 2 \cos(x - \hat{x})\right)^{[\beta/2]} \sim |x - \hat{x}|^{2[\beta/2]} \text{ over } (0, 2\pi]$$

with

$$\beta \geq \beta_{\text{min}} = \min \left\{ i \left| \lim_{x \rightarrow x^0} \frac{|x - x^0|^{2i}}{f(x)} < +\infty \right. \right\},$$

$$0 < |p|^2(x) + |p|^2(\pi + x \text{ (mod } 2\pi)) \text{.}$$

If $f$ has more than one zero in $(0, 2\pi]$, then we consider a polynomial $p$ which is the product of the basic polynomials of type (3.3), satisfying the condition (3.4) for any single zero and, globally, the condition (3.5).

Furthermore, the following set of simple observations is of crucial importance.
Relations (3.4) and (3.5) impose some restrictions on the zeros of \( f \). First, the zeros of \( f \) should be of finite order by (3.4), and this is always the case when \( f \) is a not identically zero polynomial. Secondly, if \( x^0 \) is a zero of \( f \), then \( f(\pi + x^0) \) must be positive; otherwise relationship (3.5) cannot be satisfied with any polynomial \( p \). However, the second restriction depends on the fact that we halve the dimension, so that if \( f \) has some zeros in \((0, 2\pi]\) that do not fulfill (3.5), then we have to change the “form” of the projector, that is, its smaller dimension. Compare, for instance, [12] and [6] concerning the case of symmetric Toeplitz structures: indeed in [6] for the generating function \( f(x) = x^2(\pi^2 - x^2) \), the authors consider a “block form” of the projector proposed in [12]. This new choice works much better and overcomes a problem due to the position of the zeros of \( f(x) = x^2(\pi^2 - x^2) \). Finally, we recall that a more general solution to the problem of the position of the zeros can be found in [20], where the author proposes to change the proportionality factor between the matrix sizes of the “finer” and of the “coarser” levels. The quoted choice of the polynomial \( p \) allows us to prove that the function \( \hat{f}/2 \), generating the circulant matrix \( A_k = S_k(\hat{f}/2) \), \( k = n/2 \), is nonnegative, together with some other useful properties as established in the following proposition.

**Proposition 3.1** (see [28]). Let \( k = n/2 \), \( p_n^k = S_n(p)T_n^k \), with \( p \) the polynomial product of the basic polynomial of type (3.3), with \( \beta \) being a constant independent of \( n \), satisfying the condition (3.4) for any zero of \( f \) and globally the condition (3.5), and let \( \hat{f} \) be nonnegative. Then we have the following:

1. The matrix \( 2(p_n^k)^H S_n(f)p_n^k \) coincides with \( S_k(\hat{f}) \), where \( \hat{f} \) is nonnegative and \( \hat{f}(x) = f(x/2)|p|^2(x/2) + f(\pi + x/2)|p|^2(\pi + x/2) \) for \( x \in (0, 2\pi] \). If \( f \) is a polynomial, then \( \hat{f} \) is a polynomial with a fixed degree depending only on the orders of the zeros of \( f \). In particular, if \( f \) has a unique zero, then the degree of \( \hat{f} \) is at most the same as the degree of \( f \).
2. If \( x^0 \) is a zero of \( f(x) \), then \( y^0 = 2x^0 \pmod{2\pi} \) is a zero of \( \hat{f} \).
3. The order of the zero \( y^0 \) of \( \hat{f} \) is exactly the same as the one of the zero \( x^0 \) of \( f \).

Notice that claim 1 also shows that the computational complexity of a \( V \) cycle iteration grows linearly with respect to the dimension of the problem: this is true in general if \( \beta \) is independent of \( n \) and of the level, but is no longer true if we allow \( \beta \) to depend on \( n \). However, as we will see in the following sections, conditions (3.4) and (3.3) are crucial for the convergence analysis. If condition (3.5) is not satisfied, then the new generating functions \( \hat{f} \) can have more zeros than \( f \), while if \( \beta \) is less than \( \beta_{\text{min}} \) (see (3.4)), then the order of the corresponding zero is not preserved at the lower levels.

Conversely, under assumptions (3.3), (3.4), and (3.5), the crucial data regarding the “nature” of \( \hat{f} \), namely, the position and order of the zeros of \( \hat{f} \), are formally established in claims 2 and 3, so that at the lower level the new projector is analogously defined by means of the same set of conditions.

Lastly, we have to define the smooting iterations of steps 0 and 6 of the TGM algorithm. In (2.1) we set \( V_n = I_n - \omega A_n \) (the Richardson method), so that \( V_n = S_n(1 - \omega f) \). In such a way, we consider the relaxed method

\[
x_n^{(j+1)} = x_n^{(j)} - \omega \left( A_n x_n^{(j)} - b_n \right),
\]

with \( \omega = \omega_{\text{pre}} \) for the presmoothing in step 0 and \( \omega = \omega_{\text{post}} \) for the postsmoothing in step 6.
4. Proof of convergence: The two-grid method. Here and in section 6, the main goal will be the verification of validity of assumptions (2.2), (2.3), and (2.4) in Theorem 2.1. In this way, according to the Ruge–Stüben algebraic theory [22], we will prove the optimality of the two-grid method, that is, its convergence rate independent of the size \( n \) of the algebraic problem.

**Lemma 4.1.** Let \( A_n := S_n(f) \), with \( f \) being a nonnegative trigonometric polynomial (not identically zero), and let \( V_n := I_n - A_n/\|f\|_\infty \) \((\omega = (\|f\|_\infty)^{-1}) \). If we choose \( \alpha_{\text{post}} \) so that \( \alpha_{\text{post}} \leq \|f\|_1/\|f\|_\infty \), then relation (2.2) holds true and the best value of \( \alpha_{\text{post}} \) is \( \alpha_{\text{post,best}} = \|f\|_1/\|f\|_\infty \).

**Proof.** First, we recall that since \( A_n = S_n(f) \), where \( f \) is a nonnegative polynomial, the entries of \( A_n \) are constants independent of \( n \). In particular we have \( D_n = a_0 I_n \), with \( a_0 = (2\pi)^{-1} \int_Q f = \|f\|_1 \) being positive since \( f \) does not vanish identically.

Now, by setting \( V_n = I_n - A_n/\|f\|_\infty \), the relation given in (2.2) is equivalent to writing

\[
(I_n - A_n/\|f\|_\infty) A_n (I_n - A_n/\|f\|_\infty) \leq A_n - \alpha_{\text{post}} A_n D_n^{-1} A_n,
\]

with \( \alpha_{\text{post}} > 0 \) independent of \( n \), that is, \( (I_n - A_n/\|f\|_\infty)^2 \leq I_n - \alpha_{\text{post}} A_n^{1/2} D_n^{-1} A_n^{1/2} \), or, more precisely,

\[
(I_n - A_n/\|f\|_\infty)^2 \leq I_n - (\alpha_{\text{post}}/a_0) A_n.
\]

By making some algebraic manipulations, the quoted relation can be rewritten as

\[
A_n^2/\|f\|_\infty^2 + [\alpha_{\text{post}}/a_0 - 2/\|f\|_\infty] A_n \leq 0,
\]

where the latter is equivalent to requiring that the inequality

\[
\lambda^2/\|f\|_\infty^2 + [\alpha_{\text{post}}/a_0 - 2/\|f\|_\infty] \lambda \leq 0
\]

hold for any eigenvalue \( \lambda \) of the Hermitian matrix \( A_n \) with \( \alpha_{\text{post}} > 0 \) independent of \( n \).

Since the eigenvalue function of \( \{A_n\}_n \) is \( f \), it follows that the topological closure of the union over all \( n \) of all the eigenvalues of \( A_n \) is exactly the interval \([\min f, \|f\|_\infty] \). Therefore a necessary and sufficient condition such that (4.1) holds for any \( n \) is that \( \alpha_{\text{post}} \leq a_0/\|f\|_\infty = \|f\|_1/\|f\|_\infty \).

The results of the preceding lemma can be easily generalized to the case of a generic \( \omega \) and with regard to both presmoothing and postsmoothing.

**Lemma 4.2.** Let \( A_n := S_n(f) \), with \( f \) being a nonnegative trigonometric polynomial (not identically zero), and let \( V_n := I_n - \omega A_n \), \( 0 < \omega < 2/\|f\|_\infty \). If we choose \( \alpha_{\text{post}} \) so that \( \alpha_{\text{post}} \leq a_0 \omega (2 - \omega \|f\|_\infty) \), then relation (2.2) holds true. Moreover, if we choose \( \alpha_{\text{pre}} \) so that \( \alpha_{\text{pre}} \leq \omega a_0 \min \{2, (2 - \omega \|f\|_\infty)/(1 - \omega \|f\|_\infty)\} \), then relation (2.3) holds true.

**Proof.** The proof is analogous to that of Lemma 4.1 (see also [22, p. 84] with \( D_n = a_0 I_n \) and \( \gamma = \|f\|_\infty \)).

**Lemma 4.3.** Let \( A_n := S_n(f) \), with \( f \) being a nonnegative trigonometric polynomial (not identically zero), and let \( p_n := S_n(p) T_n^k \) be the projector operator, with \( T_k \) defined in (3.1) and with \( p \) the polynomial product of the basic polynomial of type (3.3), satisfying the condition (3.4) for any zero of \( f \) and, globally, the condition (3.5). Then there exists a positive value \( \gamma \) independent of \( n \) so that inequality (2.4) holds true.
Proof. First, we recall that the main diagonal of $A_n$ is given by $D_n = a_0 I_n$ with $a_0 = (2\pi)^{-1} \int_0^\pi x = \|f\|_1 > 0$, so that $\| \cdot \|_{D_n} = a_0 \| \cdot \|_2$.

In order to prove that there exists $\gamma > 0$ independent of $n$ such that for any $x_n \in \mathbb{C}^n$

$$
\min_{y_k \in \mathbb{C}^n} \|x_n - p_n y_k\|_{D_n}^2 = a_0 \min_{y_k \in \mathbb{C}^n} \|x_n - p_n y_k\|_2^2 \leq \gamma \|x_n\|_{A_n}^2,
$$

we choose a special instance of $y_k$ in such a way that the previous inequality is reduced to a matrix inequality in the sense of the partial ordering of the real space of the Hermitian matrices. For any $x_n \in \mathbb{C}^n$, let $\overline{y}_k = y_k(x_n) \in \mathbb{C}^n$ be defined as

$$
\overline{y}_k = [(p_n^k)^H p_n^k]^{-1} (p_n^k)^H x_n.
$$

Therefore, (2.4) is implied by

$$
\|x_n - p_n^k \overline{y}_k\|_2^2 \leq (\gamma/a_0) \|x_n\|_{A_n}^2 \quad \forall x_n \in \mathbb{C}^n,
$$

where the latter is equivalent to the matrix inequality

$$
W_n(p)^H W_n(p) \leq (\gamma/a_0) S_n(f),
$$

with $W_n(p) = I_n - p_n^k [(p_n^k)^H p_n^k]^{-1} (p_n^k)^H$. Since, by construction, $W_n(p)$ is a Hermitian unitary projector, it holds that $W_n(p)^H W_n(p) = W_n^2(p) = W_n(p)$. As a consequence, the preceding matrix inequality can be rewritten as

(4.2) 

$$
W_n(p) \leq (\gamma/a_0) S_n(f).
$$

Let $F_n$ be the unilevel Fourier matrix of size $m$. Following the decomposition in (3.2), we find that $p_n^k = S_n(p) T_n^k$ can be expressed according to

$$
(p_n^k)^H = \frac{1}{\sqrt{2}} F_k \begin{bmatrix}
p_{0,n} & 0 & \ldots & 0 \\
p_1,n & p_{k,n} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & p_{k-1,n} & 0 & p_{n-1,n}
\end{bmatrix} F_n^H
$$

where $p_j,n = p(x_j^{[n]})$, $x_j^{[n]} = 2\pi j/n$.

Now, setting $\mu' = \mu + k$, $\mu = 0, \ldots, k - 1$, there exists a suitable permutation by rows and columns of $F_n^H S_n(f) F_n$ such that we can obtain a $2 \times 2$ block diagonal matrix whose $\mu$th diagonal block (\(\mu\)th and $\mu'$th rows and columns) is given by

$$
I_2 - \frac{1}{|p_{\mu,n}|^2 + |p_{\mu',n}|^2} \begin{bmatrix}
p_{\mu,n} & p_{\mu',n} \\
p_{\mu',n} & -p_{\mu,n}
\end{bmatrix}
$$

Therefore, by considering the same permutation by rows and columns of $F_n^H S_n(f) F_n$, the condition (4.2) is equivalent to requiring that there exist $\gamma > 0$ independent of $n$ such that

(4.3) 

$$
\gamma \begin{bmatrix}
f_{\mu,n} & 0 \\
0 & f_{\mu',n}
\end{bmatrix} \geq \frac{a_0}{|p_{\mu,n}|^2 + |p_{\mu',n}|^2} \begin{bmatrix}
|p_{\mu',n}|^2 & -p_{\mu,n} p_{\mu',n} \\
-p_{\mu,n} p_{\mu',n} & |p_{\mu,n}|^2
\end{bmatrix}
$$

holds true uniformly for any $\mu$ and for any $n$. Due to the continuity of $p$ and $f$, it is clear that the preceding set of inequalities can be reduced to a unique inequality involving $2 \times 2$ Hermitian matrix-valued functions:

$$
\gamma \begin{bmatrix}
f(x) & 0 \\
0 & f(\pi + x)
\end{bmatrix} \geq \frac{a_0}{|p(x)|^2 + |p(\pi + x)|^2} \begin{bmatrix}
|p(\pi + x)|^2 & -p(x) p(\pi + x) \\
-p(x) p(\pi + x) & |p(x)|^2
\end{bmatrix}
$$

with $x \in (0, 2\pi)$. 

The latter inequality is satisfied if and only if (a) the entry in position (1, 1) of the difference between the two members is nonnegative, and (b) the determinant of the difference matrix is nonnegative uniformly with regard to \( x \). We recall that, under conditions (3.4) and (3.5), the functions \(|p|^2(\pi + x)/f(x)\) and \(1/(|p|^2(x) + |p|^2(\pi + x))\) are bounded. Therefore, setting \( z_1 = \| |p|^2(\pi + x)/f(x) \|_\infty \) and \( z_2 = \| 1/(|p|^2(x) + |p|^2(\pi + x)) \|_\infty \), by direct calculation it follows that (a) holds true if \( \gamma \geq a_0 z_1 z_2 \) and (b) holds if \( \gamma \) is nonpositive or \( \gamma \geq 2a_0 z_1 z_2 \). Finally, a sufficient condition for fulfilling (2.4) is \( \gamma \geq 2a_0 z_1 z_2 \).

4.1. A critical analysis of Lemmas 4.1, 4.2, and 4.3. A critical view of Lemmas 4.1, 4.2, and 4.3 offers evidence that the Ruge–Stüber algebraic theory requires that \( A_n \) be nonsingular. This is equivalent to assuming that the zeros of \( f \) are not grid points of the circulant algebra. For the sake of simplicity we assume that \( f \) has a unique zero \( x_0 \). Therefore \( A_n = S_n(f) \) is singular if \( x_0 = x_{\lfloor n/j \rfloor} \) for some \( j \in \{0, \ldots, n - 1\} \), and then (recall relation (1.4)) instead of considering a linear system with coefficient matrix \( A_n = S_n(f) \), we solve the “stabilized” linear system with invertible coefficient matrix

\[
\tilde{S}_n(f) = S_n(f) + c f^{[n]}_j j^{[n]}_j H, \quad c > 0.
\]

The next step is to determine conditions on the parameter \( c \) such that Lemmas 4.1, 4.2, and 4.3 work unchanged for \( A_n = \tilde{S}_n(f) \).

For Lemmas 4.1 and 4.2, a direct check should convince the reader that the only constraint is

\[
(4.4) \quad c \in (0, \| f \|_\infty].
\]

Indeed we want \( A_n \) to be nonsingular; consequently we have \( c > 0 \), and in the proofs of these results the only key fact is that the spectrum of \( A_n \) is contained in

\[
[\min f, \| f \|_\infty] = [0, \| f \|_\infty]
\]

uniformly with regard to \( n \): therefore condition (4.4) is necessary and sufficient in order to make Lemmas 4.1 and 4.2 work unchanged with \( A_n = \tilde{S}_n(f) \).

For Lemma 4.3, in considering \( A_n = \tilde{S}_n(f) \) instead of \( A_n = S_n(f) \), the only modification concerns relation (4.3), where we have to replace the block

\[
\gamma \begin{bmatrix}
    f_{\mu,n} & 0 \\
    0 & f_{\mu',n}
\end{bmatrix}
\]

with the block

\[
\gamma \begin{bmatrix}
    f^{(c)}_{\mu,n} & 0 \\
    0 & f^{(c)}_{\mu',n}
\end{bmatrix},
\]

where \( f^{(c)}_{t,n} = f_{t,n} \) if \( t \neq j \) and \( f^{(c)}_{t,n} = c, f_{t,n} = 0 \) for \( t = j \).

Since \( c > 0 \) by (4.4), it follows that

\[
\gamma \begin{bmatrix}
    f^{(c)}_{\mu,n} & 0 \\
    0 & f^{(c)}_{\mu',n}
\end{bmatrix} \geq \gamma \begin{bmatrix}
    f_{\mu,n} & 0 \\
    0 & f_{\mu',n}
\end{bmatrix},
\]
and therefore
\[
\gamma \begin{bmatrix} f_{\mu,n} & 0 \\ 0 & f_{\mu',n} \end{bmatrix} \geq \frac{a_0}{|p_{\mu,n}|^2 + |p_{\mu',n}|^2} \begin{bmatrix} |p_{\mu',n}|^2 & -p_{\mu,n}p_{\mu',n} \\ -p_{\mu,n}p_{\mu',n} & |p_{\mu,n}|^2 \end{bmatrix}
\]
is directly implied by (4.3). This means that Lemma 4.3 stands with \(A_n = \tilde{S}_n(f)\) in place of \(A_n = S_n(f)\) and with the same choice of the constant \(\gamma\), i.e., \(\gamma \geq 2a_0z_1z_2\), where \(a_0, z_1,\) and \(z_2\) are the quantities defined at the end of the proof of Lemma 4.3.

For the sake of completeness we recall that in the numerical experiments, in analogy with (1.4), we define \(c\) as
\[
\left( \min_{\|t-j\| = 1} f\left( x^{[n]}_t \right) \right).
\]

We observe that this choice has also a “physical” motivation, since the resulting matrix \(A_n = \tilde{S}_n(f) + cf_j^{[n]} [f_j^{[n]}]^H\) shows the same asymptotical condition number as \(T_n(f)\).

Finally we want to point out that no further difficulties can be introduced by the recursive application of the multigrid procedure. In actuality, if \(y^0 = 2x^0\) is a grid point of \(\tilde{f}\), i.e.,
\[
y^0 = x^{[k]}_j
\]
for some \(j \in \{0, \ldots, k - 1\}\), then the zero \(x^0\) of the function \(f\) at the higher level is still a grid point since
\[
x^0 = y^0/2 + \epsilon\pi = (2j\pi/k)/2 + \epsilon\pi = (j + \epsilon k)\pi/n = x^{[n]}_j + \epsilon k, \quad \epsilon \in \{0, 1\},
\]
and consequently all the difficulties (if any) are inherited by the highest level. In conclusion, if we “stabilize” the original linear system, then all the matrices defined in the recursive application of any multigrid sweep are all invertible and “stabilized” in the same way. To see this it is enough to observe that
\[
A_n = \tilde{S}_n(f) = S_n(f) + cf_j^{[n]} [f_j^{[n]}]^H
\]
with \(c \in \{0, ||f||_\infty\}, j \in \{0, \ldots, n - 1\}\), and \(x^0 = x_j^{[n]}\), implies
\[
A_k = (p_k^{[n]})^H A_n p_k^{[n]} = S_k(\tilde{f}/2) + c' f_j^{[k]} [f_j^{[k]}]_{mod \ k}^H = \tilde{S}_k(\tilde{f}/2),
\]
with \(c' = c|p(x^0)|^2/2 \in (0, ||\tilde{f}/2||_\infty\), \(j \mod k \in \{0, \ldots, k - 1\}\), and \(y^0 = 2x^0 = x_j^{[k]} \mod k\).

### 4.2. Optimal smoothing iterations and asymptotic behavior

It is well known that the spectral radius of the iteration matrix of a stationary iterative method measures the asymptotic convergence rate of the iterative method [33]. The following simple result informs us that the positions of the presmoothing and postsmoothing can be interchanged or cumulated at the beginning or at the end of the whole TGM iteration without changing the asymptotic rate of convergence.

**Lemma 4.4.** With regard to the notation of the TGM algorithm, the following facts are true. The spectra of \(TGM(V_{n,pre}^{\nu_{post}}, V_{n,post}^{\nu_{pre}}, I, p_{k}^{N}), TGM(I, V_{n,pre}^{\nu_{pre}}, V_{n,post}^{\nu_{post}}, I, p_{k}^{N})\), and \(TGM(V_{n,pre}^{\nu_{pre}}, V_{n,post}^{\nu_{post}}, I, p_{k}^{N})\) coincide for any choice of \(V_{n,pre}, \nu_{pre}, V_{n,post}, \nu_{post},\) and \(p_{k}^{N}\).
Moreover, if the iterations of presmoothing and postsMOOTHing are of Richardson type (3.6), then \( V_{n, \text{pre}} \) and \( V_{n, \text{post}} \) commute and the spectra of \( TGM(V_{n, \text{pre}}, V_{n, \text{post}}, p_n) \) and \( TGM(V_{n, \text{post}}, V_{n, \text{pre}}, p_n) \) coincide.

Proof. From standard linear algebra (see, e.g., [3]) we know that the spectra of \( AB \) and \( BA \) are the same for every pair of complex square matrices \( A \) and \( B \) (spectral commutative property). Now, by setting \( C = [I_n - p_n^k ((p_n^k)^H A_n p_n^k)^{-1} (p_n^k)^H A_n] \), it holds that

\[
TGM(V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k) = V_{n, \text{post}}^\nu V_{n, \text{pre}}^\nu
\]

Therefore, by setting \( A = V_{n, \text{post}}^\nu C \) and \( B = V_{n, \text{pre}}^\nu \), the spectra of

\[
AB = TGM(V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k) \quad \text{and} \quad BA = TGM(I, V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k)
\]

coincide.

Analogously, by setting \( A = V_{n, \text{post}}^\nu \) and \( B = CV_{n, \text{pre}}^\nu \), the spectra of

\[
AB = TGM(V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k) \quad \text{and} \quad BA = TGM(V_{n, \text{post}}, V_{n, \text{pre}}, I, p_n^k)
\]

coincide. Finally, if the iterations of presmoothing and postsMOOTHing are of Richardson type, then \( V_{n, \text{pre}} \) and \( V_{n, \text{post}} \) are linear polynomials of the matrix \( A_n \), and therefore they commute. Consequently

\[
TGM(I, V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k) = TGM(I, V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k),
\]

so that a further application of the spectral commutative property leads to the desired result, i.e., the spectra of \( TGM(V_{n, \text{pre}}, V_{n, \text{post}}, p_n^k) \) and \( TGM(V_{n, \text{post}}, V_{n, \text{pre}}, p_n^k) \) coincide.

However, a more careful analysis is required. We are interested in optimal methods, i.e., methods converging within a given accuracy in a number of steps bounded by a constant independent of the size of the algebraic problem. Therefore, the asymptotic behavior, which is determined by the spectral radius of the iteration matrix, has a limited interest since the usual number of iterations is small (see section 8). Hence a pointwise control of the error at each iteration is more useful: in this respect we refer the reader to [15] for a beautiful analysis of iterative methods in the “transient phase,” i.e., before that phase the spectral radius begins to be crucial in the convergence behavior. In the following result and with the help of Lemma 4.2, we characterize (in the sense of the optimal majorization) the optimal parameters \( \omega_{\text{pre}} \) and \( \omega_{\text{post}} \) for the choice of the best presmoothing and postsMOOTHing iterations of type (3.6).

Lemma 4.5. With regard to the notation of the TGM algorithm, of Theorem 2.1, and of relation (3.6), the following facts are true.

1. If \( \nu_{\text{pre}} = 0 \) and \( \nu_{\text{post}} = 1 \) (only one step of postsMOOTHing), then

\[
\omega_{\text{post}, \text{best}} = 1/\|f\|_\infty, \quad \alpha_{\text{post}, \text{best}} = \|f\|_1/\|f\|_\infty.
\]

2. If \( \nu_{\text{pre}} = 1 \) and \( \nu_{\text{post}} = 0 \) (only one step of presmoothing), then

\[
\omega_{\text{pre}, \text{best}} = 3/[2\|f\|_\infty], \quad \alpha_{\text{pre}, \text{best}} = 3\|f\|_1/\|f\|_\infty.
\]

3. If \( \nu_{\text{pre}} + \nu_{\text{post}} = 1 \) (only one step of smoothing) and \( \delta = \alpha_{\text{post}, \text{best}}/\gamma \in [0, 2/3] \), then the optimal rate of convergence is reached with a single step of presmoothing and with \( \omega_{\text{pre}, \text{best}} = 3/[2\|f\|_\infty] \).
4. If \( \nu_{\text{pre}} + \nu_{\text{post}} = 1 \) (only one step of smoothing) and \( \delta = \alpha_{\text{post, best}}/\gamma \in [2/3, 1] \), then the optimal rate of convergence is reached with a single step of post-smoothing and with \( \omega_{\text{post, best}} = 1/\|f\|_\infty \).

Proof. We observe that the rate of convergence

\[
\sqrt{\frac{1 - \alpha_{\text{post}}/\gamma}{1 + \alpha_{\text{pre}}/\gamma}},
\]

indicated in Theorem 2.1, is decreasing as \( \alpha_{\text{post}} \) increases in \([0, \gamma]\) and as \( \alpha_{\text{pre}} \) increases in \([0, \infty)\). Therefore parts 1 and 2 of Lemma 4.5 are proved by looking for the maximum \( \alpha_{\text{post}} \) in \([0, \gamma]\) and of \( \alpha_{\text{pre}} \) in \([0, \infty)\). By taking into account Lemma 4.2, we know that

\[
\alpha_{\text{post}} \leq a_0 \omega (2 - \omega \|f\|_\infty),
\]

with \( \omega \in (0, 2\|f\|_\infty^{-1}) \), and therefore

\[
\alpha_{\text{post, best}} = \max_{\omega \in (0, 2\|f\|_\infty^{-1})} a_0 \omega (2 - \omega \|f\|_\infty) = a_0 / \|f\|_\infty = \|f\|_1 / \|f\|_\infty,
\]

which is reached for \( \omega_{\text{post, best}} = 1 / \|f\|_\infty \). Moreover, again by Lemma 4.2, we have

\[
\alpha_{\text{pre}} \leq \omega_0 \min \{2, (2 - \omega \|f\|_\infty) / (1 - \omega \|f\|_\infty)^2\},
\]

with \( \omega \in (0, 2\|f\|_\infty^{-1}) \), and therefore

\[
\alpha_{\text{pre, best}} = \max_{\omega \in (0, 2\|f\|_\infty^{-1})} \omega_0 \min \{2, (2 - \omega \|f\|_\infty) / (1 - \omega \|f\|_\infty)^2\}
\]

\[
= \max_{\omega \in (0, 2\|f\|_\infty^{-1})} \left\{ \begin{array}{ll}
2 \omega_0 & \text{if } \omega \leq 3/(2\|f\|_\infty) \\
\omega_0 (2 - \omega \|f\|_\infty) / (\omega \|f\|_\infty - 1)^2 & \text{if } 3/(2\|f\|_\infty) < \omega
\end{array} \right\}
\]

\[
= 3\|f\|_1 / \|f\|_\infty,
\]

which is reached for \( \omega_{\text{pre, best}} = 3/(2\|f\|_\infty) \). Consequently parts 1 and 2 are proved.

For demonstrating parts 3 and 4 we have to compare the rates of convergence in two cases: \( \nu_{\text{pre}} = 1 \) and \( \nu_{\text{post}} = 1 \). In the first case the best situation is

\[
r_{\text{pre}} = \frac{1}{\sqrt{1 + \alpha_{\text{pre, best}}/\gamma}} = \frac{1}{\sqrt{1 + \omega_{\text{pre, best}}/\gamma}},
\]

while in the second case the best is

\[
r_{\text{post}} = \sqrt{1 - \alpha_{\text{post, best}}/\gamma}.
\]

Setting \( \delta = \alpha_{\text{post, best}}/\gamma \), by simple manipulation, we deduce that \( r_{\text{pre}} > r_{\text{post}} \) if \( \delta \in (2/3, 1) \), \( r_{\text{pre}} = r_{\text{post}} \) if \( \delta = 2/3 \), and \( r_{\text{pre}} < r_{\text{post}} \) if \( \delta \in (0, 2/3) \), and the proof is complete.

4.3. Some remarks on the V cycle. In order to prove the convergence of the V cycle multigrid (recursive application of the TGM procedure) for circulant matrices, the first step is to show that the constants \( \gamma \) (see Lemma 4.3) and \( \alpha_{\text{pre}}, \alpha_{\text{post}} \) (see Lemma 4.2) can be chosen universally at each level of the recursive application of the procedure (see, e.g., [6]).

For instance, calling \( f_s \) the generating functions of the reduced circulant matrix at the level \( s, s = 0, \ldots, \log_2(n) - 1 \), this means that the constants \( \alpha_{s,\text{post, best}} = \|f_s\|_1 / \|f_s\|_\infty \), \( \alpha_{s,\text{pre, best}} = 3\|f_s\|_1 / \|f_s\|_\infty \) can be all bounded from below by a positive
constant independent of the level \( s \) and of the dimension \( n \). Finally, the proof of level independence requires that the constants \( \gamma_s \) can be bounded from above by a universal constant depending on neither \( s \) nor \( n \). This task can be done by following the same ideas as in section 3.1 of [26] (concerning the twin case of the \( \tau \) algebra), and its detailed analysis, in both the unilevel and multilevel settings, will be the subject of future work: we can just anticipate that condition (3.3) should be replaced by the more restrictive relation

\[
(4.5) \quad p(x) = (2 - 2 \cos(x - \hat{x}))^{[(\beta+1)/2]} \sim |x - \hat{x}|^{2[(\beta+1)/2]} \quad \text{over} \quad (0, 2\pi),
\]

where \( \hat{x} \) is the shifted point of \( x^0 \) and \( \beta \) is defined in (3.4), at least in the case of a generating function \( f \) having a unique zero at zero of finite order. A first answer in this direction was given in [1] with a condition stronger than (4.5), where \( [(\beta+1)/2] \) was simply replaced by \( \beta \).

However, in section 8 the numerical tests are performed for both the two-grid and the \( V \) cycle iterations, and they give evidence that our algorithm, with condition (4.5) in place of condition (3.3), shows an optimal rate of convergence in the \( V \) cycle case too.

5. The multilevel case. The multilevel case is handled in a natural manner by using tensorial arguments. The coefficient matrix \( A_n = S_n(f) \) is a multilevel circulant generated by a multivariate polynomial. The projector is constructed as \( P_n = P_n U_n \text{k} \), where \( P_n \) is a multilevel circulant with \( n = (n_1, \ldots, n_d) \) and the operator \( U_n \text{k} \) is defined as \( T_{n_1} \otimes \cdots \otimes T_{n_d} \), with \( k_r = n_r/2 \) and \( T_{n_r} \) being the unilevel operator given in (3.1) for any \( r = 1, \ldots, d \).

As in the unilevel case the key step is in highlighting the link between the space of frequencies of sizes \( n = (n_1, \ldots, n_d) \) and the corresponding space of frequencies of sizes \( k = (k_1, \ldots, k_d) \). From the definition of \( U_n \text{k} \) and from (3.2), it holds that

\[
[U_n \text{k}]^T F_n = (T_{n_1} \otimes \cdots \otimes T_{n_d})^T F_{n_1} \otimes \cdots \otimes F_{n_d}
= ((T_{n_1})^T F_{n_1}) \otimes \cdots \otimes ((T_{n_d})^T F_{n_d})
= (\sqrt{2})^{-d}([1, 1] \otimes F_{k_1}) \otimes \cdots \otimes ([1, 1] \otimes F_{k_d})
= (\sqrt{2})^{-d}F_k(([1, 1] \otimes I_{k_1})) \otimes \cdots \otimes (F_k([1, 1] \otimes I_{k_d}))
= (\sqrt{2})^{-d}F_k(([1, 1] \otimes I_{k_1}) \otimes \cdots \otimes ([1, 1] \otimes I_{k_d})),
\]

where \( F_k = F_{k_1} \otimes \cdots \otimes F_{k_d} \).

The choice of the circulant matrix \( P_n \) is performed by considering its eigenvalue function \( p \), according to the following notation: For a given point \( y \in (0, 2\pi]^d \) and for a given vector \( \delta \neq 0 \) of nonnegative integers, set \( g_{\delta, \gamma}(x) = \sum_{r=1}^{d} |x_r - y_r|^{\delta_r} \). If the function \( f \) has a unique zero \( x^0 \in (0, 2\pi]^d \), then we set \( P_n = S_n(p) \), where \( p \) is the polynomial defined as

\[
(5.1) \quad p(x) \sim \prod_{x \in C(x^0)} g_{2, 2|\gamma/2|}(x) \quad \text{over} \quad (0, 2\pi]^d,
\]

where

\[
(5.2) \quad \beta \geq \beta_{\min} = \min \left\{ \beta \left| \lim_{x \rightarrow x^0} \frac{g_{0, 2\beta}(x)}{f(x)} < +\infty \right. \right\},
\]

\[
(5.3) \quad 0 < \sum_{x \in C(x^0) \cup \{x^0\}} p^2(\hat{x}),
\]

MULTIGRID FOR MULTILEVEL CIRCULANTS
67
with $C(x^0)$ being the set of the “shifted points” of $x^0$. A formal definition is the following: $\hat{x} \in C(x^0)$ if and only if $\hat{x} \neq x^0$, and for any $r = 1, \ldots, d$ it holds that $\hat{x}_r \in \{x^0_r, \pi + x^0_r \mod 2\pi\}$.

For $d = 1$ it is evident that the unique shifted point is $\pi + x^0 \mod 2\pi$. For $d = 2$ we observe three shifted points, while in the general case the cardinality of $C(x^0)$ is $2^d - 1$. Notice that for any $\hat{x} \in C(x^0)$ we have $C(\hat{x}) = \{C(x^0) \setminus \{\hat{x}\}\} \cup \{x^0\}$. Concerning condition (5.2) we observe that it reduces to a minimization problem in $N^d$, and thus, it seems that more than one solution is possible; however, since the condition to be satisfied is a limit condition, i.e., $\lim_{x \to x^0} \frac{g_{x^0,x}(x)}{f(x)} < +\infty$, it follows that the minimal vector $\beta_{\text{min}}$ is unique, or, in other words, $i \geq \beta_{\text{min}}$ (in the componentwise partial ordering of $N^d$) for every $i$ satisfying the above condition.

If $f$ has more than one zero in $(0, 2\pi]^d$, then the corresponding polynomial $p$ will be the product of the basic polynomials of type (5.1), satisfying the condition (5.2) for any zero and, globally, the condition (5.3).

The quoted choice of $p$ induces some useful properties on the function $\hat{f}$, generating the circulant matrix $A_k = S_n(\hat{f}/2^d)$, $k = n/2$, at the lower level.

**Proposition 5.1.** Let $k = n/2$, $p_n^k = S_n(p)U_n^k$, with $p$ the polynomial product of the basic polynomial of type (5.1), with $\beta$ being a constant independent of $n$, satisfying the condition (5.2) for any zero of $f$ and, globally, the condition (5.3), and let $f$ be nonnegative. Then the following hold:

1. The matrix $2^d(p_n^k)^H S_n(f)p_n^k$ coincides with $S_k(\hat{f})$, where $\hat{f}$ is nonnegative and

\begin{equation}
\hat{f}(x) = f(x/2)p^2(x/2) + \sum_{y \in C(x/2)} f(y)p^2(y)
\end{equation}

for $x = (x_1, \ldots, x_d) \in (0, 2\pi]^d$. If $f$ is a polynomial, then $\hat{f}$ is a polynomial with a fixed degree depending only on the orders of the zeros of $f$.

2. If $x^0$ is a zero of $f(x)$, then $y^0 = 2x^0 \mod 2\pi$ is a zero of $\hat{f}$.

3. The order of the zero $y^0$ of $\hat{f}$ is exactly the same as that of the zero $x^0$ of $f$.

**Proof.** On the basis of the previous relation $[U_n^k]^{\top} F_n = (\sqrt{2})^{-d} F_k \Theta_n^k$ with $\Theta_n^k = ([1, 1] \otimes I_k_1) \otimes \cdots \otimes ([1, 1] \otimes I_k_d)$, the projected matrix $(p_n^k)^H S_n(f)p_n^k$ can be spectrally decomposed explicitly. Indeed, we have

\[
(p_n^k)^H S_n(f)p_n^k = [U_n^k]^{\top} S_n^H(p)S_n(f)S_n(p)U_n^k \\
= [U_n^k]^{\top} S_n(|p|^2 f)U_n^k \\
= [U_n^k]^{\top} F_n D_{|p|^2 f} F_n^{\top} U_n^k \\
= (2^{-d}) F_k \Theta_n^k D_{|p|^2 f} [\Theta_n^k]^{\top} F_k^{\top}.
\]

Due to the structure of $\Theta_n^k$, it holds that

\[
\Theta_n^k D_{|p|^2 f} [\Theta_n^k]^{\top} = \sum_{b \in \{0, 1\}^d} D_{b \cdot |p|^2 f}^{[n]},
\]

with $b = (b_1, \ldots, b_d)$, $b_r \in \{0, 1\}$, and

\[
D_{b \cdot |p|^2 f} = \text{Diag}_{k \leq j \leq k + e^T} (|p|^2 f)(x^0_j),
\]
with $\odot$ being the Hadamard componentwise product \cite{3}. In order to clarify the above multi-index notation we explicitly show the structure of $D^{[n]}_{\alpha \beta}$ in the case of $d = 2$: let $n = (n_1, n_2)$ and $k = (k_1, k_2)$ with $n_j = 2k_j$ and $k_j \geq 1$ for $j = 1, 2$; then

$$D^{[n]}_{\alpha \beta} = D^{[n]}_{\alpha \beta 00} + D^{[n]}_{\alpha \beta 01} + D^{[n]}_{\alpha \beta 10} + D^{[n]}_{\alpha \beta 11},$$

where

$$
\begin{align*}
D^{[n]}_{\alpha \beta 00} &= \text{Diag}_{0 \leq j_1 \leq k_1 - 1, 0 \leq j_2 \leq k_2 - 1} (|\alpha|^2 f)(x^{[n]}_{j_1, j_2}), \\
D^{[n]}_{\alpha \beta 01} &= \text{Diag}_{0 \leq j_1 \leq k_1 - 1, k_2 \leq j_2 \leq n_2} (|\alpha|^2 f)(x^{[n]}_{j_1, j_2}), \\
D^{[n]}_{\alpha \beta 10} &= \text{Diag}_{k_1 \leq j_1 \leq n_1 - 1, 0 \leq j_2 \leq k_2 - 1} (|\alpha|^2 f)(x^{[n]}_{j_1, j_2}), \\
D^{[n]}_{\alpha \beta 11} &= \text{Diag}_{k_1 \leq j_1 \leq n_1 - 1, k_2 \leq j_2 \leq n_2 - 1} (|\alpha|^2 f)(x^{[n]}_{j_1, j_2}).
\end{align*}
$$

Therefore, coming back to the general $d$-dimensional case, we conclude

$$
(p_n^k)^H S_n(f)p_n^k = \frac{1}{2^d} F_k \left( \sum_{b \in \{0,1\}^d} D^{|n|}_{b \alpha \beta} \right) F_k^H.
$$

Now, since $x^{[n]}_{j} = x^{[n]}_{j}/2 + \pi \odot b (\text{mod } 2\pi)$ for $0 \leq j \leq k - e^T$ with $j' = j + k \odot b$, it follows that the matrix $\hat{D}^{|n|}_{b \alpha \beta} = \text{Diag}_{0 \leq j_1 \leq k_1 - 1, 0 \leq j_2 \leq k_2 - 1} (|\alpha|^2 f)(x^{[n]}_{j_1, j_2})$ can be seen as $S_b(\hat{f})$, where $\hat{f}(x) = f(x/2)p^2(x/2) + \sum_{y \in C(x/2)} f(y)p^2(y)$ for $x \in (0, 2\pi]^d$.

From the expression of $\hat{f}$, and since $p(x) = 0$ for any $x \in C(x^0)$ by (5.1), it directly follows that $y^0 = 2x^0$ (mod $2\pi$) is a zero of $\hat{f}$, i.e., item 2 of the proposition is proved.

Moreover, by (5.3) we deduce that $|\alpha|^2 f(x^0) > 0$ since $|\alpha|^2 f(\hat{x}) = 0$ for any $\hat{x} \in C(x^0)$. Therefore, the order of the zero $y^0$ of $(|\alpha|^2 f)(x/2)$ is the same as the order of the zero $x^0$ of $f(x)$. However, by (5.2), we can see that $\sum_{y \in C(x/2)} f(y)p^2(y)$ has at $y^0$ a zero of order at least equal to the one of $f(x)$ at $x^0$. Since both the contributions in $\hat{f}(x)$ are nonnegative, the thesis of item 3 follows.

Finally we have to demonstrate the last part of item 1. Suppose that $f$ is a nonnegative trigonometric polynomial (and then real-valued) of degree $c$. Consequently, by looking at $f$ and $|\alpha|^2$ as $d$-variate Laurent polynomials on the Cartesian product of $d$ unit circles, we have $f(z) = \sum_{j = -c}^c a_j z^j$, $|\alpha|^2(z) = \sum_{j = -c}^c b_j z^j$, with $z^j = e^{i(j,x)}$, $x = (x_1, \ldots, x_d) \in (0, 2\pi]^d$, $j = (j_1, \ldots, j_d)$, $(j,x) = \sum_{r=1}^d j_r x_r$, and $a_j = a_{-j}$, $b_j = b_{-j}$. By a straightforward calculation we deduce the representations

$$(|\alpha|^2 f)(x/2) = (|\alpha|^2 f)(a_{j} z^{j/2}) = \sum_{j = -(c+l)}^{c+l} g_j z^{j/2},$$

with $g_j = g_{-j}$, and for any $y \in C(x/2)$

$$(|\alpha|^2 f)(y) = \sum_{j = -c+l}^{c+l} g_j (-1)^{\alpha y,j} z^{j/2},$$

where $\alpha y,j = \sum_{r:y_r = x_r/2 + \pi (\text{mod } 2\pi)} j_r$. In such a way, $\hat{f}$ is a polynomial since

$$\hat{f}(x) = \sum_{j = -(c+l); j = 2i}^{c+l} (1 + #C(x/2)) g_j z^{j/2} = \sum_{j = -(c+l)/2}^{(c+l)/2} (1 + #C(x/2)) g_{2j} z^{j}.$$
Now, in the multidimensional case if $f$ has an isolated zero, then the Laurent polynomial $f$ can have a zero of order at most $2c_r$ with respect to the single variable $z_r$, and therefore $\beta_r \leq c_r$ so that \( \text{degree}((p)^2) \leq l_{\text{max}} = 2 \lfloor c/2 \rfloor (2^d - 1) \). In the general case it holds that \( \text{degree}((p)^2) \leq l_{\text{max}} = 2 \lfloor c/2 \rfloor (2^d - 1) N_z \), $N_z$ being the number of zeros of the function $f$. Consequently we have \( \lfloor (c+l)/2 \rfloor \leq \max\{c,l-1\} \leq l_{\text{max}} - 1 \), so that the second part of item 1 is proved.

As in the unilevel case, relations (5.2) and (5.3) impose some restrictions on the zeros of $f$. First, the zeros of $f$ should be of finite order by (5.2). Second, if $x^0$ is a zero of $f$, then $f(\hat{x}) > 0$ for any $\hat{x} \in C(x^0)$; otherwise relationship (5.3) cannot be satisfied with any polynomial $p$. This second restriction can be removed by changing the “form” of the projector, that is, its smaller dimension.

From the point of view of the computational cost in a ($V$ cycle) multigrid implementation, we point out that there is no “band-explosion” at the lower levels. More precisely, calling $c(s)$ the degree of the generating function at the level $s$ and $l$ the degree of the square of projector polynomial (which is independent of $s$ by item 3), we observe the following cases: If $c(0) \geq l - 1$, then the sequence $c(s)$ is nonincreasing and therefore $c(s) \leq c(0)$; if $c(0) < l - 1$, then the sequence $c(s)$ is nondecreasing, but the following bound holds true: $l - 1 \geq c(s+1) \geq c(s) \geq c(0)$.

6. Proof of convergence: The multilevel case. As in section 4, we verify the validity of the key assumptions (2.2)–(2.4) in order to prove the optimality of the two-grid method, that is, its convergence rate independent of the size $n$ of the algebraic problem.

**Lemma 6.1.** Let $A_n := S_n(f)$, with $f$ being a $d$-variate nonnegative polynomial (not identically zero), and let $V_n := I_n - A_n/\|f\|_\infty$. If we choose $\alpha_{\text{post}}$ so that $\alpha_{\text{post}} \leq a_0/\|f\|_\infty$, then relation (2.2) holds true and the best value of $\alpha_{\text{post}}$ is $\alpha_{\text{post,best}} = \|f\|_1/\|f\|_\infty$.

**Proof.** The proof of Lemma 4.1 works unchanged in this case too.

**Lemma 6.2.** Let $A_n := S_n(f)$, with $f$ being a $d$-variate nonnegative trigonometric polynomial (not identically zero), and let $V_n := I_n - \omega A_n$, $0 < \omega < 2/\|f\|_\infty$. If we choose $\alpha_{\text{post}}$ so that

\[
\alpha_{\text{post}} \leq a_0 \omega (2 - \omega \|f\|_\infty),
\]

then relation (2.2) holds true. Moreover, if we choose $\alpha_{\text{pre}}$ so that

\[
\alpha_{\text{pre}} \leq \omega a_0 \min \left\{2, (2 - \omega \|f\|_\infty)/(1 - \omega \|f\|_\infty)^2\right\},
\]

then relation (2.3) holds true.

**Proof.** The proof of Lemma 4.2 works unchanged in this case too.

Now, before we discuss the verification of condition (2.4) with the chosen projector, we want to stress that the content of subsection 4.2 can be repeated verbatim in the multilevel setting as well.

**Lemma 6.3.** Let $A_n := S_n(f)$, with $f$ being a $d$-variate nonnegative polynomial (not identically zero), and let $p_n^k := S_n(p) U_n^k$ with $p$ the product of the basic polynomial of type (5.1), satisfying the condition (5.2) for any zero of $f$ and globally the condition (5.3). Then there exists a positive value $\gamma$ independent of $n$ so that inequality (2.4) holds true.

**Proof.** The remarks given in the first part of the proof of Lemma 4.3 can be applied analogously here, so that relation (2.4) is implied by the matrix inequality

\[
W_n(p)^H W_n(p) \leq (\gamma/a_0) S_n(f),
\]
with \( \gamma > 0 \) independent of \( n \) and where \( W_n(p) = I_n - p_n^k \left[ (p_n^k)^H p_n^k \right]^{-1} (p_n^k)^H \). Since, by construction, \( W_n(p) \) is a Hermitian unitary projector, the previous matrix inequality can be rewritten as

\[
W_n(p) \leq (\gamma/a_0) S_n(f).
\]

Let \( \mu = (\mu_1, \ldots, \mu_d) \), with \( 0 \leq \mu_i \leq k_i - 1 \), and let us denote by \( p_\mu \) and \( f_\mu \) the quantities \( p(x_\mu) \) and \( f(x_\mu) \), respectively, where \( x_\mu \in (0, 2\pi)^d \) with \( (x_\mu)_r = 2\mu_r \pi/n_r \). By performing the same (block) diagonalization procedure considered in Lemma 4.3 and by taking into account the tensor structure of all the matrices, we have that (6.1) is equivalent to the following set of 2\( d \times 2\( d \) matrix inequalities:

\[
I_{2^d} - \frac{1}{\|p[\mu]\|_2^2} (p[\mu](p[\mu])^T) \leq (\gamma/a_0) \text{Diag}(f[\mu]),
\]

where \( p[\mu] \) and \( f[\mu] \) are 2\( d \)-dimensional vectors whose entries are given by the evaluations of \( p \) and \( f \), respectively, over the points belonging to \( C(x_\mu) \cup \{ x_\mu \} \). Due to the continuity of \( f \) and \( p \), the previous inequalities can be reduced to the following unique inequality involving Hermitian 2\( d \times 2\( d \) matrix-valued functions:

\[
I_{2^d} - \frac{1}{\|p[x]\|_2^2} p[x](p[x]^T) \leq (\gamma/a_0) \text{Diag}(f[x]),
\]

with \( p[x] = (p(y_1), p(y_2), \ldots, p(y_{2^d}))^T \) and \( f[x] = (f(y_1), f(y_2), \ldots, f(y_{2^d}))^T \), where \( y_1 \equiv x \) and \( y_i \in C(x) \) for \( i = 2, \ldots, 2^d \). Now, by setting

\[
R(x) = \text{Diag}^{-\frac{1}{2}}(f[x]) \left( I_{2^d} - \frac{1}{\|p[x]\|_2^2} p[x](p[x]^T) \right) \text{Diag}^{\frac{1}{2}}(f[x])
\]

and multiplying by \( \text{Diag}^{-\frac{1}{2}}(f[x]) \) to the left and to the right of both sides of (6.2), the Sylvester inertia law tells us that (6.2) is equivalent to finding \( \gamma > 0 \) independent of \( n \) such that \( R(x) \leq (\gamma/a_0) I_{2^d} \).

Therefore, the lemma is proved if we show that the Hermitian matrix-valued function \( R(x) \) is uniformly bounded in the spectral norm, the latter being implied by the \( L^\infty \) boundedness of the generic entry of \( R(x) \).

For \( s \neq t \) we have

\[
R_{s,t}(x) = -\frac{p(y_s)p(y_t)}{\sqrt{f(y_s)f(y_t)} \|p[x]\|_2^2},
\]

where (5.3) implies that \( 1/\|p[x]\|_2^2 \in L^\infty \) and (5.2) implies that \( p(y_s)/\sqrt{f(y_t)} \) and \( p(y_t)/\sqrt{f(y_s)} \) belong to \( L^\infty \). Finally,

\[
R_{s,s}(x) = \sum_{y \in C(y_s)} \frac{p^2(y)}{f(y_s)} \|p[x]\|_2^2,
\]

which is again bounded by the simultaneous application of (5.2) and (5.3).

\[ \square \]

Remark 6.1. As observed in section 4.3, the optimal convergence of the (\( V \) cycle) multigrid algorithm requires slightly more severe conditions that can be resumed in the following statement: the constant vector \( \beta \) defined in (5.1) must be replaced by \( \beta + e^T \).

The proof of this fact uses the monotonicity of the circulant operator with regard to the symbol and can be directly translated from analogous arguments for the \( \tau \) class discussed in [26].
7. Extensions of the proposal. This section is devoted to some variations on the theme and extensions of the proposed algorithms with a twofold aim: we would like to accelerate the convergence, and we would like to include new cases of ill-conditioned problems where the methods can be successfully applied.

7.1. Complex-valued, nonpolynomial symbols. The first extension is really simple and concerns the multilevel banded case when the generating polynomial is not real-valued so that the matrix \( S_n(f) \) is not Hermitian. This situation occurs when discretizing differential operators not of elliptic type by finite differences and when the definition domain (with respect to the space variables) is multidimensional (see, e.g., [2, 18]).

If \( S_n(f) \) is not singular, then a system \( S_n(f)y_n = \gamma_n \) can be equivalently written as \( S_n^H(f)S_n(f)y_n = \gamma_n' \), where the cost of the computation of \( \gamma_n' \) is \( O(N(n)) \) ops due to the bandedness of \( S_n^H(f) \). Moreover, the matrix \( S_n^H(f)S_n(f) = S_n(|f|^2) \), so that we have reduced the problem to the case of multilevel circulants generated by a multivariate real-valued nonnegative polynomial.

The proposed technique can be also extended to the case where

\[
\hat{M}_n = M_n + \sum_{i=1}^{s} \vartheta_i f_i^{[n]} \left[f_i^{[n]}\right]^H,
\]

\( M_n \) being a banded circulant and \( \sum_{i=1}^{s} \vartheta_i f_i^{[n]} \left[f_i^{[n]}\right]^H \) being a special s-rank correction, with \( f_i^{[n]} \) denoting the \( q \)th Fourier column of \( F_n \). It is enough to point out that \( \left[T_n^k\right]^T M_n T_n^k = M_k + C_k \), where \( M_k \) is a multilevel circulant and \( C_k \) is still a low-rank circulant correction matrix of rank at most \( s \). Therefore, since the structure is preserved at the lower level, the recursive application of the multigrid is possible and the cost is asymptotically the same as in the case of \( s = 0 \), since \( s \) is a constant independent of \( n \). We recall that this extension is of interest since it allows us to treat the case of the modified Strang preconditioner defined in (1.4).

Finally, it is clear that the proposal can be extended to the dense case where \( f \) is continuous with isolated zeros of finite order. However, in this case the applicative interest is less relevant, since the computational cost is \( O(N(n) \log N(n)) \) with a constant hidden in the big “\( O \)” which can be substantially bigger than in the case of the multidimensional FFT.

7.2. A glimpse of multi-iterative methods. As proved in Lemma 4.2, the choice of \( \omega = 2||f||_\infty^{-1} \) is not acceptable since \( \alpha_{\text{pre}} = \alpha_{\text{post}} = 0 \) and the rate of convergence in Theorem 2.1 is 1. In other words, the iteration (3.6) with such an \( \omega \) is not a smoother. However, in [12] and [13], this iteration has been used in connection with a real smoother defined by \( \omega = ||f||_\infty^{-1} \). In the next result we prove that the combination of the two iterations leads to a smoother so that the cumulated iteration defined by the matrix \( V_n := (I_n - A_n/||f||_\infty)(I_n - 2A_n/||f||_\infty) \) is still in the framework of the algebraic multigrid theory.

**Lemma 7.1.** Let \( A_n := S_n(f) \), with \( f \) being a nonnegative trigonometric polynomial (not identically zero), and let \( V_n := (I_n - A_n/||f||_\infty)(I_n - 2A_n/||f||_\infty) \). If we choose \( \alpha_{\text{post}} \) so that \( \alpha_{\text{post}} \leq a_0/||f||_\infty \), then relation (2.2) holds true and the best value of \( \alpha_{\text{post}} \) is \( \alpha_{\text{post}, \text{best}} = ||f||_1/||f||_\infty \).

**Proof.** Since \( A_n = S_n(f) \), where \( f \) is a nonnegative polynomial, the entries of \( A_n \) are constants independent of \( n \). In particular we have \( D_n = a_0 I_n \), with \( a_0 = (2\pi)^{-1} \int_Q f = ||f||_1 \) being positive since \( f \) does not vanish identically.
Now, by setting $V_n = (I_n - A_n/\|f\|_\infty)(I_n - 2A_n/\|f\|_\infty)$, the relation given in (2.2) is equivalent to writing $V_nA_nV_n \leq A_n - \alpha_{\text{post}} A_n D^{-1} A_n$, with $\alpha_{\text{post}} > 0$ independent of $n$, that is, $V_n^2 \leq I_n - \alpha_{\text{post}} A_n^{1/2} D^{-1} A_n^{1/2}$, or, more precisely,

\[
V_n^2 \leq I_n - (\alpha_{\text{post}}/a_0) A_n.
\]  

(7.1)

By making some algebraic manipulations, the quoted relation can be rewritten as

\[
\frac{4}{\|f\|_\infty} A_n^4 - \frac{12}{\|f\|_\infty} A_n^3 + \frac{13}{\|f\|_\infty} A_n^2 + \left( \frac{\alpha_{\text{post}}}{a_0} - \frac{6}{\|f\|_\infty} \right) A_n \leq 0,
\]

where the latter is equivalent to requiring that the inequality

\[
\frac{4}{\|f\|_\infty} \lambda^4 - \frac{12}{\|f\|_\infty} \lambda^3 + \frac{13}{\|f\|_\infty} \lambda^2 + \left( \frac{\alpha_{\text{post}}}{a_0} - \frac{6}{\|f\|_\infty} \right) \lambda \leq 0
\]

hold for any eigenvalue $\lambda$ of the Hermitian matrix $A_n$, with $\alpha_{\text{post}} > 0$ independent of $n$. Since the eigenvalue function of $\{A_n\}_n$ is $f$, it follows that the topological closure of the union over all $n$ of all the eigenvalues of $A_n$ is exactly the interval $[\min f, \|f\|_\infty]$. Therefore a necessary and sufficient condition such that (7.1) holds for any $n$ is that

\[
\alpha_{\text{post}} \leq a_0 \min_{\lambda \in [\min f, \|f\|_\infty]} \left( \frac{4}{\|f\|_\infty} \lambda^4 + \frac{12}{\|f\|_\infty} \lambda^3 - \frac{13}{\|f\|_\infty} \lambda + \frac{6}{\|f\|_\infty} \right)
\]

\[
= \|f\|_1/\|f\|_\infty.
\]

However, the preceding result is in some sense not satisfactory. Indeed, it seems that the presence of the iteration (3.6) with $\omega = 2\|f\|_\infty^{-1}$ is totally useless since the convergence rate (defined by the parameter $\alpha_{\text{post}}$) is exactly the same as that found in Lemma 4.1 by using the iteration (3.6) with $\omega = \|f\|_\infty^{-1}$ alone. In other words, we double the arithmetic cost of step 6 of the algorithm, but the estimate of the convergence rate remains the same.

On the other hand, in [12, 13] and in our section 8 we observe that the presence of this further iteration accelerates the convergence substantially. This means that the real explanation of this phenomenon falls outside the convergence theory of the algebraic multigrid and indeed pertains to multi-iterative methods [23].

For instance, in the case of the $d$-dimensional discrete Laplacian, it is easy to prove that the postsmoothing $\omega = \|f\|_\infty^{-1}$ is strongly converging in the subspace of the high frequencies since $1 - \omega f$ vanishes at $\pi$ and that the coarse grid correction defined in [12, 13] (and analogous to the choice in this paper) strongly reduces the error in the low frequencies subspace. Therefore, if we look at the combination of the two iterations, which separately are slowly convergent on the global space $C^d$, we discover that the complementarity leads to a fast convergent two-grid method.

However, a finer analysis tells us that the global error is now essentially localized in the middle frequencies: the iteration (3.6) with $\omega = 2\|f\|_\infty^{-1}$ is not a smoother, but it is fast convergent just in the middle frequencies subspace since $1 - \omega f$ vanishes at $\pi/2$. Therefore its further use in step 6 or equivalently in step 0 increases very much the “spectral complementarity,” so that we obtain a real multi-iterative method whose convergence rate is really small. We will call an iteration having a spectral behavior complementary to both the coarse grid correction and the smoother an “intermediate iteration.” In this sense iteration (3.6) with $\omega = 2\|f\|_\infty^{-1}$ is an example of an “intermediate iteration” at least in the case of the discrete Laplacian.
7.3. Enriching the smoothing iterations dynamically. In this subsection we propose a modification of the multigrid algorithm for generic matrices. In steps 0 and 6 a fixed number, $\nu_{\text{pre}}$ and $\nu_{\text{post}}$, respectively, of basic iterations is used. Here we suggest applying $\nu_{\text{pre}}(j)$ and $\nu_{\text{post}}(j)$ iterations, where $j$ is the actual level in a single multigrid iteration. We call $\nu(j)$ the sum $\nu_{\text{pre}}(j) + \nu_{\text{post}}(j)$. Therefore the computational cost $C(r)$ of the modified multigrid procedure at dimension $r = (r_1, \ldots, r_d) \leq n = (n_1, \ldots, n_d)$ is described by the multi-index recursion

\[
\begin{aligned}
C(r) &\leq C(r/2) + \nu(j)cN(r), \\
C(n_{\min}) &= C_{\min},
\end{aligned}
\]

where $c$ is a constant taking into account the bandedness of the involved matrices in the Richardson method. Moreover, the term $C(r/2)$ follows from the recursive call at step 4, $n_{\min}$ is the minimal dimension where the problem is exactly solved, and $\nu(j)cN(r)$ is an upper bound for the cost of steps 0–3 and steps 5–6.

Our aim is to determine conditions on $\nu(j)$ for which the total cost $C(n)$ grows linearly with respect to the dimension, i.e., $O(N(n))$. To this end we must express $j$ as a function of the variable $r$ and of the maximal multi-index dimension $n$. By the vector relation $r = n/2^j - 1$ we have

\[ j = \log_2 \left( \frac{n_1}{r_1} \right) + 1, \]

and therefore the recursion (7.2) is equivalent to

\[
\begin{aligned}
C(r) &\leq C(r/2) + \nu \left( \log_2 (n_1/r_1) + 1 \right)cN(r), \\
C(n_{\min}) &= C_{\min}.
\end{aligned}
\]

By observing that $N(n/2^j) = N(n)/(2^d)^{j-1}$ and by taking into account (7.3), we deduce that

\[
C(n) \leq C(n/2) + \nu(1)cN(n)
\]

\[
\leq C(n/4) + \nu(2)cN(n/2) + \nu(1)cN(n)
\]

\[
\leq \cdots
\]

\[
\leq C_{\min} + c \sum_{j=1}^{\#(\text{levels})-1} \nu(j)N(n/2^{j-1})
\]

\[
= C_{\min} + cN(n) \sum_{j=1}^{\#(\text{levels})-1} \nu(j)/(2^d)^{j-1}.
\]

Finally, if $\nu(j)$ has any polynomial growth as $j$, then the quantity

\[
\sum_{j=1}^{\#(\text{levels})-1} \nu(j)/(2^d)^{j-1}
\]

is bounded by the sum of the convergent series $\sum_{j=1}^{\infty} \nu(j)/(2^d)^{j-1}$, and consequently $C(n) = O(N(n))$: see Table 8.7, where $\nu_{\text{pre}}(j) = 1 + \theta j$ with $\theta = 1, 2$, and Table 8.13, where $\nu_{\text{pre}}(j) = \nu_{\text{post}}(j) = 3 + \theta j$ with $\theta = 2$.

In order to complete the picture concerning the practical analysis of our technique, we have to discuss the memory requirements. Indeed, it is worth pointing out that
the storage requirements are practically negligible: the circulant structure can be stored by means of a stencil whose size is only related to the degree of the generating function, and consequently the “computation” of the reduced matrices simply consists in computing the new stencil through convolutions between $O(1)$ stencils.

Finally, in order to show all the involved constants hidden in the big “$O$” of the linear cost $O(N(n))$, we give a detailed expression of $C(n)$ in the case where $d = 2$ and $\nu(j)$ is a linear polynomial of the variable $j$. First we consider the computational cost of the setup phase, i.e., the computation of the stencils of all the restricted matrices, the factorization of the last matrix of size $n_{\min}$, and the computation of the parameters $\omega$ in (3.6), which is related to the evaluation of the infinity norm of all the generating functions. These computations have to be performed only once and their cost is

$$
(7.4) \quad K_1(b_{A,max} + 2b_{P,max}) \log(b_{A,max} + 2b_{P,max})(\#(\text{levels}) - 1) + n_{\min}^3/3
$$

$$
+ O(n_{\min}^2) + K_2b_{A,max}(\#(\text{levels}) - 1),
$$

with $K_1$ and $K_2$ positive small constants and where $b_{M,n}$ denotes the maximum number of nonzero entries in the rows of the matrix $M_n$, and $b_{M,max}$ denotes the maximum number of nonzero entries in the rows of the matrices $M_{n/2^j}$.

The computational cost per iteration in the solution phase is given at the biggest level $r = n$ by

$$
C(r) = (1 + \nu_{\text{pre},r} + \nu_{\text{post},r}) \prod(A_r) + N(r)(\nu_{\text{pre},r} + \nu_{\text{post},r}) + 2\prod(P_r) + C(r/2),
$$

and at the subsequent levels, in the case of recursion, is given by

$$
C(r) = (\nu_{\text{pre},r} + \nu_{\text{post},r}) \prod(A_r) + N(r)(\nu_{\text{pre},r} + \nu_{\text{post},r}) + 2\prod(P_r) + C(r/2).
$$

Here $N(r)$ is the actual dimension, $\nu_{\text{pre},r}$ and $\nu_{\text{post},r}$ are the number of presmoothing and postsMOOTHing iterations at size $N(r)$, and $\prod(M_r)$ is the cost of the matrix-vector product with matrix $M_r$.

More precisely, for $d = 2$, we obtain

$$
C(n) \leq \frac{N(n)}{9} \left\{ 9(1 + \nu_{\text{pre}} + \nu_{\text{post}})b_{A,n} + 4\left[ 3(2b_{P,n} + \nu_{\text{pre}} + \nu_{\text{post}}) + \theta_{\text{pre}} + \theta_{\text{post}} \right] + \left[ 3(\nu_{\text{pre}} + \nu_{\text{post}}) + 4(\theta_{\text{pre}} + \theta_{\text{post}}) \right] b_{A,max} + \left[ 3(\nu_{\text{pre}} + \nu_{\text{post}}) + 4(\theta_{\text{pre}} + \theta_{\text{post}}) \right] b_{P,max} \right\} - \frac{2N(n_{\min})}{9} \left\{ \left[ 3(\nu_{\text{pre}} + \nu_{\text{post}}) + 4(\theta_{\text{pre}} + \theta_{\text{post}}) \right] b_{A,max} + \left[ 3(\nu_{\text{pre}} + \nu_{\text{post}}) + 4(\theta_{\text{pre}} + \theta_{\text{post}}) \right] b_{P,max} \right\} + (4 + 3u)(\theta_{\text{pre}} + \theta_{\text{post}}) + N(n_{\min})^2.
$$

Here $N(n) = (2^d)^t$ is the original dimension, $N(n_{\min}) = (2^d)^{t_{\min}}$ is the dimension at which the system is resolved by the direct method, and $u = t - t_{\min} - 1 = \#(\text{levels}) - 1$ is the number of performed projections. Finally, $\theta_{\text{pre}}, \theta_{\text{post}}$ denote the optional constant increments of $\nu_{\text{pre}}, \nu_{\text{post}}$ at each sublevel, i.e., $\nu_{\text{pre}}(j) = \nu_{\text{pre}} + \theta_{\text{pre}}j$ and $\nu_{\text{post}}(j) = \nu_{\text{post}} + \theta_{\text{post}}j$.

8. Numerical experiments and conclusions. In this section we give numerical evidence of the theoretical results contained in the previous sections. In subsection 8.1 we consider the solution of circulant linear systems coming from one-dimensional and two-dimensional discretized differential problems. We show the optimality
of our multigrid procedure and of the multi-iterative procedure (“intermediate iteration” plus multigrid). Moreover, we test the multi-iterative technique with dynamic smoothing and dynamic intermediate iterations: the numerical results show that this modified procedure stabilizes the number of iterations to the same value as that of the two-grid procedure (which is the best possible result!).

In subsection 8.2 we consider a class of differential problems with Dirichlet-periodic boundary conditions. The resulting linear systems possess an interesting two-level structure, where the inner structure is \( \tau \) and the outer structure is circulant. Therefore we apply our multigrid/multi-iterative idea where the projector has a tensor structure as well: the inner level is \( \tau \) and is taken from [26] and the outer level is circulant and is defined according to the results of this paper. Moreover, the quoted examples show severe difficulties since the differential problems are semielliptic; i.e., the function \( a(x_1, x_2) \) has some isolated zeros. Therefore the resulting matrices have two distinct sources of ill-conditioning: the first is related to the order of the differential operator (and consequently to the order of the zero at zero of the generating function) and the second is originated by the order of the zeros of \( a(x_1, x_2) \). The interest of the related experiments relies on the existence of examples where a PCG method with standard preconditioner and the geometric multigrid [16] alone are theoretically not optimal and practically not efficient. Therefore we propose two effective strategies which are both of multi-iterative type: the first is a PCG where the preconditioner is of circulant-tau times diagonal structure and where the circulant-tau part is the constant coefficient discretization operator (for this part the basic multigrid in subsection 8.1 is employed); the second is a \( V \) cycle method in which the projector is exactly chosen as in subsection 8.1 but where we enrich the iterations in steps 0 and 6 of the algorithm in such a way that a very fast convergence is achieved. In the difficult semielliptic case both the techniques are optimal in time (i.e., a constant number of iterations is required with a total linear cost in order to reach the solution within a preassigned accuracy): the second technique is more effective both practically and theoretically from the point of view of the cost in time; the first is more like a black box (simpler for a user with less experience with multigrids) and is more efficient from the point of view of the memory requirements, since at every level of the multigrid iteration we need only a constant space for saving the current circulant matrix (in this way only a few vectors for storing the iterations are needed).

Finally, in subsection 8.3 we discuss how a discretized integral equation coming from an image restoration problem [14] with periodic boundary conditions can be reduced to a discretized differential problem from the viewpoint of our multigrid/multi-iterative algorithm.

**8.1. Circulant preconditioning.** Let us consider the precision order 2 finite differences discretizations of boundary value problems of the forms

\[
(8.1) \quad (-1)^q \left( \frac{d^q}{dx^q} \left( a(x) \frac{d^q}{dx^q} u(x) \right) \right) = h(x) \text{ on } \Omega = (0, 1)
\]

and

\[
(8.2) \quad (-1)^q \left( \sum_{i=1}^{2} \frac{\partial^m}{\partial x_i^m} \left( a(x_1, x_2) \frac{\partial^m}{\partial x_i^m} u(x_1, x_2) \right) \right) = h(x_1, x_2) \text{ on } \Omega = (0, 1)^2,
\]

with proper homogeneous boundary conditions on \( \partial \Omega \) and with \( a \) being uniformly elliptic and uniformly bounded in its definition set. Classical circulant preconditioners
for the mentioned finite differences discretization of (8.1) and (8.2) are defined by

\[ \tilde{S}_n(f) = S_n(f) + \left( \min_{|j| < n} f(x_j^n) \right) \frac{c e^T}{N(n)}, \]

where \( f(x) = f_q(x) = [2 - 2 \cos(x)]^q \) in the unilevel case and \( f(x_1, x_2) = f_q(x_1, x_2) = f_q(x_1) + f_q(x_2) \) in the two-level case. Here we consider the solution of linear systems of the form

\[ A_n x_n = b_n, \quad A_n = \tilde{S}_n(f) \]

by the proposed two-grid, multigrid, and multi-iterative procedures for different values of \( q \) and for several choices of the dimension \( n \) in both the unilevel and two-level settings. More specifically, in the unilevel case we take \( q = 1, 2, 3 \) and \( p(x) \) defined as \( p_{1,w}(x) = [2 + 2 \cos(x)]^w \) according to the conditions in (3.3) and (3.5). Moreover, in order to satisfy (3.4) it is clear that \( w \) should be at least 1 if \( q = 1, 2 \) and at least 2 if \( q = 3 \). It is obvious that the lower the value of \( w \) is, the greater the advantage will be from a computational viewpoint.

Notice that, by the consistency condition, the quoted generating functions have a zero at \( x^0 = 0 \), so that \( \varphi^0 = x^0 = 0 \), and consequently the function \( p(x) \) at all the lower levels is exactly the same as at the first level. This property is of great help for a simplified implementation of the related multigrid algorithm.

According to the previous definition of \( T_n^k \), the size of the coarse grid is \( N(k) = N(n)/2^k \), and in the (V cycle) multigrid procedure the basic dimension at which the system is solved by a direct method is \( N(n) = 16^d \) [11]. It is worth stressing that in the reported numerical experiments we have considered both an “intermediate iteration” (in place of a presmoothing iteration and with \( \nu_{\text{pre}} = 1 \)) defined by \( V_{n,\text{pre}} = S_n(1 - f/\|f\|_\infty) \) and a postsmoothing iteration (with \( \nu_{\text{post}} = 1 \)) defined by \( V_{n,\text{post}} = S_n(1 - f/\|f\|_\infty) \). Finally, the considered stopping criterion is \( \|b_n - A_n x_n\|_2/\|b_n\|_2 < \varepsilon \) with \( \varepsilon = 10^{-7} \).

The results in Table 8.1 confirm the optimality of the corresponding two-grid iteration in the sense that the number of iterations is uniformly bounded by a constant not depending on the size \( N(n) \), indicated in the first column. Following the suggestions in section 4.3 (see (4.5)), in order to have a (V cycle) multigrid optimality we must choose \( w \) at least equal to 1 if \( q = 1 \) and at least equal to 2 if \( q = 2, 3 \). Indeed, Table 8.2 confirms the necessity of these constraints since the only case where we observe a growth in the number of iterations with respect to \( n \) is the fourth column, where \( q = 2 \) and \( \beta = w = 1 \), so that (4.5) is not fulfilled, while (3.3) is.

Lastly, the effectiveness of the “intermediate iteration” is evident when we compare the previous results with those reported in Tables 8.3 and 8.4, where only two postsmoothing iterations with \( V_{n,\text{post}} = S_n(1 - f/\|f\|_\infty) \) are performed.

### Table 8.1: Number of iterations of the two-grid method in the one-dimensional case.

<table>
<thead>
<tr>
<th>( N(n) )</th>
<th>( q = 1 )</th>
<th>( q = 2 )</th>
<th>( q = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w = 1 )</td>
<td>( w = 2 )</td>
<td>( w = 3 )</td>
</tr>
<tr>
<td>32</td>
<td>2</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>64</td>
<td>2</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>128</td>
<td>2</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>256</td>
<td>2</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>512</td>
<td>2</td>
<td>15</td>
<td>19</td>
</tr>
</tbody>
</table>
Table 8.2
Number of iterations of the V cycle method in the one-dimensional case.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
<th>$q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 1$</td>
<td>$w = 2$</td>
<td>$w = 1$</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>2</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>64</td>
<td>7</td>
<td>15</td>
<td>41</td>
</tr>
<tr>
<td>128</td>
<td>8</td>
<td>15</td>
<td>77</td>
</tr>
<tr>
<td>256</td>
<td>8</td>
<td>15</td>
<td>134</td>
</tr>
<tr>
<td>512</td>
<td>8</td>
<td>15</td>
<td>224</td>
</tr>
</tbody>
</table>

Table 8.3
Number of iterations of the two-grid ($\nu_{\text{pre}} = 0$ and $\nu_{\text{post}} = 2$) method in the one-dimensional case.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
<th>$q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 1$</td>
<td>$w = 2$</td>
<td>$w = 1$</td>
</tr>
<tr>
<td>32</td>
<td>11</td>
<td>25</td>
<td>26</td>
</tr>
<tr>
<td>64</td>
<td>11</td>
<td>24</td>
<td>26</td>
</tr>
<tr>
<td>128</td>
<td>11</td>
<td>24</td>
<td>26</td>
</tr>
<tr>
<td>256</td>
<td>11</td>
<td>24</td>
<td>26</td>
</tr>
<tr>
<td>512</td>
<td>11</td>
<td>24</td>
<td>26</td>
</tr>
</tbody>
</table>

Table 8.4
Number of iterations of the V cycle ($\nu_{\text{pre}} = 0$ and $\nu_{\text{post}} = 2$) method in the one-dimensional case.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
<th>$q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 1$</td>
<td>$w = 2$</td>
<td>$w = 1$</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>11</td>
<td>25</td>
<td>26</td>
</tr>
<tr>
<td>64</td>
<td>12</td>
<td>24</td>
<td>48</td>
</tr>
<tr>
<td>128</td>
<td>13</td>
<td>24</td>
<td>89</td>
</tr>
<tr>
<td>256</td>
<td>13</td>
<td>24</td>
<td>159</td>
</tr>
<tr>
<td>512</td>
<td>13</td>
<td>24</td>
<td>269</td>
</tr>
</tbody>
</table>

With respect to the two-level setting, we will consider two different choices of the polynomial $p_w(x_1, x_2)$. First, we apply the polynomial

\[
p_w(x_1, x_2) = [(4 + 2 \cos(x_1) + 2 \cos(x_2))(4 - 2 \cos(x_1) + 2 \cos(x_2))]
\cdot (4 + 2 \cos(x_1) - 2 \cos(x_2))^w\]

with $w = 1, 2, 3$. The comments concerning Tables 8.5 and 8.6 apply quite analogously to the unilevel case. The only remarkable difference pertains to the V cycle multigrid iterations in the cases $q = 1$, $w = 1$ and $q = 3$, $w = 2$. Moreover, the convergence behavior can be greatly improved by considering the trick described in subsection 7.3. We will consider $\nu_{\text{pre}}(j) = \nu_{\text{post}}(j) + 1 + \theta j$. More explicitly, at each projection on a coarser grid the number of smoothing iterations performed at that level is increased by a fixed constant $\theta$. Notice that this strategy keeps the $O(N(n))$ computational cost unaltered, as proved in subsection 7.3, since the function $\nu_{\text{pre}}(j) + \nu_{\text{post}}(j)$ is linear as $j$ and therefore polynomial as $j$. In Table 8.7 we report the number of iterations of the V cycle multigrid where such increase of $\theta$ is, respectively, equal to 1 or 2: it is important to note the improvement with respect to the case of $\theta = 0$ (see Table 8.6) and the fact that, when the V cycle conditions are satisfied (see (5.2)–(5.3) and the modification of (5.1) reported in Remark 6.1), the number of iterations is the best possible since it equals the number of iterations of the two-grid method reported in
Table 8.5

Number of iterations of the two-grid method in the two-dimensional case.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
<th>$q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 1$</td>
<td>$w = 2$</td>
<td>$w = 3$</td>
</tr>
<tr>
<td>$32^2$</td>
<td>16</td>
<td>23</td>
<td>27</td>
</tr>
<tr>
<td>$64^2$</td>
<td>16</td>
<td>23</td>
<td>27</td>
</tr>
<tr>
<td>$128^2$</td>
<td>16</td>
<td>23</td>
<td>27</td>
</tr>
</tbody>
</table>

Table 8.6

Number of iterations of the $V$ cycle method in the two-dimensional case.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$q = 1$</th>
<th>$q = 2$</th>
<th>$q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 1$</td>
<td>$w = 2$</td>
<td>$w = 3$</td>
</tr>
<tr>
<td>$16^2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$32^2$</td>
<td>16</td>
<td>23</td>
<td>27</td>
</tr>
<tr>
<td>$64^2$</td>
<td>24</td>
<td>23</td>
<td>86</td>
</tr>
<tr>
<td>$128^2$</td>
<td>33</td>
<td>23</td>
<td>255</td>
</tr>
<tr>
<td>$256^2$</td>
<td>43</td>
<td>23</td>
<td>646</td>
</tr>
<tr>
<td>$512^2$</td>
<td>54</td>
<td>23</td>
<td>983</td>
</tr>
</tbody>
</table>

Table 8.7

Number of iterations of the $V$ cycle method with increasing $\nu$ of a constant $\theta$ in the two-dimensional case.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$q = 1$</th>
<th>$q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 1$</td>
<td>$w = 2$</td>
</tr>
<tr>
<td>$16^2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$32^2$</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$64^2$</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>$128^2$</td>
<td>18</td>
<td>16</td>
</tr>
<tr>
<td>$256^2$</td>
<td>19</td>
<td>16</td>
</tr>
<tr>
<td>$512^2$</td>
<td>29</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 8.5. We notice also that the same strategy applied in the case $q = 2$, $w = 1$ does not lead to a number of iterations constant with respect to the dimension $N(n)$. Nevertheless, it can be observed that such a number of iterations depends only on the number of considered sublevels.

An alternative choice is given by considering the most trivial extension of the unilevel projector to the two-level setting as

$$p_n^k = [S_{n_1}(p_w^{1d}(x_1))T_{n_1}^k] \otimes [S_{n_2}(p_w^{1d}(x_2))T_{n_2}^k],$$

with $p_w^{1d}(z) = [2 + 2 \cos(z)]^w$. In fact, $p_n^k = (S_{n_1} \otimes S_{n_2})U_n^k$, where $S_{n_1} \otimes S_{n_2} = S_n(p_w)$ with

$$p_w(x_1, x_2) = [(2 + 2 \cos(x_1))(2 + 2 \cos(x_2))]^w, \quad w = 1, 2, 3.$$  

Notice that such a choice is less expensive from a computational point of view, since the degree of the former is $3w$, while the degree of the latter is just $w$. Moreover, the infinite number of zeros of $p_w$ in (8.4) does not increase the number of zeros in $\hat{f}$ in (5.4). In such a case the comments concerning Tables 8.8 and 8.9 are of the same type as those in the unilevel case.
8.2. Elliptic and semielliptic PDEs. Let us consider the precision order 2 finite differences discretizations of the two-dimensional boundary value problem

\[
- \left( \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( a(x_1, x_2) \frac{\partial}{\partial x_i} u(x_1, x_2) \right) \right) = h(x_1, x_2) \text{ on } \Omega = (0, 1)^2,
\]

\[
u(x, 0) = u(x, 1) = 0, \quad x \in (0, 1),
\]

\[
u(0, y) = u(1, y), \quad y \in (0, 1),
\]

\[
\frac{\partial u(0, y)}{\partial y} = \frac{\partial u(1, y)}{\partial y}, \quad y \in (0, 1),
\]

with \( a \) being uniformly (semi)elliptic and uniformly bounded in its definition set, and with \( A_n(a) \) denoting its standard finite differences discretization matrix. In the following we will consider two multi-iterative techniques.

8.2.1. Circulant+tau preconditioning. We consider a PCG method with preconditioner defined as \( M_n(a) = D_n^{1/2}(a)A_n(1)\tilde{D}_n^{1/2}(a) \) or \( M_n(1) = A_n(1) \), where \( A_n(1) \) is the matrix \( A_n(a) \) in the case \( a(x_1, x_2) = 1 \), \( \text{i.e.}, A_n(1) = I_n \otimes S_n(2 - 2 \cos(x_1)) + \tau_n(2 - 2 \cos(x_2)) \otimes I_n \), and \( \tilde{D}_n^{1/2}(a) \) is the suitably scaled main diagonal of \( A_n(a) \) [24, 27, 29]. The results in Table 8.10 confirm the optimality of the proposed PCG technique with \( A_n(a) \) also in the case of \( a \geq 0 \), while the optimality with \( M_n(1) \) is guaranteed only under the assumption of \( a > 0 \) (the stopping criterion is \( \|b - A_n(a)x\|_2/\|b\|_2 < \varepsilon = 10^{-7} \)). For the resolution of the matrix \( A_n(1) \) we propose a multigrid technique designed as follows: The projector is constructed as \( P_n^k = P_n^\nu \), where \( P_n = I_n \otimes S_n(2 - 2 \cos(x_1)) + \tau_n(2 - 2 \cos(x_2)) \otimes I_n \), and \( \tilde{T}_n^k = T_{n_1}^k \otimes T_{n_2}^k \), with \( T_{n_1}^{k_1}, k_1 = n_1/2 \), defined as in (3.1) and \( \tilde{T}_{n_2}^{k_2}, k_2 = (n_2 - 1)/2 \), defined as

\[
(\tilde{T}_{n_2}^{k_2})_{i,j} = \begin{cases} 1 & \text{for } i = 2j, \ j = 1, \ldots, k_2, \\ 0 & \text{otherwise}, \end{cases}
\]

while the “intermediate iteration” (with \( \nu_{\text{pre}} = 1 \)) is defined by \( V_{n,\text{pre}} = I_n - 2A_n(1)/\|f\|_\infty \) and the postsmoothing iteration (with \( \nu_{\text{post}} = 1 \)) is defined by \( V_{n,\text{post}} = \)

---

**Table 8.8**
Number of iterations of the two-grid method in the two-dimensional case: “alternative projector.”

<table>
<thead>
<tr>
<th>( N(n) )</th>
<th>( q = 1 )</th>
<th>( w = 1 )</th>
<th>( q = 2 )</th>
<th>( w = 2 )</th>
<th>( q = 3 )</th>
<th>( w = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>15</td>
<td>23</td>
<td>24</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>15</td>
<td>23</td>
<td>24</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>15</td>
<td>23</td>
<td>24</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
</tbody>
</table>

**Table 8.9**
Number of iterations of the V cycle method in the two-dimensional case: “alternative projector.”

<table>
<thead>
<tr>
<th>( N(n) )</th>
<th>( q = 1 )</th>
<th>( w = 1 )</th>
<th>( q = 2 )</th>
<th>( w = 2 )</th>
<th>( q = 3 )</th>
<th>( w = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>32</td>
<td>15</td>
<td>23</td>
<td>24</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>15</td>
<td>23</td>
<td>24</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>15</td>
<td>23</td>
<td>44</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>15</td>
<td>23</td>
<td>66</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>512</td>
<td>15</td>
<td>23</td>
<td>89</td>
<td>44</td>
<td>44</td>
<td></td>
</tr>
</tbody>
</table>
Number of iterations of the PCG method: case $A_n(a)$, $a_1(x_1, x_2) = \sin(2\pi x_1) + \sin(2\pi x_2) + 4 > 0$, $a_2(x_1, x_2) = \sin(10\pi x_1) + \sin(10\pi x_2) + 4 > 0$, $a_3(x_1, x_2) = \sin(2\pi x_1 + 3\pi/2) + \sin(2\pi x_2 + 3\pi/2) + 2 \geq 0$.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$a_1(x_1, x_2)$</th>
<th>$M_n(a)$</th>
<th>$M_n(1)$</th>
<th>$a_2(x_1, x_2)$</th>
<th>$M_n(a)$</th>
<th>$M_n(1)$</th>
<th>$a_3(x_1, x_2)$</th>
<th>$M_n(a)$</th>
<th>$M_n(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 · 15</td>
<td>6</td>
<td>12</td>
<td>11</td>
<td>7</td>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32 · 31</td>
<td>6</td>
<td>11</td>
<td>12</td>
<td>7</td>
<td>63</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64 · 63</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>7</td>
<td>120</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128 · 127</td>
<td>6</td>
<td>10</td>
<td>11</td>
<td>7</td>
<td>227</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256 · 255</td>
<td>5</td>
<td>10</td>
<td>11</td>
<td>7</td>
<td>404</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>512 · 511</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>7</td>
<td>758</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of iterations of the V cycle method: case $A_n(a)$, $a_1(x_1, x_2) = \sin(2\pi x_1) + \sin(2\pi x_2) + 4 > 0$, $a_2(x_1, x_2) = \sin(10\pi x_1) + \sin(10\pi x_2) + 4 > 0$, $a_3(x_1, x_2) = \sin(2\pi x_1 + 3\pi/2) + \sin(2\pi x_2 + 3\pi/2) + 2 \geq 0$.

<table>
<thead>
<tr>
<th>$N(n)$</th>
<th>$a_1(x_1, x_2)$</th>
<th>$a_2(x_1, x_2)$</th>
<th>$a_3(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 · 15</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>32 · 31</td>
<td>41</td>
<td>35</td>
<td>1390</td>
</tr>
<tr>
<td>64 · 63</td>
<td>37</td>
<td>39</td>
<td>4095</td>
</tr>
<tr>
<td>128 · 127</td>
<td>35</td>
<td>41</td>
<td>&gt;5000</td>
</tr>
<tr>
<td>256 · 255</td>
<td>34</td>
<td>38</td>
<td>&gt;5000</td>
</tr>
</tbody>
</table>

$I_n - A_n(1)/\|f\|_{\infty}$, with $f(x_1, x_2) = 4 - 2\cos(x_1) - 2\cos(x_2)$. The number of V cycle-multi-iterative iterations required in order to satisfy the stopping criterion $\|b_n - A_n x_n\|_2/\|b_n\|_2 < \varepsilon = 10^{-7}$ is constant with respect to $N(n)$ and equals 16. It is worth stressing that the number of PCG iterations displayed in Table 8.10 is exactly the same in the case where the matrix $A_n(1)$ is directly dealt with by means of an LU factorization [11]; i.e., the required precision $\varepsilon$ is enough to guarantee the correct PCG convergence behavior.

Lastly, in Table 8.11, we show the number of iterations required to achieve the previous stopping criterion when the V cycle is directly applied to the matrix $A_n(a)$: the projector operator is defined as in the case $A_n(1)$; instead of a pre-smoothing iteration (with $\nu_{\text{pre}} = 1$), we use an “intermediate iteration” defined by $V_n = I_n - 2A_n(a)/(\max(a)\|f\|_{\infty})$, and in addition we consider the post-smoothing iteration (with $\nu_{\text{post}} = 1$) defined by $V_{n,\text{post}} = I_n - A_n(a)/(\max(a)\|f\|_{\infty})$, with $f(x_1, x_2) = 4 - 2\cos(x_1) - 2\cos(x_2)$.

It is evident that the use of our blending of multigrid for the constant coefficient case and PCG method for preconditioning the nonconstant case by $M_n(a)$ is crucial in the case of semi-elliptic problems: neither the multigrid nor the PCG methods alone are effective in handling such a case. In the strictly elliptic case our technique is as optimal as the multigrid alone. However, the computational cost is slightly higher, while the memory requirements are smaller since the matrices are never formed explicitly except at the lowest level.

From the computational point of view, it is evident that by directly applying the multigrid procedure to the finite differences matrix $A_n(a)$, the storage requirements are sensitively higher, since the whole sparse matrix has to be stored at each level. Moreover, in the PCG+multigrid strategy, the computational cost is given by

$$NIT_{\text{PCG}}[7N(n)+3+b_{A_n}N(n)+\text{RISOL}(M_n)]+b_{A_n}N(n)+\text{RISOL}(M_n)+\text{SETUP}(M_n),$$
where $\text{SETUP}(M_n)$ is of the same type as in (7.4) and

$$
\text{RISOL}(M_n) \leq \frac{N(n)}{9} \left[ 9(1 + \nu_{\text{pre}} + \nu_{\text{post}}) b_{A_n} + 4(3b_{P_n} + \nu_{\text{pre}} + \nu_{\text{post}}) \right] \\
+ \left[ (3(\nu_{\text{pre}} + \nu_{\text{post}})) b_{A,\text{max}} \right] - \frac{2N(n_{\text{min}})}{9} \left[ (3(\nu_{\text{pre}} + \nu_{\text{post}})) b_{A,\text{max}} \right] \\
+ 3(2b_{P_n} + \nu_{\text{pre}} + \nu_{\text{post}}) + 2N(n_{\text{min}}) b_{A,\text{max}},
$$

with $b_{A,\text{max}} = b_{A_n} = 5$, $b_{P_n} = 9$, $\nu_{\text{pre}} = \nu_{\text{post}} = 1$, and $\theta_{\text{pre}} = \theta_{\text{post}} = 0$.

On the other hand, in the multigrid strategy for $A_n(a)$, the computational cost of the setup phase has the same structure as in (7.4), but the convolution contribution $(b_{A,\text{max}} + 2b_{P,\text{max}}) \log(b_{A,\text{max}} + 2b_{P,\text{max}}) (\#(\text{levels}) - 1)$ is replaced by the substantially more expensive restriction of band matrices. We recall also that the memory requirements grow since the matrices have to be stored as band matrices ($O(N(n))$ bytes) instead of stencils ($O(1)$ bytes).

Finally, the computational cost of the solution phase is given by a part substantially equivalent to $\text{RISOL}(M_n)$ with the proper number of iterations $NIT_{\text{Multigrid}}$.

### 8.2.2. V cycle with enriched pre- and postsmoothing iterations.

Here we consider a V cycle method in which the projector is exactly chosen as in subsection 8.1, but where we enrich the iterations in steps 0 and 6 in such a way that a very fast convergence is achieved; therefore the technique is again of multi-iterative type. More precisely, we have the following set of choices for steps 0 and 6: $(i, j)$, $i, j \in \{a, b, c, d, e\}$. The label $a$ means one step of Jacobi (which is a smoother), $b$ means one step of damped Jacobi (which is an intermediate iteration), $c$ means one step of Gauss–Seidel (which is a smoother), $d$ means one step of CG (which is an intermediate iteration), and $e$ means one step of PCG with incomplete Choleski as preconditioner (which acts as intermediate iteration, but also helps the coarse grid correction operator to converge in the space where $A_n(a)$ is ill-conditioned).

As shown in Table 8.12, in all the cases including the difficult semielliptic case we observe asymptotic optimality (i.e., a constant number of iterations is required with a total linear cost in order to reach the solution within a preassigned accuracy). The considered technique (especially with the choices $cd$ and $ce$) is more effective than the one in subsection 8.1 both practically and theoretically from the point of view of the cost in time, while the procedure in subsection 8.1 is simpler for a user with less experience in multigrids and is more efficient from the point of view of the cost in space.

### 8.3. A look into a discretized integral problem.

As a final example of application, it is interesting to consider the case of a two-variate function possessing a
unique zero at \((\pi, \pi)\). This situation is of interest in image restoration problems when we consider periodic boundary conditions (refer, e.g., to [14]), i.e., when the data outside the domain of consideration are periodically extended. In this way (see [14, p. 258]) the resulting blurring matrix is a two-level circulant matrix. The interesting fact is that the choice of the proper projector according to (3.3) or (4.5) and (3.4)–(3.5) in the unilevel case produces at the lower level a circulant matrix \(A_k\) associated with a generating function having a unique zero at \(x = 0\). The same is true in the multilevel case. Therefore, starting from that level, the multigrid strategy is the same as in the previously considered cases, from the point of view of both practical and theoretical issues.

We consider a 256-by-256 image \(S\) representing a satellite and the two-level circulant blurring operator related to the symbol \(f(x_1, x_2) = \frac{1}{2} \psi(x_1, x_2)(1 + \psi(x_1, x_2))\) with

\[
\psi(x_1, x_2) = \left[\frac{2 + \cos(x_1) + \cos(x_2)}{4}\right]^3.
\]

We notice that the function \(f\) is nonnegative, has nonnegative Fourier coefficients, and has a unique zero of order 6 at \((\pi, \pi)\): therefore its one-rank correction \(\tilde{S}_n(f)\) is invertible, and it is a suitable blurring operator. The related linear system \(\tilde{S}_n(f)S = B\) is very ill-conditioned (\([N(n)]^3\) with \(N(n) = 256^2\)), and therefore we consider its Tikhonov regularized version

\[
([\tilde{S}_n(f)]^2 + \mu I)S = \tilde{S}_n(f)B.
\]

The original image \(S\) and its blurred version are displayed in Figure 8.1, while in Figure 8.2 we report the restored image with Tikhonov parameter \(\mu = 0\).

We stress that at level \(j\) we are applying \(\nu_{\text{post}}(j)\) steps of the ordinary conjugate gradient method as postsmoother (it is an intermediate iteration) and \(\nu_{\text{pre}}(j)\) steps of the Richardson method as presmoother (it is a real smoothing iteration); moreover the polynomial defining the projector at the highest level is \(p(x_1, x_2) = [(2 - 2 \cos(x_1))(2 - 2 \cos(x_2))]^2\) and \(\nu_{\text{pre}}(j) = \nu_{\text{post}}(j) = 3 + 2j\) (\(\theta = 2\)).

Finally, we remark that in the case of noise the regularized systems with \(\mu > 0\) (\(\mu\) Tikhonov parameter) have a better conditioning than in the case of \(\mu = 0\); therefore, our multigrid procedure, which is optimal for \(\mu = 0\), is robust since the number of iterations is bounded by a constant independent both of \(N(n)\) and of \(\mu\), as is clearly evident from Table 8.13, where the constant is 22 with a relative residual stopping criterion of \(10^{-7}\).
Fig. 8.2. Restored image with periodic boundary conditions.

Table 8.13
Number of iterations with residual in $\|\cdot\|_2$ norm less than $10^{-7}$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>#(Iter.)</th>
<th>Relative error w.r.t. $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4</td>
<td>4.99E-01</td>
</tr>
<tr>
<td>0.01</td>
<td>12</td>
<td>7.337E-02</td>
</tr>
<tr>
<td>0.001</td>
<td>20</td>
<td>4.668E-02</td>
</tr>
<tr>
<td>0.0001</td>
<td>21</td>
<td>3.125E-02</td>
</tr>
<tr>
<td>0.000001</td>
<td>22</td>
<td>8.466E-03</td>
</tr>
<tr>
<td>0</td>
<td>22</td>
<td>3.090E-06</td>
</tr>
</tbody>
</table>

REFERENCES


