A Small Surrogate for the Golden Angle in Time-Resolved Radial MRI Based on Generalized Fibonacci Sequences

Stefan Wundrak, Jan Paul, Johannes Ulrici, Erich Hell, and Volker Rasche

Abstract—In golden angle radial MRI a constant azimuthal radial profile spacing of 111.246... guarantees a nearly uniform azimuthal profile distribution in k-space for an arbitrary number of radial profiles. Even though this profile order is advantageous for various real-time imaging methods, in combination with balanced SSFP sequences the large azimuthal angle increment may lead to strong image artifacts, due to the varying eddy currents introduced by the rapidly switching gradient scheme. Based on a generalized Fibonacci sequence, a new sequence of smaller irrational angles is introduced (49.750..., 23.628..., 27.198..., 23.628..., ...). The subsequent profile orders guarantee the same sampling efficiency as the golden angle if at least a minimum number of radial profiles is used for reconstruction. The suggested angular increments are applied for dynamic imaging of the heart and the temporomandibular joint. It is shown that for balanced SSFP sequences, trajectories using the smaller golden angle surrogates strongly reduce the image artifacts, while the free retrospective choice of the reconstruction window width is maintained.

Index Terms—golden angle, real-time imaging, radial trajectory.

I. INTRODUCTION

In radial MRI the k-space is sampled with a set of radial profiles that each pass through the k-space center [1]. Radial k-space trajectories are less sensitive to artifacts introduced by motion and therefore are often used for imaging of dynamic physiological processes [2]. Additionally, compared to Cartesian trajectories, radial trajectories are known to be robust to a certain level of azimuthal undersampling, which is used in time-resolved imaging to increase the temporal resolution. Sliding window reconstruction [2] is used to further increase the image update rate. Using the normal radial sampling scheme, the width of the reconstruction window is fixed and must be selected before the acquisition is started. In most cases however, the optimal acquisition window size is not known a priori. Changing the acquisition window size requires rescanning of the subject with a differently spaced radial trajectory.

It was shown in [3] that if radial profiles are placed using the golden angle profile order which uses a consecutive profile spacing of 111.246...(golden angle), a nearly uniform profile distribution is guaranteed for an arbitrary number of successively acquired profiles. This means that the number of profiles in the reconstruction window and thus the degree of undersampling can be adjusted retrospectively to adapt to different levels of motion. During the last years the golden angle ordering scheme has found widespread use in various applications from real-time imaging, over self-gated acquisition, to single scan T1- and T2-mapping [4]–[10]. In particular, radial trajectories with a golden angle ordering scheme have been used in combination with parallel imaging and compressed sensing [11]–[13] due to the intrinsic properties of this trajectory. First, the variable density sampling in k-space that oversamples the k-space center, second the flexibility in the degree of retrospective undersampling, and last the incoherent aliasing artifacts that are essential for compressed sensing [14], [15].

In this work we introduce a new sequence of angles that is based on a generalized Fibonacci sequences. We show that the smaller tiny golden angles exhibit properties that are very similar to the original golden angle.

We show the advantages of the newly introduced angles for dynamic MR imaging in combination with fully balanced steady-state free precession sequences (b-SSFP) [16]. The large angular increment of the golden angle profile order leads to a permanently and abruptly changing gradient scheme, which leads to rapidly changing eddy currents in the conducting part of the magnet, and in-turn to rapidly varying field inhomogeneities. The equilibrium of the b-SSFP sequence is especially sensitive for these varying eddy current effects and may exhibit strong signal fluctuations [17]. These image artifacts are avoided by the smaller angle increments of the new surrogate angles while preserving the inherent properties of the golden angle ordering scheme for dynamic imaging.

II. THEORY

A. Golden Angle Trajectory

The golden angle \( \psi_{\text{gold}} = \pi / \tau \) divides the semicircle by the golden ratio \( \tau = (1 + \sqrt{5})/2 \). The golden angle trajectory is the radial k-space trajectory where successive radial profiles are placed using a successive angle increment of \( \psi_{\text{gold}} \). Note, that in this work a full revolution is defined by \( \pi \) instead of \( 2\pi \) due to the symmetry of the center-through trajectory that is used for most radial MRI sequences. The uniform angle increment is defined by \( \psi_{\text{uni}} = \pi / P \) and provides the most uniform radial sampling trajectory for a given number of radial

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profiles $P$. If $P$ cannot be predetermined prior to the acquisition, it was shown that the golden angle trajectory guarantees a near-optimal radial distribution for an arbitrary number of profiles [3]. The golden angle increment successively divides one of the largest remaining azimuthal gaps by the golden ratio.

To compare the quality of different sampling trajectories the sampling efficiency is calculated. A non-uniform profile distribution has a non-uniform azimuthal sampling density, which affects the signal-to-noise ratio (SNR) of the reconstructed image. The SNR for a given trajectory can be derived directly from the sampling density of the $k$-space sampling schemes. The sampling efficiency $SE_P^\psi$ for a given $\psi$ and for $P$ radial profiles equals the ratio of the SNR of the uniform sampling scheme $SNR_{uni}$ and the SNR of the $\psi$ sampling scheme $SNR_\psi$ [18]. This leads to the following definition of the sampling efficiency proposed by Winkelmann et al. (see [3] for details):

$$SE_P^\psi = \frac{SNR_\psi}{SNR_{uni}} = \sqrt{\frac{\pi^2/P}{\sum_{i=0}^{P-1} (\Delta \Phi_i)^2}} \tag{1}$$

with $\Delta \Phi_i$ being the mean azimuthal distance of the $i^{th}$ profile to its two adjacent profiles. The sampling efficiency of the golden angle $SE_P^{\psi_{\text{golden}}}$ falls never below 0.9732 for all $P$ and has its local maxima if $P$ is a Fibonacci number [3].

### B. Tiny Golden Angles

The aim of this work is to find smaller angles that exhibit similar properties as the golden angle. A geometric construction is used for a new sequence of angles (Fig. 1). If angle increments $0 < \psi < \pi$ are used the $(N + 1)^{th}$ radial profile will have an angle larger than $\pi$ if $N = \lfloor \pi/\psi \rfloor$. We define all angles $\psi_N$ that divide the last azimuthal gap before reaching $\pi$ by the golden ratio $\tau$ using the relation

$$\frac{\psi_N}{\pi - N \psi_N} = \tau. \tag{2}$$

Solving (2) for $\psi_N$ using the known relation $1/\tau + 1 = \tau$ leads to the sequence of angles

$$\psi_N = \frac{\pi}{\tau + N - 1}. \tag{3}$$

The first two members for $N = 1$ and $N = 2$ of this sequence are the well-known golden angle and the complementary small golden angle

$$\psi_1 = \pi/\tau; \quad \psi_2 = \pi/(\tau + 1) = \pi - \psi_1. \tag{4}$$

In this work we will call all $\psi_N$ for $N > 2$ tiny golden angles. The first ten members of the sequence $\psi_N$ are shown in Table 1. Some of these angles are known to be found in the process of phyllotaxis of various plants [19], [20]. Figure 2 shows the profile placement for a selection of angles and values for $P$. The sampling efficiency $SE_P^{\psi_N}$ has its local maxima if $P$ is a member of the generalized Fibonacci sequence $G^N$ [21], that is defined by

$$G_1^N = 1; G_2^N = N; G_{n+1}^N = G_n^N + G_{n-2}^N \tag{5}$$

where $G^1$ is the well known Fibonacci sequence. In fact, the sequence $\psi_N$ could be directly derived from the generalized Fibonacci sequences itself (see Appendix A).

If radial profiles are added successively spaced by the constant increment $\psi_N$, each additional profile after $P > N$ divides one of the currently largest azimuthal gaps by the golden ratio (see Appendix B for proof). The sampling efficiency is comparable to the sampling efficiency of the golden angle trajectory (Fig. 3) for $P > 2N$ and the bounds defined by the local minima and maxima of $SE_P^{\psi_N}$ converge to the bounds of $SE_P^{\psi_{\text{golden}}}$. The guaranteed bounds for $\psi_N$ were calculated numerically for the window $P \in [2N+1, 10^4]$ using (1) and are shown in Table 1.

### C. Beyond the Tiny Golden Angles

A radial trajectory using any constant irrational angle increment will exhibit at most three different gap sizes, and will successively divide one of the largest gaps (proven by Knuth [22]). Furthermore, any irrational number $x$ can be expressed by a regular continued fraction [23] of the form

### Table I

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\psi_N$</th>
<th>$G_3^N$</th>
<th>$G_4^N$</th>
<th>$G_5^N$</th>
<th>$G_6^N$</th>
<th>$SE_P^{\psi_N}$</th>
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<td>1</td>
<td>111.24611...</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>0.973, 0.996</td>
</tr>
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<td>13</td>
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<tr>
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<td>4</td>
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<td>11</td>
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<td>5</td>
<td>9</td>
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<td>32</td>
<td>53</td>
<td>0.973, 0.999</td>
</tr>
</tbody>
</table>

Fig. 1. Geometric construction of $\psi_N$ for $N = 5$. The last gap before the profile angle gets larger than $\psi_i$, between the $5^{th}$ and the $1^{st}$ profile, is divided by the golden ratio. The sixth profile divides the next gap also by the golden ratio. Solving the relation $a/b = \tau$ leads to $\psi_N$. 

G1N = 1; G2N = N; Gn+1N = GnN + Gn−2N

where $G^1$ is the well known Fibonacci sequence. In fact, the sequence $\psi_N$ could be directly derived from the generalized Fibonacci sequences itself (see Appendix A).
Fig. 2. The placements of the radial profiles using the tiny golden angle increments \( \psi_N \) for \( N = 1, 3, 5, \) and 7. The sampling efficiency \( SE_\psi^N \) is printed under each profile distribution. Profile distributions where \( P \) is an element of the generalized Fibonacci sequence \( G^N \), are marked using the * sign.

\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_0, a_1, a_2, a_3, \ldots].
\]

Using this notation and the well known definition of \( \tau = [1, 1] \) \cite{24}, the angles \( \psi_N \) as defined in (3) can be expressed by the regular continued fraction

\[
\psi_N = \frac{\pi}{N - 1 + [1, 1]} = \pi[0, N, 1].
\]

Figure 4 shows a plot of the minimal sampling efficiency for all angle increments between 0 and \( \pi \) using the function

\[
f(\alpha) = \min_{P\in [21;400]} SE_\psi^P.
\]

The arbitrary choice of \( P \geq 21 \) ensures on the one hand that for the first ten angles \( N > 2^P \) (see Section B.) which covers angles \( > 17^\circ \), and on the other hand \( P = 21 \) is still a small enough reconstruction window for most real-time applications. The plot shows clearly that all \( \psi_N \) are located at high local maxima of \( f \). Note that the maxima for \( \psi_{N>10} \) are not visible in this plot due to the choice of \( P \geq 21 \).

However, additional local maxima are visible in between the locations of \( \psi_N \). Analog to the original geometric construction of \( \psi_N \) (Fig. 1), we define the set of angles \( \psi_{N,M} \) by dividing the last segment using the ratio \( M-1+\tau \) instead of \( \tau \). Analog to (2) the relation \( a/b = M-1+\tau \) leads to the definition of

\[
\psi_{N,M} = \frac{\pi}{N + \frac{1}{M-1+\tau}} = \pi[0, N, M, 1].
\]

The angles \( \psi_{2,2}, \psi_{2,3}, \psi_{2,4}, \psi_{3,2}, \psi_{3,3}, \) and \( \psi_{3,4} \) specify the prominent maxima below \( 90^\circ \) shown in Figure 4 (dotted green lines) that were not covered by any \( \psi_N \). Analog to (5) the generalized Fibonacci sequence that defines the peak sampling efficiencies for \( \psi_{N,M} \) is

\[
G^N_{N,M} = 1 + (M-1)N; G^N_{2} = N;
\]

Theoretically, one could extend this scheme to use more coefficients of the continued fraction to explore more angles in the set \( \psi_{N,M_1,\ldots,M_k} = [0, N, M_1,\ldots, M_k, 1] \). One would find more sub-optimal angles, but with increasing \( M \) and \( k \) the local maxima get less distant from each other and exhibit worse uniformity. Practically, two coefficients and \( M < 5 \) are sufficient to reach all relevant local maxima (compare Fig. 4).

III. Material & Methods

A. Experiments

1) Point Spread Function Analysis: Time-resolved sequences of two-dimensional point spread functions (PSF) were created using radial sampling patterns with a constant azimuthal increment of \( \psi_1 \) and \( \psi_5 \). For both the PSF were created using the optimal resolution for \( \psi_1 \) \((G^1_{10} = 55)\) and
ψ₃ = 49.7508...°
ψ₁ extremaxbounds
ψ₇ = 23.6281...°
ψ₁ = golden angle

Fig. 3. The sampling efficiency for (a) ψ₃ and (b) ψ₇ compared to the golden angle increment ψ₁ = 111.246...°. A logarithmic scale is used on the x-axis. The maximum sampling efficiencies are reached if P is a member of the generalized Fibonacci sequence \( G_N^T = 1; G_N^N = N; G_{N+1}^N = G_{N+1}^N + G_{N+2}^N \). For \( P > 2N \), the sampling efficiency stays practically within the extremal bounds of the golden angle. The plots for ψ₃ to ψ₁₀ are provided as supplementary material.

for ψ₅ (\( G_8^5 = 73 \)). For all four sampling schemes the temporal maximum intensity projection (t-MIP) [15] was created by taking the maximum value of each pixel of the PSF sequence over time. The point spread functions were generated by an inverse gridding reconstruction followed by a successive gridding reconstruction of an image of 256 x 256 pixel that contained one single pixel with a signal intensity greater than zero.

2) Phantom Imaging: A standard test phantom (Philips, Best, Netherlands) was scanned using a radial trajectory with a constant azimuthal increment using the golden angle ψ₁, as well as the tiny golden angles ψ₂ to ψ₇. In addition, a reference trajectory with a small azimuthal sampling increment of 0.5° was used that leads to a perfect uniform trajectory if a multiple of \( P = 360 \) radial profiles are used for the image reconstruction window. The acquisition parameters for the balanced SSFP sequence were TR / TE = 4.6 / 2.3 ms, flip angle = 48°, pixel bandwidth of 934.8 Hz, a spatial resolution of 0.75 x 0.75 x 8 mm, and a 320 x 320 pixel acquisition matrix.

3) Real-Time Cardiac Imaging: Dynamic short axis real-time cardiac data sets were acquired from a healthy volunteer using a radial trajectory with a constant azimuthal increment using the golden angle ψ₁, and the angles ψ₃, ψ₁₀, ψ₇. A reference image was acquired using a constant sampling increment of 3° which leads to an uniform trajectory if a multiple of \( P = 60 \) radial profiles are used for the image reconstruction window. A 32 channel cardiac coil was used during breathhold. The acquisition parameters for the balanced SSFP sequence were TR / TE = 2.3 / 1.15 ms, flip angle = 43°, pixel bandwidth of 2688 Hz, spatial resolution of 2.74 x 2.74 x 8 mm, and a 124 x 124 pixel acquisition matrix.

4) Dynamic Temporomandibular Joint Imaging:Dynamic images of the temporomandibular joint (TMJ) were acquired under a slow and continuous opening and closing motion of the mandibular. The volunteer was instructed to open and close...
the mouth continuously and as uniformly as possible within 20 seconds. The fully balanced SSFP sequence was used due to its T1/T2 contrast at short echo times which is essential for the fast imaging of the discus articularis [25]. A radial k-space trajectory with a constant angular increment using the tiny golden angle \( \psi_1 \) was used. The scan was repeated using the balanced SSFP sequence as well as a FLASH sequence [26]. A 2x4 channel carotid coil (Chenguang Medical Technologies, Shanghai, China) was used. The acquisition parameters for the balanced SSFP sequence were: in-phase, TE / TR = 2.3 / 4.6 ms, flip angle = 45°, pixel bandwidth of 949 Hz, and spatial resolution of 0.75 x 0.75 x 5 mm with an acquisition matrix of 256 x 256 pixel. The acquisition parameters for the FLASH sequence were the same, except that the flip angle was reduced to 15°.

All datasets were acquired on a Philips 3T Achieva system (Philips Healthcare, Best, The Netherlands). Written informed consent was obtained in all cases prior to examination.

B. Reconstruction

For all experiments the images were reconstructed using gridding convolution interpolation with a Kaiser-Bessel-window width of 5 pixel [27]. The density compensation was adapted to compensate the variations of azimuthal gaps [3]. Prior to reconstruction, the k-space data was corrected for phase errors by identifying pairs of profiles that were measured in approximately opposite directions. For each pair a trajectory shift was estimated using linear regression of the profiles’ phase deviation in image space as described in [28], [29]. The cardiac image sequence was filtered using a total variation filter along the temporal dimension to remove the remaining incoherent streak artifacts [30], all other image sequences are unfiltered. All images were reconstructed using an in-house software package implemented with MATLAB (The MathWorks, Massachusetts, USA).

IV. RESULTS

A. Point Spread Function Analysis

Figure 5 shows the result of the point spread function analysis. For 55 radial profiles \( \psi_1 \) shows lower aliasing lobes than \( \psi_5 \) (Fig. 5a). This is explicable, since \( P = 55 = G_1^0 \) is a Fibonacci number which is optimal for the golden angle \( \psi_1 \) profile order, but not for the \( \psi_5 \) profile order. In case of \( P = 73 \) profiles, \( P \) is the generalized Fibonacci Number \( G_5^2 \), thus yielding lower aliasing lobes for the \( \Psi_5 \) acquisition order (Fig. 5b).

B. Phantom Imaging

The images taken from the test phantom using the tiny golden angles were compared to the reference image that were taken with a small angle increment of 0.5°. The normalized root mean square error (NRMSE) was used for comparison. Figure 6a shows that with a decreasing azimuthal angle the error also tends to decrease. Despite the small local maxima at \( \psi_2 \) and \( \psi_5 \), for \( N > 7 \) the error falls below 1.4%. Figure 6b shows a selection of the images that were used to create the plot in Figure 6a. The image artifacts are shown accentuated by scaling the image intensity by a factor of two. The image artifacts are clearly strongest with the large and the small golden angle \( \psi_1 \) and \( \psi_2 \) (NRMSE > 6%), and are reduced for \( \psi_7 \) to a visually acceptable level in comparison to the reference image (NRMSE = 1.4%).

C. Real-Time Cardiac Imaging

Figure 7a shows short axis views of the heart in systolic state. The images were reconstructed from continuously acquired radial profiles using a retrospectively chosen reconstruction window width of \( P = 60 \) radial profiles. The image acquired with the golden angle azimuthal increment \( \psi_1 \) shows strong image artifacts, which decrease with decreasing angular increment. The image artifacts are reduced if the angle gets smaller (\( \psi_3, \psi_5 \)). The image that was acquired with the smallest angle \( \psi_7 \) exhibits an artifact level visually comparable to the reference image. Since the resting phase of the heart in end-diastolic phase is significantly longer than the resting phase of the systole, the reconstruction window width could be increased to 120 radial profiles without introducing temporal
Fig. 7. Cardiac short-axis view scanned multiple times with different constant angle increments $\psi_N$ and $3^\circ$ using a b-SSFP sequence during breathhold. The systole was reconstructed with 60 radial profiles (a), the diastole with 120 profiles (b). The golden angle $\psi_1$ shows strong image artifacts due to the large angle increment and the resulting rapidly changing eddy currents.

Fig. 6. The normalized root mean square error (NRMSE) for images acquired with radial angle increments $\psi_N$ and a b-SSFP sequence, in comparison to a reference image acquired with uniform sampling at $0.5^\circ$ (a). The reconstructed images for a selection of radial angles $\psi_N$ exhibit strong image artifacts for $N < 7$ (b). The NRMSE is given in percent for each image. The signal intensity was scaled by a factor of two to accentuate the image artifacts.

D. Dynamic Temporomandibular Joint Imaging

The resulting images of the moving TMJ during its opening phase are shown in Fig. 8. The image that was reconstructed using the tiny golden angle $\psi_7$ has comparable image quality to the reference image that was acquired with the $3^\circ$ uniform sampling scheme (middle row). The borders of the discus articularis are clearly visible (arrow) due to the T1/T2 contrast of the b-SSFP sequence. In contrast to the uniform sampling scheme, the tiny golden angles profile order allows the reconstructing using an arbitrary reconstruction window width without introducing additional artifacts due to a non-uniform sampling scheme (top row). Using the original golden angle profile order $\psi_1$ introduces strong image artifacts (bottom right). These perturbations are not visible if a FLASH sequence is used, but at the cost of reduced SNR and the reduced visibility of the discus articularis (bottom left, white arrows).

V. DISCUSSION

In this work a modified golden ratio acquisition order using smaller angular increments has been introduced for radial MRI. The angular spacing is chosen according to a generalized Fibonacci sequence. It has been shown that the proposed order yields similar properties as the well-known golden angle acquisition order [3], but shows improved performance for balanced SSFP sequences with respect to eddy-current related artifacts. The existence of the tiny golden angles became evident in a numerical simulation of the sampling efficiency for all possible angular increments (Fig. 4). Similar results were reported previously in the field of theoretical biology using the distribution uniformity of leaves [19] and a shadow cast model [20] as objective functions.

It was shown previously that the golden angle increment guarantees a near uniform profile distribution for an arbitrary number of radial profiles. Using numerical simulation it was shown that for the practically relevant parameters the proposed tiny golden angles guarantee a similar optimal sampling efficiency if at least a minimum of $2N+1$ radial profiles are used, thus enabling free choice of the retrospective reconstruction window.
In the same work it was suggested to pair profiles that do not involve with the golden angle ordering scheme. Our experience was not sufficient to fully compensate the large eddy current effects. An abrupt change of the angle increments may lead to strong image artifacts induced by rapidly changing eddy currents. An abrupt change of the angle increments may lead to strong image artifacts induced by rapidly changing eddy currents. An abrupt change of the angle increments may lead to strong image artifacts induced by rapidly changing eddy currents. An abrupt change of the angle increments may lead to strong image artifacts induced by rapidly changing eddy currents. An abrupt change of the angle increments may lead to strong image artifacts induced by rapidly changing eddy currents.

SSFP sequences. In combination with a b-SSFP sequence, large eddy currents were examined in this work. Time-resolved imaging with b-SSFP sequences is likely higher on 3T systems, which may explain the effects observed in this work. To our experience, eddy current effects due to large angle increments are also observable in b-SSFP images at 1.5T, but to a much lesser degree.

The experiments in this work showed that using the smaller azimuthal angle increments of the tiny golden angles significantly reduces the eddy current induced artifacts. The experiments showed that the strength of the image artifacts is difficult to predict. For instance, the artifacts in the cardiac experiment appeared much stronger than in the TMJ experiment. Furthermore, the phantom experiment showed that the image artifact level does not necessarily decrease strictly monotonic with decreasing angles (Fig. 6b). However, in all cases an angle increment between \( \psi_7 \) and \( \psi_7 \) seemed sufficient to suppress most image artifacts to an acceptable level, which enables the combination of b-SSFP sequences with the tiny golden angle profile ordering.

In case of TMJ imaging, this result is important, since the T1/T2 contrast of the b-SSFP sequence is essential for imaging the discus articularis which in most cases is the main interest of the physician. No fast sequence with a T2 dependent contrast is known that could replace the b-SSFP in this case.

In this work, gridding reconstruction was used. In practical applications, parallel imaging [32], [33] and compressed sensing with an iterative reconstruction [14] could be used to increase the temporal resolution. The quasi-random property of the golden angle profile ordering with its incoherent aliasing artifacts [15] is essential for achieving a high temporal resolution in combination with compressed sensing bases approach, e.g. k-t SPARSE SENSE [4], [34]. The PSF analysis showed that the tiny golden angles exhibit similar incoherent point spread functions as the golden angle (Fig. 5) and should therefore work well with compressed sensing based approaches.

It should be possible to use most other applications that depend on the golden angle in conjunction with the introduced golden angle surrogates, e.g. time-resolved contrast enhancing [3], [10], single shot T1- or T2-mapping [9], [35], or self-gating [6], [7]. In future, other applications may be identified where a smaller surrogate for the golden angle may be beneficial, e.g. for the reduction of the acoustic noise during time-resolved acquisitions.

**APPENDIX A**

This work initially derived the tiny golden angles from a geometric construction. Alternatively, the angles could be derived directly from the generalized Fibonacci sequence using approximately have the same angle. For time-resolved imaging, this would however lead to a by a factor of two lowered temporal resolution. The original publication by Winkelmann et al. also applied the golden angle ordering scheme to cardiac b-SSFP imaging on a 1.5T system. Despite the observations made by Bieri et al. [17], no severe image perturbations were observed in their work [3]. However, it is known that eddy-current effects are highly dependent on the MRI system and the actual gradient switching scheme. Their impact on the image quality in b-SSFP sequences is likely higher on 3T systems, which may explain the effects observed in this work. To our experience, eddy current effects due to large angle increments are also observable in b-SSFP images at 1.5T, but to a much lesser degree.
\[
\psi_N = \lim_{n \to \infty} \frac{\pi G_n^2}{N_{n+1}}.
\]

**APPENDIX B**

Hypothesis: If \( n \) radial profiles are successively added spaced by the constant increment \( \psi_N \), each additional profile after \( n > N \) divides one of the currently largest azimuthal gaps by the golden ratio \( \tau \).

**Lemma 1:** All members of the sequence \( \psi_N \) are irrational.

**Proof.** All \( \psi_N \) can be expressed using an infinite continuous fraction and are thus irrational, see equation (7).

**Lemma 2:** If \( \psi_N \) is irrational, only 2 or 3 different gap sizes appear, and each additional profiles after \( P > N \) divides one of the largest gaps.

The proof for Lemma 2 can be derived directly from the proof given by Knuth for any irrational angle \( \psi \) [22]. The following proofs the hypothesis using induction:

**Invariant 1:** Either three gap sizes \( a_n, b_n, c_n \) exist with \( a_n/b_n = \tau \) and \( b_n/c_n = \tau \) (which implies \( a_n = b_n + c_n \)), or two gap sizes \( a_n, b_n \) exist with \( a_n/b_n = \tau \).

Proof by induction that Invariant 1 is valid for all \( n \geq N \).

**Induction start:** \( n = N \): Two gap sizes exist, and Invariant 1 is clearly valid due to the definition of \( \psi_n \) through equation (2).

**Inductive step:** If for the profile distribution \( n \) Invariant 1 is valid, show that for the profile distribution \( n + 1 \) Invariant 1 is still valid.

**Case 1:** Three gap sizes \( a_n, b_n, c_n \) exist and \( a_n \) is not the last gap of its size.

Then the \((n + 1)\text{th}\) profile divides one of the largest gaps of size \( a_n \) (Lemma 2) into \( a_n = b_n + c_n \) (no \( 4^{\text{th}} \) gap size allowed due to Lemma 2 and \( a_n \neq b_n + b_n \) due to Invariant 1).

\[
\Rightarrow \text{Three gap sizes } a_{n+1} = a_n, b_{n+1} = b_n, \text{ and } c_{n+1} = c_n \text{ exists and Invariant 1 is valid for } n + 1.
\]

**Case 2:** Three gap sizes \( a_n, b_n, c_n \) exist and \( a_n \) is the last gap of its size.

Then the \((n + 1)\text{th}\) profile divides the largest gap of size \( a_n \) (Lemma 2) still into \( a_n = c_n + b_n \), since \( a_n = d_n + b_n \) would imply \( d_n = c_n \), and \( a_n = d_n + c_n \) is not allowed due to Lemma 2.

\[
\Rightarrow \text{Two gap sizes } a_{n+1} = b_n \text{ and } b_{n+1} = c_n \text{ are left and Invariant 1 is valid for } n + 1.
\]

**Case 3:** Two gap sizes \( a_n, b_n \) exist and \( a_n \) is not the last gap of its size.

Then the \((n + 1)\text{th}\) profile divides the largest gap of size \( a_n \) (Lemma 2) into \( a_n = b_n + c_{n+1} \), since \( a_n \neq b_n + b_n \) due to Invariant 1 and \( a_n \neq c_{n+1} + c_n \) since \( \psi_n \) irrational.

Since \( a_{n+1} = a_n \) and \( b_{n+1} = b_n \) follows

\[
\frac{b_n}{c_{n+1}} = \frac{b_n}{a_n - b_n} = \frac{b_n}{\left(\frac{a_n}{b_n} - 1\right)}
\]

\[
\Rightarrow \text{Three gap sizes } a_{n+1} = a_n, b_{n+1} = b_n, \text{ and } c_{n+1} = b_n + 1 \text{ exists and Invariant 1 is valid for } n + 1.
\]

**Case 4:** Two gap sizes \( a_n, b_n \) exist and \( a_n \) is the last gap of its size.

This case never happens. For \( n \) to be of Case 3, \( n - 1 \) had to be of Case 2. But Case 2 will always create at least two gaps of the size \( a_n \).

\[
\Rightarrow \text{The Invariant 1 is valid for } n + 1 \text{ in all 3 cases. Together with Lemma 2 this proves the hypothesis.}
\]

**REFERENCES**


