On parallel asset-liability management in life insurance: a forward risk-neutral approach

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**Abstract**

In this paper we discuss the development of a valuation system of asset-liability management of portfolios of life insurance policies on advanced architectures. According to the new rules of the Solvency II project, numerical simulations must provide reliable estimates of the relevant quantities involved in the contracts; therefore, valuation processes have to rely on accurate algorithms able to provide solutions in a suitable turnaround time. Our target is to develop an effective valuation software. At this aim we first introduce a change of numéraire in the stochastic processes for risks sources, thus providing estimates under the forward risk-neutral measure that result in a gain in accuracy. We then parallelize the Monte Carlo method to speed-up the simulation process.

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1. Introduction

This work focuses on the development of parallel algorithms for the evaluation of profit-sharing life insurance policies (PS policies). This research activity is mainly motivated by the Solvency II project [12], the European project involving the outstanding Control Authorities, which aims to establish a revised set of capital requirements and risk management standards for insurance companies. The new rules of the Solvency II Directive Proposal are increasing more and more the request of stochastic Asset-Liability Management (ALM) models. The ALM, in the Professional Actuarial Specialty Guide [29], is “the practice of managing a business so that decisions on assets and liabilities are coordinated; it can be defined as the ongoing process of formulating, implementing, monitoring and revising strategies related to assets and liabilities in an attempt to achieve financial objectives for a given set of risk tolerances and constraints”. This can be obtained by stochastic modelling and simulation.

In this context, we investigate the computational issues in the ALM of PS policies. In these contracts, the benefits which are credited to the policyholder are indexed to the annual return of an investment portfolio: the company invests the reserve in a fund, called the segregated fund, and shares the yearly return with the policyholder. A profit-sharing policy is then a derivative contract, with underlying the segregated fund. In Italian insurance market, the crediting mechanism typically guarantees a minimum to the policyholder. It is worth emphasizing that PS policies have been widely analysed in the Solvency II report, since the minimum guarantee feature results in a risk mitigation which allows insurers to reduce the sum needed at the beginning of each year in order to meet the future liabilities. Profit-sharing policies require mark-to-market valuations in order to properly compute all the quantities related to risk management, thus obtaining reliable estimates. The literature on this topic is very rich, we recall [1–3,16,17,20,22] among the others. In particular, we refer to [16,17]. The
numerical simulation of these financial instruments leads to large-scale computational problems. Our target is to develop a valuation system capable of being properly scaled in order to balance accuracy and efficiency.

The typical computational kernels in the application we consider are Stochastic Differential Equations (SDE) and multidimensional integrals. In [13] we investigated the use of different methods for the numerical solution of the mentioned kernels. Starting from the analysis we carried out in [14] we focused on the development of a parallel algorithm for the evaluation of participating life insurance policies in distributed environments.

In the present paper, we deal with the numerical simulation of a real ALM portfolio; in this framework, we analyse a change of numéraire in the stochastic processes for risk sources, since the flexibility of this approach can be particularly valuable in a model with stochastic interest rates. In particular, we analyse the use of the numéraire which defines the forward risk-neutral measure [6,18,19,21]. Pricing under the forward measure can provide considerable gains in accuracy, since it allows to discount at a deterministic price deflator, even though the short rate is stochastic [18]. Moreover, we use parallel computing environments to obtain efficient simulation processes.

We propose parallel algorithms for asset-liability management of PS policies portfolios, under both risk-neutral and forward risk-neutral measure, based on the parallelization of Monte Carlo method.

This paper is organized as follows. In Section 2 we outline the asset-liability framework for the evaluation of PS policies; in Section 3 we introduce the stochastic processes for the risk sources; in particular, in Section 3.1 we present the risk-neutral setting, while in Section 3.2 we discuss the change of numéraire, describing the mathematical framework under the forward risk-neutral measure. In Section 4 we discuss the parallel Monte Carlo algorithm. In Section 5 we report the numerical results of a valuation of a real portfolio, in terms of accuracy and efficiency. We test both sequential algorithms based on risk-neutral and forward measure respectively, and the parallel ones implemented on a blade server with twelve processors. Finally, in Section 6 we give some conclusions.

2. Valuation framework

In this section we describe the main features of the mathematical formalization of the Italian contractual standard for profit-sharing policies. Here we briefly report the fundamental elements which are necessary to our discussion. We address to [16,17] for a complete description on the matter.

In a typical asset-liability framework, the basic elements for the evaluation of a PS policies portfolio are:

- the evaluation date $t$;
- the payment dates $t := \{t_1, t_2, \ldots, t_m\}$;
- the stream of premiums $X := \{X_1, X_2, \ldots, X_m\}$;
- the stream of benefits $Y := \{Y_1, Y_2, \ldots, Y_m\}$;
- the cash-flow stream generated by the segregated fund $Z := \{Z_1, Z_2, \ldots, Z_m\}$.

From the insurance company point of view, vectors $X$ and $Z$ are on the asset side, while $Y$ is on the liability one. The core of the evaluation problem is the computation of the assets and the liabilities at time $t$, for the control of the financial equilibrium between them. Then, at time $t$, the value of the assets:

$$V(t; Z) + V(t; X)$$

and the value of the liabilities:

$$V(t; Y)$$

must be computed. For providing estimates which are reliable and market-consistent, a mark-to-market stochastic model has to be considered. In such a framework, the value:

$$V_t := V(t; Y) - V(t; X)$$

that is, the value in $t$ of the difference between the obligations of the company and the obligations of the policyholders, gives the market value in $t$ of the outstanding net liabilities of the company: then, it actually represents the amount required to the company at time $t$ in order to meet the future liabilities. For this reason this quantity, called stochastic reserve, plays a crucial role for the insurance company.

In order to describe a simplified evaluation framework, we consider a single premium pure endowment insurance contract. Let the policy be written in $t_0 = 0$ for a life of age $x$; we denote with $T$ the term in years of the contract and with $F_t$ the market value at time $t$ of the segregated fund. Its rate of return in the period $[t - 1, t]$ is the random variable

$$l_t = \frac{F_t}{F_{t-1}} - 1$$

According to a typical interest crediting mechanism, the benefits are readjusted at the end of the year $t$ according to:

$$C_t = C_{t-1}(1 + \rho_t), \quad t = 1, \ldots, T$$

where $\rho_t$ is the readjustment rate defined as:
\[ \rho_t := \frac{\max\{\beta_t, \bar{i}\} - \bar{i}}{1 + \bar{i}} \]  
(2)

\( \bar{i} \) is the technical interest rate. The number \( \beta \in (0, 1) \) is the so-called participation coefficient: the product \( \beta_t \) in (2) represents the portion of the fund return which is credited to the policyholder, the remaining portion \( (1 - \beta) \bar{i} \) represents the company gain. Both \( \beta \) and \( \bar{i} \) are contractually defined. The final benefit is given by the insured sum \( C_0 \), raised at the financial readjustment factor \( \Phi_T \):
\[ C_T = C_0 \Phi_T \]  
(3)

Therefore, from (1)–(3), it follows:
\[ \Phi_T = \prod_{t=1}^{T} (1 + \rho_t) = (1 + \bar{i})^{-T} \prod_{t=1}^{T} (1 + \max\{\beta_t, \bar{i}\}) \]

At the purpose of discussing the evaluation of \( V_n \), we consider, for instance, \( V(t; Y) \) in \( t_0 \) since the generalization is straightforward. If we denote by \( \bar{\varepsilon}(x, T) \) the event "the aged \( x \) insured is alive at time \( T \)", then the liability of the company in \( T \) is given by:
\[ Y_T = C_0 \Phi_T \text{I}_{\bar{\varepsilon}(x, T)} \]
where \( \text{I}_{\bar{\varepsilon}(x, T)} \) is the indicator function of \( \bar{\varepsilon}(x, T) \), defining actuarial uncertainty. Both actuarial and financial uncertainty have to be taken into account, anyway these risk sources can be supposed to be mutually independent, thus they can be treated separately. Under the risk-neutral probability measure \( Q \), the functional \( V \) is expressed by the conditional expectation:
\[ V(0, Y_T) = C_0 \mathbb{E}_Q\left[ e^{-\int_0^T r(t) dt} \Phi_T \right] \]  
(4)

where \( r(t) \) is the spot rate and, following the usual actuarial notation, the symbol \( \mathbb{E}_p \) denotes the technical expectation. Therefore, the computation of the expected value of the readjustment factor, discounted at the risk-free deflator, is required.

3. Stochastic processes for risks

This section is devoted to the description of the risk sources involved in the ALM portfolio we consider. Risk sources time evolution is modelled by stochastic differential equations, under a certain probability measure. It is well-known that any positive martingale, with initial value one, defines a change of probability measure, thus, through the Radon–Nikodym derivative we can switch between suitable probability measures. This process is usually referred to as a change of numéraire \([6, 19]\), where the numéraire is a non-dividend paying asset with respect to which a probability measure is defined. In this paper, we compare the classical approach based on the risk-neutral measure to an approach based on the forward risk-neutral one. We start by introducing the models for risks in a general framework, then we deal with the specific choices of the numéraire.

The segregated fund is typically composed of stocks and bonds, thus, risk sources related to them both have to be taken into account. We refer to a market model with three sources of uncertainty: interest rate, stock-market and inflation risk. In the following of this section, we denote by
\[ \mathbf{W} = (W_r, W_p, W_i) \]  
(5)
a three-dimensional standard Brownian motion driving the time evolution processes of the short rate, inflation and stock-market respectively.

For the interest rate risk, we refer to the one-factor CIR model \([15]\). Denoted by \( r(t) \) the market short rate at time \( t \), it is assumed to follow the square-root mean-reverting diffusion process:
\[ dr(t) = \alpha(\gamma - r(t))dt + \sigma_r \sqrt{r(t)}dW_r(t) \]  
(6)
\( \gamma, \alpha \) and \( \sigma_r \) are positive constant parameters, with \( 2\alpha\gamma > \sigma_r^2 \), that ensures the positivity of the process. The function \( q \) which gives the market price of interest rate risk is supposed to satisfy the relation:
\[ q(r(t), t) = \frac{\pi \sqrt{r(t)}}{\sigma_r} \]  
with \( \pi \in \mathbb{R} \) a constant parameter. Stock market and inflation uncertainties are described via log-normal processes; in particular, we suppose inflation risk to evolve as \([10]\):
\[ \frac{dp(t)}{p(t)} = y_i dt + \sigma_i dW_i(t) \]  
(7)
where, denoted by \( y_0, y_\infty \) the levels of current and long-period expected inflation respectively, it is:
\[ y_t = y_\infty + (y_0 - y_\infty)e^{-\lambda t} \]

Finally, we refer to the Black and Scholes model for the stock-market risk [4]:
\[
\frac{dS(t)}{S(t)} = (\mu - \lambda)dt + \sigma_d dW_S(t)
\]  
where \( \lambda \) is the dividend yield.

### 3.1. Risk-neutral measure

The numéraire corresponding to the risk-neutral measure is the money market account:
\[
\beta(t, T) = e^{\int_t^T r(u)du}
\]  
In this framework, pricing at time \( t \) a security with payoff \( V(T) \) requires the computation of the expected value of the payoff, discounted at a stochastic price deflator given by the related numéraire (9).

It is well-known that the risk-adjusted parameters satisfy the following relations [19]:
\[
\bar{\alpha} = \alpha - \pi, \quad \bar{\gamma} = \frac{\gamma}{\bar{\alpha}}, \quad \bar{y}_t = y_t - \sigma_p^2
\]

We denote by
\[
\tilde{W} = (\tilde{W}_r, \tilde{W}_p, \tilde{W}_S)
\]
the vector containing the risk-neutral Girsanov transformations of the Brownian motions in (5).

We suppose only short rate and stock-market to be correlated; in particular, we consider a constant correlation between \( r \) and \( S \). We denote by \( \mathbf{L} \) the Cholesky factor of the correlation matrix; from the mentioned hypotheses it follows that the matrix \( \mathbf{L} \) has the following sparsity pattern:
\[
\mathbf{L} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
l_{13} & 0 & 1
\end{pmatrix}
\]  
with \( l_{33} = \sqrt{1 - l_{13}^2} \). Let \( \mathbf{W} = (\mathbf{W}_r, \mathbf{W}_p, \mathbf{W}_S) \) be the correlated Brownian motion; the following relation holds:
\[
d\mathbf{W} = \mathbf{L} \cdot d\mathbf{W} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
l_{13} & 0 & 1
\end{pmatrix} \begin{pmatrix}
d\mathbf{W}_r(t) \\
d\mathbf{W}_p(t) \\
l_{13}d\mathbf{W}_r(t) + l_{33}d\mathbf{W}_S(t)
\end{pmatrix}
\]

The risk-neutral dynamics of the correlated state variables (6)–(8) is:
\[
\begin{align*}
    dr(t) &= \bar{\alpha} \bar{\gamma} - r(t)dt + \sigma_r \sqrt{r(t)}d\mathbf{W}_r(t) \\
    dp(t) &= \bar{y}_t dt + \sigma_p d\mathbf{W}_p(t) \\
    dS(t) &= (r(t) - \lambda)dt + \sigma_S d\mathbf{W}_S(t)
\end{align*}
\]  

### 3.2. Forward risk-neutral measure

The forward risk-neutral measure for maturity \( T \) (the expression has been proposed by Jamshidian in [21]) is the probability measure associated with taking as numéraire a zero-coupon bond maturing at \( T \), with unitary face value [6,18,19,21]. In this case, it can be shown that the pricing formula of a security becomes:
\[
V(t) = B(t, T) E^F[V(T)]
\]
where \( B(t, T) \) is the value in \( t \) of the numéraire bond and \( E^F \) denotes expectation under the forward measure. As a consequence, the stochastic value of liabilities in (4), under the forward measure becomes:
\[
V(0, Y_T) = C_0 B(0, T) E^F[\Phi_T, T, P_x]
\]  
Pricing under the forward measure can provide considerable gains in accuracy, since it allows to discount at the deterministic price deflator \( B(t, T) \), even though the short rate is stochastic; indeed the forward measure is considered the right probability measure when evaluating a future random cash-flow in a stochastic interest rate environment [18].

The bond price dynamics is given, when \( r \) evolves according to the CIR model, under the risk-neutral measure, by [19]:
\[
\frac{dB(t,T)}{B(t,T)} = r(t)dt - A(t,T)\sigma_r \sqrt{r(t)}d\tilde{W}_r(t)
\]

where

\[
A(t,T) = \frac{2(e^{d(T-t)} - 1)}{(d + \sigma_r)\sqrt{r(T-t)}} + 2d
\]

and \(d = \sqrt{2^2 + 2\sigma_r^2}\). Applying Girsanov’s Theorem, it can be shown that the process \(W_t^f\) defined by

\[
dW_t^f = d\tilde{W}_r + \sigma_r \sqrt{r(t)}A(t,T)dt
\]

is a standard Brownian motion under the forward measure. It can be proved that, in general, a change of numéraire affects the drift of the processes only \([6,19]\). In the following, we briefly discuss how stochastic processes for inflation and stock-market are modified under the forward measure.

The components of the vector \(\mathbf{W}^f = (W_r^f, W_p^f, W_s^f)\), defined by

\[
\begin{align*}
    dW_r^f(t) & = d\tilde{W}_r(t) + \sigma_r \sqrt{r(t)}A(t,T)dt \\
    dW_p^f(t) & = dW_p^f(t) \\
    dW_s^f(t) & = dW_s^f(t)
\end{align*}
\]

are independent Brownian motions under the forward measure \([6]\). From (15) it follows:

\[
\begin{align*}
    d\tilde{W}_r(t) & = dW_r^f(t) - \sigma_r \sqrt{r(t)}A(t,T)dt \\
    dW_p^f(t) & = dW_p^f(t) \\
    dW_s^f(t) & = dW_s^f(t)
\end{align*}
\]

Combining (15) and (16), we obtain:

\[
d\tilde{W} = L \cdot dW = \begin{pmatrix} dW_r^f(t) - \sigma_r \sqrt{r(t)}A(t,T)dt \\ dW_p^f(t) \\ l_{13}(dW_r^f(t) - \sigma_r \sqrt{r(t)}A(t,T)dt) + l_{13}dW_s^f(t) \end{pmatrix}
\]

The dynamics of the correlated state variables becomes, thus, under the forward risk-neutral measure:

\[
\begin{align*}
    dr(t) & = \tilde{\gamma} - r(t)dt + \sigma_r \sqrt{r(t)}dW_r^f(t) - \sigma_r \sqrt{r(t)}A(t,T)dt \\
    dp(t) & = \tilde{\gamma} dt + \sigma_p dW_p^f(t) \\
    dS(t) & = (r(t) - \lambda) dt + \sigma_S[l_{13}(dW_r^f(t) - \sigma_r \sqrt{r(t)}A(t,T)dt) + l_{13}dW_s^f(t)]
\end{align*}
\]

from which it follows:

\[
\begin{align*}
    dr(t) & = [\tilde{\gamma} - \tilde{\gamma} + \sigma_r^2 A(t,T)] r(t)dt + \sigma_r \sqrt{r(t)}dW_r^f(t) \\
    dp(t) & = \tilde{\gamma} dt + \sigma_p dW_p^f(t) \\
    dS(t) & = (r(t) - \lambda - l_{13}\sigma_S \sqrt{r(t)}A(t,T)) dt + \sigma_S[l_{13}dW_r^f(t) + l_{13}dW_s^f(t)]
\end{align*}
\]

Let \(\mathbf{W}^f = (W_r^f, W_p^f, W_s^f)\) be the correlated Brownian motion under the forward measure; the following relation holds:

\[
d\tilde{W} = L \cdot d\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{13} & 0 & l_{13} \end{pmatrix} \begin{pmatrix} dW_r^f(t) \\ dW_p^f(t) \\ dW_s^f(t) \end{pmatrix} = \begin{pmatrix} dW_r^f(t) \\ dW_p^f(t) \\ l_{13}dW_r^f(t) + l_{13}dW_s^f(t) \end{pmatrix}
\]

The forward risk-neutral dynamics of the correlated state variables (6)-(8) is:

\[
\begin{align*}
    dr(t) & = [\tilde{\gamma} - (\tilde{\gamma} + \sigma_r^2 A(t,T))] r(t)dt + \sigma_r \sqrt{r(t)}dW_r^f(t) \\
    dp(t) & = \tilde{\gamma} dt + \sigma_p dW_p^f(t) \\
    dS(t) & = (r(t) - \lambda - l_{13}\sigma_S \sqrt{r(t)}A(t,T)) dt + \sigma_S dW_s^f(t)
\end{align*}
\]
Note that the dynamics of the state variables under the forward measure depends on the bond maturity $T$. In ALM models, where the valuation of $V(0,Y_T)$, by means of (14), has to be performed at least for $T \leq t$, that is at least at all the payment dates, this dependence implies that the simulation process of the dynamics of state variables changes with respect to the specific date of payment. This affects the computational complexity of the numerical evaluation process, as it will be discussed in Section 5.

4. The parallel algorithms

The simulation of an asset-liability portfolio results in a large-scale computational problem. Let $0$ and $T$ be respectively the beginning and the end of the simulation period, expressed in years. The interval $[0, T]$ is typically decomposed into $K$ periods, the period length $T/K$ is often equal to one month. According to the ALM strategy, then, every month the investment strategy is reviewed in dependence from assets and liabilities current value. Therefore, all the involved quantities, expressed by complex functions of random variables, have to be evaluated at each month up to year $T$. Moreover, at least at the end of each year the balance sheet and the statutory reserve have to be computed. Given the complexity of the profit-sharing rule, the expectations in (4) and (14) are computed by means of Monte Carlo method; in Fig. 1 an outline of a procedure for asset-liability management of a portfolio is shown.

In order to develop a parallel version of the simulation algorithm, we introduce parallelism in Monte Carlo method. Monte Carlo method for multivariate integration is based on the replacement of a continuous average with a discrete one over randomly selected points. Denoted by

$$I(f) = \int_{[0,1]^d} f(x)dx$$

the integral of function $f$ over the $d$-dimensional unit cube, then

$$I(f) \approx I_N(f) = \frac{1}{N} \sum_{i=1}^{N} f(Z_i)$$

with $Z_i, i = 1, \ldots, N$ uniform in $[0,1]^d$. Note that in Monte Carlo evaluation of an expectation involving a stochastic process with $K$ time periods, the resulting integral is $K$-dimensional [8]. In our application, we have $d = 3K$, since we consider three risk sources.

Monte Carlo methods have become the standard de facto for the computation of multidimensional integrals in financial applications. In particular, much effort has been devoted to the development of parallel MC algorithms for the evaluation of financial derivatives, as a rich scientific literature shows; we recall [11,24,30] and references therein among the others. The natural strategy for parallelizing Monte Carlo method is to distribute trajectories among processors; processors work

```
begin ALM portfolio
% N is the number of trajectories in MC scheme
% T_end is the end of the simulation period
for i = 1 to N do
  simulate risk factors ⇔ integrate (6), (7), (8) in [0,T_end]
  for each month:
    evaluate assets and liquidity
    apply the investment strategy
    reevaluate the insured capitals according to (1)
  for each year:
    evaluate the balance sheet
    evaluate the statutory reserve
endfor
compute the averages over the trajectories
end
```

Fig. 1. Sketch of a procedure for the numerical simulation of asset-liability portfolio of PS policies.
concurrently to compute the local averages that are afterwards combined to obtain the overall sample value. Therefore, Monte Carlo is generally considered “naturally parallel”. The core of MC is the generation of pseudo-random sequences capable to mimic random samples drawn from uniform distribution. Effective pseudo-random generators (PRG) must provide long-period sequences of uncorrelated values. Moreover, generation of pseudo-random sequences in a parallel setting must deal with both inter-processor and intra-processor correlations. Parallelization schemes based on cycle parameterization are mostly employed. These methods rely on the capability of certain generators to produce different full-period streams, that is, non-overlapping sequences, given different seeds. In this way, processors concurrently generate uncorrelated streams, thus providing scalable procedures [27,28,31]. Therefore, we employ a PPRG based on a cycle parameterization technique. Different pseudo-random numbers generators are well-known in literature. The substantial difference among them consists in the recursive relation that defines the pseudo-random sequence, given the initial seed. Lagged-Fibonacci (LF) generators are widely used for implementations on advanced architectures. We use the additive lagged-Fibonacci generator (ALFG) that is based on the following basic recursion:

\[ x_n = x_{n-j} + x_{n-k} \mod 2^m, \quad j < k \]

This generator has maximal period equal to \((2^k - 1)2^{m-1} \) and \(2^{(k-1)(m-1)}\) different full-period cycles [5].

In Fig. 2 the outline of the parallel MC algorithm is shown. Communication among processors is limited to the initialization phase, where basic common information are to be exchanged, and the final phase, when partial results are combined to compute the global average which gives the MC method result.

5. Numerical simulations

In the following we show some of the numerical experiments we performed on a real portfolio. We first discuss the impact of the change of numéraire on the accuracy of the estimates. In particular, we compare approaches based on risk-neutral and forward risk-neutral measures. We then turn to analyse the performances of the parallel software based on the more accurate algorithm.

The asset-liability framework for a real portfolio is obviously much more complex than the one we described in Section 2; we address to [9] for a deep analysis. Anyway, we point out that the strategies we propose here to improve accuracy and efficiency, are related to the valuation of the risk factors and to the parallelization of Monte Carlo method – basic kernels of ALM models – thus allowing to apply them to more complex situations as well. In order to test the performances of these strategies we integrated the algorithms that we developed in the ALM software described in [9].

We simulate a real portfolio containing about 78,000 policies aggregated in 5600 fluxes. The time horizon of simulation we consider is 40 years. The segregated fund includes about 100 assets, both bonds and equities. The return of the segregated fund \( F_t \) is defined by a trading strategy on stocks and bonds:

\[ F_t = \delta S_t + (1 - \delta)B_t \]

where \( S_t \) is a stock index, \( B_t \) is a bond index and \( \delta = 5.08\% \) at time zero.

We solve the SDEs for the risk sources (11)–(13) and (17)–(19) by means of the Euler method [13,23] with a monthly discretization step; as a consequence the dimension of the involved integrals is \( 3 \cdot 480 \).

The valuation is performed using a single-factor CIR model calibrated on market data at December 30th 2005. In Table 1 the used parameters are reported. Stock market volatility is \( \sigma_S = 0.1 \), the dividend yield is \( \lambda = 0.027 \), the parameters related
...the ratio between the RSE values is always about ten; the RSE obtained when working under the forward measure and the ratio between them, for different numbers of MC simulated trajectories. We note that, as already expressed in hours, for different values of simulated trajectories is reported.

Hard and TESTU01 packages. The pseudo-random streams have been mapped to values drawn from standard normal distribution, by

dinvnr of the package dcdflib [7], available through Netlib repository. The routine approximates the inverse normal cumulative function via Newton’s method [19].

Let us start our analysis of the impact of the change of numéraire on the results accuracy. In Table 2 we compare the 95% confidence intervals obtained in the estimation of the market value of the outstanding liabilities of the company when stochastic processes for risks are modelled under the risk-neutral measure and the forward measure respectively, for different values of Monte Carlo simulated trajectories. We denote by \( \bar{V} \) the sample mean, and, as usual, by \( Z_{0.05/2} \) the 95% quantile of the standard normal distribution, by \( s \) the sample standard deviation, so that \( s/\sqrt{N} \) is the standard error. In all the cases, we observe that the confidence intervals obtained via the forward measure are contained into the corresponding ones estimated under the risk-neutral measure; moreover, the half-width of the confidence intervals estimated in the risk-neutral setting is about ten times the half-width observed working under the forward measure, thus we gain one order of magnitude in terms of accuracy.

In Fig. 3 we show the relative standard error:

\[
\text{RSE} = \frac{s}{\bar{V} \sqrt{N}}
\]

We observe that the estimated RSE is of order 10^{-4} when simulations are performed under the risk-neutral measure, 10^{-5} in the forward measure case. The two lines representing the RSE exhibit almost the same slope, that is, an almost constant reduction factor in RSE is observed when simulating under the forward measure, coherently to values reported in Table 2. On the other hand, we recall that, as already pointed out in Section 3.2, the dynamics of the state variables under the forward measure depends on the bond maturity \( T \). Therefore, for each evaluation date a different simulation has to be carried out for all the risk sources. This obviously results in a time overhead, as it can be seen in Fig. 4, where the execution time, expressed in hours, for different values of simulated trajectories is reported.

In order to go deep inside into the matter, in Table 3 we report the values of RSE obtained in the risk-neutral setting, in the forward setting and the ratio between them, for different numbers of MC simulated trajectories. We note that, as already observed, the ratio between the RSE values is always about ten; the RSE obtained when working under the forward measure...
with $N = 1000$ MC trajectories is smaller than the one concerning the MC simulation under risk-neutral measure with $N = 5000$, thus confirming that modelling risks under the forward measure results in a considerable improvement in terms on accuracy. In Table 3 we also report the execution time, expressed in hours, required in the two cases and the ratio between these values, which gives the time overhead related to the forward measure approach. We note that the simulation under the forward measure almost doubles the execution time with respect to the one performed under the risk-neutral measure, but we observe that the simulation corresponding to $N = 1000$ trajectories under the forward measure provides,
in about 17 minutes, more accurate estimates than the one corresponding to \(N = 12,000\) trajectories under the risk-neutral measure as well, which requires about one hour and forty-five minutes. This means that the forward risk-neutral approach is only apparently more time consuming, since it actually requires a significantly smaller number of Monte Carlo trajectories to provide the same accuracy as the risk-neutral approach.

<table>
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<tr>
<th>(N)</th>
<th>RSE</th>
<th>Execution time</th>
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<td>RSE</td>
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**Table 3**

Column 1: number of MC trajectories; column 2: RSE under the risk-neutral measure; column 3: RSE under the forward measure; column 4: RSE under the risk-neutral measure over RSE under the forward measure. Column 5: execution time in hours under the risk-neutral measure; column 6: execution time in hours under the forward measure; column 7: overhead of the forward measure approach.

**Fig. 5.** Execution time in hours versus number of processors involved in the simulation. The global number of simulated trajectories is fixed. For each simulation, the value of the RSE is also reported.

**Table 4**

Execution time in minutes; the global number of trajectories is fixed.

<table>
<thead>
<tr>
<th>nprocs</th>
<th>(N = 6000)</th>
<th>(N = 12,000)</th>
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</thead>
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<tr>
<td>1</td>
<td>92</td>
<td>196</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>100</td>
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<td>10</td>
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<tr>
<td>12</td>
<td>8</td>
<td>17</td>
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</tbody>
</table>
Fig. 6. RSE versus number of processors involved in the simulation. The local number of simulated trajectories is fixed. For each simulation, the value of the execution time in hours is also reported.

Fig. 7. Speed-up versus number of processors.
We now turn to analyse the performances of the parallel algorithms we developed. Since the change of numéraire does not affect parallel performances, we confine our discussion to the forward measure, which, as already pointed out, gives the more accurate results. In Fig. 5 the execution time in hours versus the number of processors involved in the computation is represented. Here the global number of trajectories is fixed; for this reason, the number of MC simulated trajectories has been chosen so to be divisible by the number of processors. Moreover, we report for each simulation time the corresponding RSE value. The execution time values, expressed in minutes, are also reported in Table 4 for the sake of readability. We observe that the RSE keeps the same order of magnitude as the number of processors increases, thus confirming the scalability of the chosen parallel pseudo-random numbers generator. In this case, if we fix a target accuracy for estimates, then parallelism allows to realize the target accuracy in a strongly reduced time: this is clearly meaningful to insurance companies.

In Fig. 6 the RSE versus the number of processors involved in the computation is shown. The results refer to simulations in which we fix the local number of simulated trajectories: thus, for instance, the point corresponding to 4 processors in the line referring to \( N = 1000 \) trajectories is the value of the RSE obtained with a global number of 4000 trajectories. Moreover, we report for each estimated RSE value the related execution time in hours. Looking at the two lines, we observe that the execution times at most vary on the second decimal digit with respect to processors, thus confirming the scalability of the parallel MC algorithm, and, obviously, the RSE is reduced. Therefore, if we fix a target time for responses, then, parallelism allows to improve the estimates reliability within the target time.

We finally show, in Fig. 7, the speed-up for the same simulations. The graphic reveals the good scalability properties of the algorithm. The same behavior was observed in all our experiments. The speed-up is almost linear: this is motivated by the lowest communication cost of the parallel algorithm, indeed, communication is only required during the initialization phase of the pseudo-random numbers generator and during the reduction phase, when the local averages are combined to compute the sample mean provided by the Monte Carlo method.

6. Conclusions

In this work we discussed the development of parallel algorithms for the simulation of portfolios of profit-sharing policies. Asset-liability management of these financial instruments requires insurance companies to be equipped with effective computational tools in order to be able to compute reliable estimates of all the relevant quantities in the contracts in suitable turnaround times. We presented parallel algorithms for the valuation of portfolios of PS policies, based on the parallelization of Monte Carlo method. We considered the standard risk-neutral setting for the simulation of stochastic processes for risk sources as well as the forward risk-neutral measure, that allows to significantly reduce the standard error in the estimate of the stochastic reserve, thus resulting in a considerable gain in terms of both accuracy and efficiency. We showed that the combined use of forward risk-neutral measure and parallel computing methodologies enables the production of effective simulation software which meets both the needs of insurance companies and the requirements of the Control Authorities. The developed software is thus a tool from which insurance companies can actually benefit.

Acknowledgments

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References