Dynamic modelling of a 4-DOF parallel kinematic machine with revolute actuators

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Abstract: Recursive matrix equations for kinematics and dynamics analysis of a 4-DOF parallel manipulator with a passive constraining leg and revolute actuators are presented in this paper. The prototype of this robot is a spatial mechanism, which has two translation degrees of freedom and also two rotation degrees of freedom. The manipulator consists of a base platform, a moving platform and a system of four connecting legs, having wide application in the fields of industrial robots, simulators, parallel machine tools and any other manipulating devices in which high mobility is required. Supposing that the position and the motion of the moving platform are known, an inverse dynamics problem is solved using the principle of virtual powers. Finally, some iterative matrix relations and graphs of the torques of all actuators are analysed and determined.

Keywords: parallel manipulator; parallel kinematic machines; dynamic modelling; recursive matrix method; revolute actuator.


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1 Introduction

A parallel manipulator, generally, comprises two platforms, which are connected by joints and several legs acting in parallel (Merlet, 2000; Tsai, 1999). One of these platforms is attached to the fixed reference frame (or base) and the other one can have arbitrary motion in its workspace. Several mobile legs or limbs, made up as serial robots, connect the moving platform to the fixed frame.

Typically, a parallel mechanism is said to be symmetrical if it satisfies the following conditions: the number of legs is equal to the number of degrees of freedom of the moving platform, one actuator, which can be mounted at or near the fixed base, controls every limb and the location and the number of actuated joints in all the limbs are the same. The links of the mechanisms are connected to one another by spherical joints, universal joints, revolute joints or prismatic joints.

The parallel robots are spatial mechanisms with supplementary characteristics, compared with the serial architecture manipulators such as: more rigid structure, important dynamic charge capacity, high orientation accuracy, stable functioning as well as good control of velocity and acceleration limits. On the other hand, parallel kinematic machines offer significant advantages over their serial counterparts: higher stiffness, lower moving masses, higher natural frequencies, simpler modular mechanical construction and possibility to locate actuators on the fixed base. However, most existing parallel manipulators have limited and complicated workspace with singularities and highly non-isotropic input – output relations.

For two decades, parallel manipulators attracted the attention of more and more researches and industries that consider them as a valuable alternative design for robotic mechanisms. Thus, parallel mechanisms can work at higher velocities and yet maintain sufficient rigidity to deliver high levels of positioning accuracy. Precision in the execution of the tasks are essential since the robot is intended to operate on fragile objects; any errors in the positioning of the tool could lead to expensive damages. They can be found in many technical applications in which it is desired to orient a rigid body in space of high speed such as: aircraft simulators (Stewart, 1965), positional trackers (Carretero et al., 2000), telescopes (Dunlop and Jones, 1999) and micro-motion devices (Lee and Arjunan, 1991). In the development of high-precision machine tools (Fedewa et al., 2000; Huang et al., 2004; Zhang and Gosselin, 2002; Zhang, 2000), parallel mechanisms have recently been used by many companies, such as Giddings and Lewis, Ingersoll, Hexel, Geodetic and Toyoda and others. The Hexapod machine tool is one of the successful applications.

While the kinematics has been studied extensively during the last two decades, fewer papers can be focused on the dynamics of parallel robots. When good dynamic performance and precise positioning under high load are required, the dynamic model is important for their control.

Dynamics analysis of parallel mechanisms is usually implemented through analytical method in classical mechanics (Dasgupta and Mruthyunjaya, 1998; Li et al., 2003),
in which projection and resolution of vector equations on the reference axes are written in a considerable number of cumbersome, scalar equations and the solutions are rendered by large-scale computations together with time-consuming computer codes.

Some systematic approaches have been developed for general-purpose parallel mechanisms analysis (Ji, 1994; Dasgupta and Choudhury, 1999; Geng et al., 1992; Wang and Chen, 1994). In comparison with an approach for kinematic modelling of robot manipulator, a dynamic modelling method could be systematised easily. Meanwhile, quite a few of special approaches have been conducted for dynamic modelling of specific parallel mechanism configurations (Lee and Shah, 1988; Tsai, 2000; Pang and Shahinpoor, 1994; Bhattacharya et al., 1998; Zaganeh et al., 1997). Kane and Levinson (1985) obtained some vector recursive relations concerning the equilibrium of generalised forces that are applied to a serial mechanism. Sorli et al. (1997) conducted the dynamic modelling for Turin parallel mechanism, though the mechanism has three identical legs, it has 6-DOF. Geng et al. (1992) developed Lagrange’s equations of motion under some simplifying assumptions regarding the geometry and inertia distribution of the manipulator. Dasgupta and Mruthyunjaya (1998) used the Newton–Euler approach to develop closed-form dynamic equations of Stewart platform, considering all dynamic and gravity effects as well as viscous friction at joints.

However, to our knowledge, there are no efficient dynamic modelling approaches available for parallel mechanisms. Furthermore, all the existing approaches are of analytical type, while the matrix recursive method, proposed in this paper, is a simple and generic one.

In this paper, the new approach, called Recursive Matrix Method (RMM), is developed. It has been proved to reduce the number of equations and computational operations significantly by using a set of matrices for kinematics and dynamic modelling (Staicu, 1999, 2000). A spatial 4-DOF parallel mechanism (Huang et al., 2004), which can be used in several applications, including machine tools, is proposed in this paper. This manipulator belongs to the type with revolute actuators and a passive leg located in the centre to improve the stiffness.

2 Geometric model of the manipulator

Some iterative matrix relations for kinematics and dynamic analysis of a 4-DOF parallel mechanism with revolute actuators are established in the paper. The proposed parallel manipulator consists of five kinematical chains, including four actuated legs with identical topology and one passive leg, connecting the fixed base to the moving platform (Figure 1). The links of these legs have given sizes and masses. In this 4-DOF parallel mechanism, the kinematical chains associated with the four identical legs consist, from base to the platform, of an actuated revolute joint, a lower moving link, a Hooke joint, an upper moving link and, finally, a spherical joint attached to the moving platform.

The aim of using the passive leg is to constrain the mobility of the platform and limit it to only four degrees of freedom. Since the weight and the external loads on the platform will induce bending or torsion in the passive leg, its mechanical design is a very important issue, which can be addressed using the dynamics model proposed here.

Let us locate the fixed Cartesian frame $Ox_0 y_0 z_0$ at the centre $O$ of the fixed base about which the parallel manipulator of known geometry moves (Figure 2). The elements of all legs have known dimensions and masses. To simplify the graphical image of the
kinematical scheme of the mechanism, in what follows we will represent the intermediate reference systems by only two axes, so that one proceeds in most of the books (Merlet, 2000; Tsai, 1999; Angeles, 2002). The \( z_k \) axis is represented, of course, for each component element \( T_k \). It is noted that the relative rotation with \( \phi_{k,k-1} \) angle or relative translation of \( T_k \) body with \( \lambda_{k,k-1} \) displacement must be always pointing about or along the direction of \( z_k \) axis.

**Figure 1** CAD model of a 4-DOF parallel mechanism with revolute actuators

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**Figure 2** Kinematical scheme of the mechanism

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One of the four active legs, for example \( A_1 A_2 A_3 A_4 \), connects first an active revolute joint \( A_1 \) attached to the base and a moving link \( A_1 x_1^t z_1^t \) (called \( T_1^4 \)) of length \( l_1 \), masse \( m_1 \) and tensor of inertia \( \hat{J} \), which has a rotation about the axis \( A_1 x_1^t \) characterised by the angle \( \phi_{10}^t \), angular velocity \( \omega_1^t = \phi_{10}^t \) and angular acceleration \( \varepsilon_{10}^t = \phi_{10}^t \). A small Hooke joint \( A_2 y_2^t z_2^t (T_2^4) \) connects the element \( T_2^4 \) to next link \( A_3 y_3^t z_3^t (T_3^4) \), which has two
concurrent orthogonal rotations about the axes \( A_i z_i^e, A_i z_i^a \) with the angles \( \phi_i^e, \phi_i^a \), so that \( \omega_{21} = \phi_{21}^e, \omega_{32} = \phi_{32}^e, \epsilon_{21} = \phi_{21}^a, \epsilon_{32} = \phi_{32}^a \). This upper link has the length \( A_4 A_4 = l_z \), mass \( m_2 \) and tensor of inertia \( \hat{J}_z \). Finally, a spherical joint \( A_4 \) is attached to the moving platform \( T_{4e} \).

The fifth chain connecting the base point \( O \) to the centre of moving platform is a passive constraining arm, which has a different architecture from other four chains. It consists of a revolute joint attached to the base and a moving lower link \( E_3 x_3 ^e \) (called \( T_{3e} \)) of length \( l_z \), mass \( m_3 \) and tensor of inertia \( \hat{J}_3 \), having a rotation angle \( \phi_6^e \), velocity \( \omega_6^e \) and acceleration \( \epsilon_6^e \) about \( E_3 z_3^e \) axis. The point \( E_3 \) is defined as the centre of another revolute joint connecting the following moving link \( E_3 E_4 = l_z \), of mass \( m_4 \) and tensor of inertia \( \hat{J}_4 \). About the axis \( E_3 z_3^e \), this last upper link has a rotation characterised by the angle \( \phi_5^e \), angular velocity \( \phi_5^e \) and angular acceleration \( \phi_5^e \).

Finally, a small Hooke joint \( E_3 E_4 x_4 ^e y_4 ^e z_4 ^e \) is attached to the moving platform \( E_3 x_3 ^e y_3 ^e z_3 ^e \) \( (T_{4e}) \). This last body can be a square plate \( A_5 B_5 C_5 D_5 \) of edge \( l \), mass \( m_5 \) and central tensor of inertia \( \hat{J}_5 \), which has two concurrent orthogonal rotations about the axes \( E_3 z_3^e, E_4 z_4^e \) of angles \( \phi_5^e \) and \( \phi_6^e \) with respect to neighbouring link.

The following angles give the position of four active revolute joints \( A_i, B_i, C_i \) and \( D_i \):

\[
\chi_A = 0, \quad \chi_B = \frac{\pi}{2}, \quad \chi_C = \pi, \quad \chi_D = -\frac{\pi}{2}, \quad \theta_0 = \frac{\pi}{3}, \quad \theta = \chi_i - \theta_0 \quad (i = A, B, C, D). \tag{1}
\]

Let us consider the angles of rotation \( \phi_{10}^e, \phi_{10}^a, \phi_{10}^e, \phi_{10}^a \) of four revolute joints as parameters giving the input vector \( \phi_{10} \) of the instantaneous position of the mechanism. But, the objective of the inverse geometric problem is to find the vector \( \phi_{10} \) and the absolute position and orientation of the robot with the given four independent coordinates \( x_0, z_0, \alpha, \beta \), where \( x_0, z_0 \) are two Cartesian coordinates of mass centre \( G \) of the moving platform and \( \alpha, \beta \) are two Euler angles of rotation about the moving axes \( G_y, G_z \).

In what follows, we apply the method of successive displacements to geometric analysis of closed-loop chains and we note that a joint variable is the displacement required to move a link from initial location to actual position. If every link is connected to at least two other links, the chain forms one or more independent closed-loops. Variable angles \( \phi_{k+1} \) of rotation about the joint axes \( z_k \) are the parameters needed to bring the next link from a reference configuration to the next configuration. We call the matrix \( a_{i-1}^e \), for example, the orthogonal transformation \( 3 \times 3 \) matrix of relative rotation with the angle \( \phi_{i-1}^e \) of link \( T_i \) around \( z_i^e \) axis.

In the study of the kinematics of robot manipulators, we are interested in deriving a matrix equation relating the location of an arbitrary \( T_k \) body to the joint variables. When the change of coordinates is taken in succession, the corresponding matrices are multiplied. So, starting at the reference origin \( O \) and pursuing the \( OA_1 A_2 A_3 A_4 \) way of first active limb, for example, the transformation matrices are given by the expressions:

\[
a_{01} = a_{0}^e a_{a1}^e a_{0}^a, \quad a_{21} = a_{2}^e a_{a1}^e a_{2}^a, \quad a_{32} = a_{3}^e a_{a1}^e a_{3}^a, \quad a_{43} = a_{4}^e a_{a1}^e a_{4}^a, \tag{2}
\]

where (Staicu et al., 2006; Staicu and Carp-Ciocadia, 2003):
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\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

\[
a_i^d = \begin{bmatrix}
\cos \theta_i & \sin \theta_i & 0 \\
-\sin \theta_i & \cos \theta_i & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad
a_u = \begin{bmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad
a_p = \begin{bmatrix}
\cos \beta & \sin \beta & 0 \\
-\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
a_{i,k}^p = \begin{bmatrix}
\cos \phi_{i,k} & \sin \phi_{i,k} & 0 \\
-\sin \phi_{i,k} & \cos \phi_{i,k} & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad
a_{i_0} = \prod_{j=1}^{k} a_{i-j+1,k-j}, \quad (k = 1, 2, 3).
\]

Some analogous relations can be written for other three active chains \(OB_1B_2B_3\), \(OC_1C_2C_3\) and \(OD_1D_2D_3\).

Pursuing the constraining passive leg \(E_1E_2E_3E_4\), we obtain the following matrices:

\[
e_i = e_i^0 \ a_i a_i, \quad e_{i+1} = e_{i+1}^0 \ a_{i+1} a_{i+1}, \quad e_{i+2} = e_{i+2}^0 \ a_{i+2} a_{i+2}, \quad e_{i+3} = e_{i+3}^0 \ a_{i+3} a_{i+3}
\]

so that the transformation matrix from fixed base \(O_0x_0y_0z_0\) to last moving frame \(E_4x_4y_4z_4\) is \(e_{i_0} = e_{i_1}e_{i_2}e_{i_3}e_{i_4}\).

Let us assume that the following relations express the simulation of absolute motion of the platform

\[
r_0^G = [x_0^G \ z_0^G \ y_0^G] \ x_0^G = x_0^G \left\{ 1 - \cos \left( \frac{2 \pi \ i}{3} \right) \right\}, \quad z_0^G = h - z_0^G \left\{ 1 - \cos \left( \frac{2 \pi \ i}{3} \right) \right\}
\]

\[
\alpha_i = \alpha_i^0 \left\{ 1 - \cos \left( \frac{2 \pi \ i}{3} \right) \right\} \quad (i = 2, 3).
\]

To solve the inverse geometric problem, we must first consider the passive constraining leg as a serial 4-DOF mechanism whose four coordinates are determined by two conditions

\[
u_i^G \ r_i^E + e_1^0 \ r_1^E + e_2^0 \ r_2^E + e_3^0 \ r_3^E = \bar{u}_i^G \ r_0^G \quad \text{or} \quad e_i^T e_{i_0} = a,
\]

where \(a = a_i a_i^0\) is the commonly known rotation matrix from the fixed referential \(O_0x_0y_0z_0\) to the platform’s frame \(Gx_Gy_Gz_G\) and we denoted

\[
r_0^F = \bar{r}, \quad r_0^E = - l_i \bar{u}_i, \quad r_{i_0}^E = l_i \bar{u}_i
\]

\[
e_{i_0}^T = \begin{bmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad \bar{u}_i = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Actually, equation (6) means that there is only one inverse geometric solution for the constraining leg of the manipulator.
l_1 \sin(\phi_1^e + \gamma) + l_4 \sin(\phi_1^e - \phi_2^e + \gamma - \delta) = x_0^e
\]
\[
l_1 \cos(\phi_1^e + \gamma) + l_4 \cos(\phi_1^e - \phi_2^e + \gamma - \delta) = z_0^e
\]
\[
\phi_3^e = \alpha_5 - \phi_5^e + \phi_2^e, \quad \phi_4^e = \alpha_3.
\]  

Once the solution to the inverse geometric model of this serial arm is found, the complete position and orientation of the platform and the rotation angles $\phi_1^e, \phi_2^e, \phi_3^e, \phi_4^e$ of four legs $A, B, C, D$ can be definitively determined using new geometric conditions of constraint

$$
\vec{r}_{10}^t + \sum_{k=4}^3 \vec{a}_{10}^t \vec{r}_{k1.5}^t - \vec{e}_{40}^t \vec{r}_4^k = \vec{r}_{10}^b + \sum_{k=4}^3 \vec{b}_{10}^t \vec{r}_{k1.5}^t - \vec{e}_{40}^t \vec{r}_4^b
= \vec{r}_{10}^c + \sum_{k=4}^3 \vec{c}_{10}^t \vec{r}_{k1.5}^t - \vec{e}_{43}^t \vec{r}_4^c
= \vec{r}_{10}^d + \sum_{k=4}^3 \vec{d}_{10}^t \vec{r}_{k1.5}^t - \vec{e}_{40}^t \vec{r}_4^d = \vec{e}_0^e.
$$

where, for example, we note

$$
\vec{r}_{10}^t = [l_0 \cos(\chi_4), \quad l_0 \sin(\chi_4), \quad 0]^T
\]
\[
\vec{r}_{4k}^t = -l_k \vec{u}_k, \quad \vec{r}_{32}^t = \vec{0}, \quad \vec{r}_{43}^t = -l_3 \vec{u}_3.
\]
\[
\vec{r}_{41}^t = [l_1 \sin(\chi_4 + \pi/4), \quad -l_2 \cos(\chi_4 + \pi/4), \quad 0]^T.
\]  

These vector equations (9) mean that there it is possible to obtain an analytical closed-form resolution of the inverse kinematical model of the manipulator:

$$
l_2 \sin \phi_2^e = p_i
\]
\[
l_1 \sin(\phi_1^e + \alpha) + l_4 \sin(\phi_1^e - \phi_2^e + \alpha - \beta) \cos \phi_3^e = q_i
\]
\[
l_1 \cos(\phi_1^e + \alpha) + l_4 \cos(\phi_1^e - \phi_2^e + \alpha - \beta) \cos \phi_4^e = r_i,
\]

where we denote

$$
p_i = -x_0^e \sin \theta_i - l_1 \sin \theta_i \cos \alpha_i \cos(\alpha_i + \theta_i) + l_4 \cos \theta_i \sin(\alpha_i + \theta_i) - l_6 \sin \theta_6
\]
\[
q_i = x_0^e \cos \theta_i + l_1 \cos \theta_i \cos \alpha_i \cos(\alpha_i + \theta_i) + l_4 \sin \theta_i \cos(\alpha_i + \theta_i) + l_6 \cos \theta_6
\]
\[
r_i = z_0^e - l_4 \sin \alpha_i \cos(\alpha_i + \theta_i) \quad (i = A, B, C, D).
\]  

### 3 Inverse kinematics analysis

The objective of the inverse kinematics solution is to determine in matrix closed-form the velocities and accelerations of the manipulator, supposing that the pose of the moving platform is considered known in a fixed Cartesian space.

The following skew-symmetrical matrices characterise the motions of the links of each leg (for example leg $A$) (Staicu, 2005):

$$
\dot{\alpha}_{10}^t = a_{1,1}^t \ddot{a}_{1,1}^t + \alpha_{1,1}^t \ddot{u}_1.
$$
These matrices are associated with absolute angular velocities expressed in recursive form
\[ \dot{\alpha}_0^k = a_{k, k-1} \dot{\alpha}_0^{k-1} + \omega_{k, k-1}^0, \quad \alpha_{k, k-1} = \phi_{k, k-1}^0. \] (14)

Following relation gives the velocity \( \ddot{v}_0^k \) of the joint \( A_k \)
\[ \ddot{v}_0^k = a_{k, k-1} \{ \dot{v}_0^{k-1} + \dot{\alpha}_0^k \omega_{k, k-1}^0 \} + v_{k, k-1}^0 \]
\[ \ddot{v}_0^k = 0, \quad (k = 1, 2, 3). \] (15)

The kinematical conditions of connectivity shall be given through constraint relations written in a forward computation between relative velocities of each independent closed-loop of the mechanism.

To establish some matrix relations for the relative angular velocities \( \omega_0^E, \omega_1^E, \omega_2^E, \omega_3^E \) of passive leg \( E_1E_2E_3E_4 \), equations of geometrical constraints (6) can be derived with respect to time
\[ \omega_0^E = \alpha_2 - \alpha_0^E + \alpha_2^E, \quad \omega_3^E = \alpha_3. \] (16)

These four analytical equations (8) of passive chain of the robot can be written as follows:
\[ 2l_2 \sin(\phi_0^E + \gamma) + 2l_3 \sin(\phi_0^E + \gamma) = x_0^G + z_0^G + l_2^2 - l_3^2 \]
\[ 2l_2 \cos(\phi_0^E + \delta) = x_0^G + z_0^G - l_2^2 - l_3^2 \] (17)

The derivative with respect to time of these conditions leads to the matrix equation
\[ J_1^E \begin{bmatrix} \phi_0^E & \phi_1^E & \phi_2^E & \phi_3^E \end{bmatrix}^T = J_2^E \begin{bmatrix} x_0^G & z_0^G & \alpha_2 & \alpha_3 \end{bmatrix}^T, \] (18)
with the following Jacobian matrices
\[ J_1^E = \begin{bmatrix} l_2 \sin(\phi_2^E + \delta) & 0 & 0 & 0 \\ 0 & l_2 \sin(\phi_2^E + \delta) & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
\[ J_2^E = \begin{bmatrix} -\sin(\phi_0^E - \phi_2^E + \gamma - \delta) & -\cos(\phi_0^E - \phi_2^E + \gamma - \delta) & 0 & 0 \\ -x_0^G & -z_0^G & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \] (19)

Knowing the motion of the platform by the relations (5), we develop the inverse kinematics problem and determine the characteristic velocities of each of the moving links. Equations of geometrical constraints (9) can be also derived with respect to time.
to obtain the following matrix conditions of connectivity for the relative angular velocities (Staicu, 2005).

\[
\begin{align*}
\omega^4_0 & \left[ \hat{u}^T_0, a_{10} \hat{u}_1, r_{12}^3 \right] + \hat{u}_0^T a_{10} \hat{u}_1 a_{12}^2 (a_2^3 r_{13}) + \omega^4_0 \left[ \hat{u}^T_0 a_{10} \hat{u}_1 a_{12}^3 r_{13}^4 \right] + \omega^4_0 \left[ \hat{u}^T_0 a_{10} \hat{u}_1 a_{12}^3 r_{13}^4 \right] + \omega^4_0 \left[ \hat{u}^T_0 a_{10} \hat{u}_1 a_{12}^3 r_{13}^4 \right] + \omega^4_0 \left[ \hat{u}^T_0 a_{10} \hat{u}_1 a_{12}^3 r_{13}^4 \right] = \hat{u}_0^{j^4} + \sum_{j^4} \hat{c}_j \theta_{j^4} r_{j^4}, \quad (j^4 = 1, 2, 3),
\end{align*}
\]

where \( \hat{u}_1, \hat{u}_2, \hat{u}_3 \) are skew-symmetric matrices associated with three orthogonal unit vectors \( \hat{u}_1, \hat{u}_2, \hat{u}_3 \).

If the other three kinematical chains of the manipulator are pursued, analogous relations can be easily obtained. From these equations, the relative angular velocities \( \omega_1^0, \omega_2^0, \omega_3^0, \omega_4^0, \omega_5^0, \omega_6^0, \omega_7^0, \omega_8^0, \omega_9^0, \omega_10^0, \omega_11^0, \omega_12^0 \) result.

Equations (20) also give the complete Jacobian of the mechanism. This square invertible matrix is an essential element for the analysis of the robot workspace and the particular configurations of singularities where the manipulator becomes uncontrollable.

Rearranging, constraint equations (11) of the spatial manipulator can be immediately written as follows:

\[
2l_i q_i \sin(\phi_i^0 + \alpha) + 2l_i r_i \cos(\phi_i^0 + \alpha) = p_i^0 + q_i^0 + r_i^0 - l_i^0, \quad (i = A, B, C, D),
\]

where the initial ‘zero’ position \( (x_i^0 = h, \alpha_i = 0, \alpha_i = 0) \) corresponds to the joints variables \( \phi_i^0 = [0 \quad 0 \quad 0]^T \). The derivative with respect to time of all conditions (21) leads to the matrix equation

\[
J_1 \dot{\phi}_i^0 = J_2 [\dot{x}_i^0, \dot{\alpha}_i, \dot{\alpha}_i]^T.
\]

Matrices \( J_1 \) and \( J_2 \) are, respectively, the inverse and forward Jacobian of the manipulator and can be expressed as

\[
J_1 = \text{diag}\{\delta_x, \delta_y, \delta_z, \delta_\alpha, \delta_\theta, \delta_\beta\}, \quad J_2 = \begin{bmatrix}
\beta_1^a & \beta_2^a & \beta_3^a & \beta_4^a \\
\beta_1^b & \beta_2^b & \beta_3^b & \beta_4^b \\
\beta_1^c & \beta_2^c & \beta_3^c & \beta_4^c \\
\beta_1^d & \beta_2^d & \beta_3^d & \beta_4^d
\end{bmatrix},
\]

with

\[
\begin{align*}
\delta_x &= -l_i \sin(\phi_{i^1}^0 + \beta) \cos \phi_{i^2}^0, \\
\beta_1^a &= \cos \theta_i \sin(\phi_i^0 - \phi_{i^1}^0 + \alpha - \beta) \cos \phi_{i^2}^0 - \sin \theta_i \sin \phi_{i^2}^0, \\
\beta_1^b &= \cos(\phi_i^0 - \phi_{i^1}^0 + \alpha - \beta) \cos \phi_{i^2}^0, \\
\beta_1^c &= -l_i (\beta_i^c \sin \alpha_i + \beta_i^c \cos \alpha_i) \cos(\alpha_i + \theta_i), \\
\beta_1^d &= -l_i (\beta_i^d \cos \alpha_i - \beta_i^d \sin \alpha_i) \sin(\alpha_i + \theta_i), \\
\end{align*}
\]

\[
\begin{align*}
+ l_i \sin(\theta_i \sin(\phi_i^0 - \phi_{i^1}^0 + \alpha - \beta) \cos \phi_{i^2}^0 + \theta_i \sin \phi_{i^2}^0) \cos(\alpha_i + \theta_i) \\
(i = A, B, C, D).
\end{align*}
\]

Now, the three kinds of singularities of four closed-loop kinematical chains can be determined through the analysis of two Jacobian matrices \( J_1 \) and \( J_2 \).
Let us assume that the mechanism has a first virtual motion determined by the angular velocities $\omega^0_1, \omega^0_2, \omega^0_3, \omega^0_4$. The virtual relative velocities expressed as functions of the position of the mechanism are given by the connectivity conditions (16) and (20). Other relations of connectivity can be obtained if we consider successively along the other active legs $B, C, D$ that $\omega^0_1 = 1, \omega^0_2 = 1, \omega^0_3 = 1$.

The relative angular accelerations $\epsilon_{10}^0, \epsilon_{21}^0, \epsilon_{32}^0, \epsilon_{43}^0$ and $\epsilon_{10}^4, \epsilon_{21}^4, \epsilon_{32}^4$ are obtained by some new conditions of connectivity given by the derivative of equations (16) and (20), as follows:

\[
\epsilon_{ij}^E = \tilde{\epsilon}_{ij}^E + \epsilon_{ij}^0, \quad \epsilon_{ij}^E = \epsilon_{ij}^0, i = 1, 2, 3.
\]

Performing the derivatives with respect to time of equations (13)–(15), we obtain the recursive form of accelerations $\gamma_{k+1}^A$, $\tilde{\epsilon}_{ij}^E$:

\[
\gamma_{k+1}^A = \gamma_{k+1}^A + (\tilde{\epsilon}_{k+1}^E + \epsilon_{k+1}^0) \tilde{\epsilon}_{k+1}^E, \\
\gamma_{k+1}^A = 0, \quad (k = 1, 2, 3).
\]

and a useful general square characteristic matrix (Staicu, 1998):

\[
\tilde{\epsilon}_{ij}^{A} = \epsilon_{k-1} - \epsilon_{k-1}^E + \epsilon_{k-1}^A + \omega_{k-1} \tilde{\epsilon}_{ij}^{A}.
\]

The matrix relations (16), (20) and (25) constitute the inverse kinematics model of the parallel mechanism.

## 4 Dynamics simulation

Although the parallel manipulators with less DOF have been investigated to some extent, works on their dynamics are relatively few. Because of the existence of multiple closed-loop chains, dynamic analysis of a parallel manipulator is quite complicated.
In the context of the real-time control, neglecting the frictional forces in the joints and considering the gravitational effects, the relevant objective of the dynamics is to determine the input torques, which must be exerted by the actuators to produce a given trajectory of the effector. There are three methods, which can provide the same results concerning the actuating torques and powers. The first one is using the Newton–Euler classic procedure (Dasgupta and Mruthyunjaya, 1998; Li et al., 2003; Dasgupta and Choudhury, 1999; Guegan et al., 2002), the second one applies Lagrange’s equations and multipliers’ formalism (Geng et al., 1992) and the third approach is based on the principle of virtual powers (Tsai, 2000; Angeles, 2002; Staicu, 1998; Zhang and Song, 1993).

Within the inverse dynamic problem, in this paper, one applies the principle of virtual powers to establish some recursive matrix relations for the torques of the four active systems.

The electric motors \( A_1, B_1, C_1, D_1 \) that generate four torques \( \bar{m}_{i0}^4 = m_{0i}^4 \bar{h}_i \), \( \bar{m}_{i0}^8 = m_{0i}^8 \bar{h}_i \), \( \bar{m}_{i0}^C = m_{0i}^C \bar{h}_i \), which have the directions of the axes \( A_1 \bar{z}_1 \), \( B_1 \bar{z}_1 \), \( C_1 \bar{z}_1 \), \( D_1 \bar{z}_1 \), control the motion of mechanism’s legs. The characteristic vectors \( \bar{f}_{i0}^4 \), \( \bar{m}_{i0}^4 \) of the wrench about \( A_1 \) evaluate the influence of the action of the weight \( m_i^4 \bar{g} \) and of other external or internal forces applied at the same link \( A_1 \) of the mechanism. On the other hand, the force of inertia and the resultant moment of forces of inertia are also computed with respect to the centre of joint \( A_1 \):

\[
\begin{align*}
\bar{f}_{i0}^{4,m} &= -m_i^4 (\bar{f}_{i0}^{4} + (\bar{\alpha}_{i0}^A + \bar{\epsilon}_{i0}^A) \bar{r}_{i0}^{(C_1)}) \\
\bar{m}_{i0}^{4,m} &= -m_i^4 \bar{r}_{i0}^{(C_1)} \bar{f}_{i0}^{4} - \bar{f}_{i0}^{4} \bar{\alpha}_{i0}^A \bar{\epsilon}_{i0}^A.
\end{align*}
\]  

(28)

Knowing the position and kinematics state of each link as well as the external forces acting on the robot, in this paper one applies the principle of virtual powers in an inverse dynamic problem. The active torques required in a given motion of the moving platform will easily be computed using a recursive procedure.

The principle of virtual powers states that a mechanism is under dynamic equilibrium if and only if the virtual powers developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the kinematic constraints imposed to the mechanism. Assuming that frictional forces at the joints are negligible, the virtual power produced by the forces of constraint at the joints is zero.

Applying the fundamental equations of the parallel robots dynamics established in compact form by Staicu (2000), the following matrix relation results:

\[
\begin{align*}
m_{i0}^{4} &= \bar{u}_i^4 \left( \bar{m}_{i0}^4 + \omega_{i0}^{z_1} \bar{m}_{10}^4 + \omega_{i0}^{z_2} \bar{m}_{20}^4 + \omega_{i0}^{z_3} \bar{m}_{30}^4 + \omega_{i0}^{z_4} \bar{m}_{40}^4 + \omega_{i0}^{x_1} \bar{m}_{10}^8 + \omega_{i0}^{x_2} \bar{m}_{20}^C + \omega_{i0}^{x_3} \bar{m}_{30}^8 + \omega_{i0}^{x_4} \bar{m}_{40}^C \right) \\
&\quad + \omega_{i0}^{x_1} \bar{m}_{10}^D + \omega_{i0}^{x_2} \bar{m}_{20}^D + \omega_{i0}^{x_3} \bar{m}_{30}^D + \omega_{i0}^{x_4} \bar{m}_{40}^D + \omega_{i0}^{y_1} \bar{m}_{10}^E + \omega_{i0}^{y_2} \bar{m}_{20}^E + \omega_{i0}^{y_3} \bar{m}_{30}^E + \omega_{i0}^{y_4} \bar{m}_{40}^E \),
\end{align*}
\]  

(29)

where

\[
\begin{align*}
\bar{f}_{k0}^{4} &= -\bar{f}_{k}^{in} - \bar{f}_{k}^{ex} \\
\bar{m}_{k0}^{4} &= -\bar{m}_{k}^{in} - \bar{m}_{k}^{ex} \\
\bar{f}_{k}^{4} &= \bar{f}_{k0}^{4} + \bar{a}_{k+1}^{C_1} \bar{f}_{k+1}^{4} \\
\bar{m}_{k}^{4} &= \bar{m}_{k0}^{4} + \bar{a}_{k+1}^{C_1} \bar{m}_{k+1}^{4} + \bar{h}_{k+1}^{C_1} \bar{f}_{k+1}^{4},
\end{align*}
\]  

(30)

\( k = 1, 2, 3, 4 \).
The relations (29) and (30) represent the inverse dynamics model of the parallel manipulator with revolute actuators.

5 Numerical example

As for application, let us consider a mechanism that has the following characteristics:

\[ x_0^{\alpha} = 0.10 \text{ m}, \quad x_0^{\beta} = 0.15 \text{ m}, \quad \alpha_0^\alpha = \pi / 36, \quad \alpha_0^\beta = \pi / 18, \quad \alpha = \pi / 6 \]

\[ l = 0.75 \text{ m}, \quad l_1 = 0.5 \text{ m}, \quad l_2 = 1.25 \text{ m}, \quad l_3 = 0.65 \text{ m}, \quad l_4 = 1 \text{ m}, \quad l_5 = l / \sqrt{2} \]

\[ l_2 \sin(\beta - \alpha) = l_1 \sin \alpha + 0.5l / \sin \theta, \quad OA = l_0 = 0.5l(1 + 1/\tan \theta_0), \]

\[ h = l_1 \cos \alpha + l_2 \cos(\beta - \alpha) \]

\[ m_1 = 2.5 \text{ kg}, \quad m_2 = 1.5 \text{ kg}, \quad m_3 = 2 \text{ kg}, \quad m_4 = 1.5 \text{ kg}, \quad m_5 = 5 \text{ kg}, \quad \Delta t = 3 \text{s}. \]

Finally, we obtain the time-history of the torques \( m_1^{\alpha} \) (Figure 3), \( m_2^{\alpha} \) (Figure 4), \( m_3^{\alpha} \) (Figure 5), \( m_4^{\alpha} \) (Figure 6) of the four revolute actuators. According to the motion laws (5), the symmetry of the graphics of the active torques \( m_1^{\alpha}, m_2^{\alpha}, m_3^{\alpha}, m_4^{\alpha} \) with respect to the maximum ascension of the platform, is verified.

Figure 3  Torque \( m_1^{\alpha} \) of first revolute actuator

Figure 4  Torque \( m_2^{\beta} \) of second revolute actuator
6 Advantage of the present method

Most of dynamical models based on the Lagrange formalism neglect the weight of intermediate bodies and take into consideration only the active forces or moments and the wrench of applied forces on the moving platform. The number of relations given by this approach is equal to the total number of the position variables and Lagrange multipliers inclusive. Also, the analytical calculi involved in these equations are very tedious, thus presenting an elevated risk of making errors.

The commonly known Newton–Euler method, which takes into account the free-body-diagrams of the mechanism, leads to a large number of equations with unknowns among which are also the connecting forces in the joints. Finally, the actuating torques could be obtained.
Within the inverse kinematics analysis, some exact relations that give in real time the position, velocity and acceleration of each element of the parallel robot have been established in closed form in this study. The dynamics model takes into consideration the masses and forces of inertia introduced by all component elements of the parallel mechanism.

Based on the principle of virtual powers, the new approach is far more efficient, can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of torques and powers required by the three actuators. Also, the above-described method is quite available in forward and inverse mechanics of serial and parallel mechanisms, the platform of which having translation, spherical evolution or more general 6-DOF motion.

7 Conclusions

In the paper, the matrix relations for the real-time computation of position, velocity and acceleration of each link of a 4-DOF parallel mechanism with revolute actuators have been established. With the example of the 4-DOF parallel mechanisms, the new method illustrated an efficient way to determine the time-history of the torques of all revolute actuators of a parallel mechanism. In a context of automatic control, the iterative matrix relations (29) and (30) given in this dynamics model can be easily transformed into a robust model for the computerised command of the most general parallel mechanisms.

References


**List of symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$\omega_{k,-1}$</td>
<td>Orthogonal transformation matrix</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>General transformation matrix of moving platform</td>
</tr>
<tr>
<td>$\bar{u}_1, \bar{u}_2, \bar{u}_3$</td>
<td>Three orthogonal unit vectors</td>
</tr>
<tr>
<td>$\phi_{k,-1}$</td>
<td>Relative rotation angle of $T_k$ rigid body</td>
</tr>
<tr>
<td>$\dot{\omega}_{k,-1}$</td>
<td>Relative angular velocity of $T_k$</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>Absolute angular velocity of $T_k$</td>
</tr>
<tr>
<td>$\dot{\omega}_{k,-1}$</td>
<td>Skew-symmetric matrix associated with the angular velocity $\dot{\omega}_{k,-1}$</td>
</tr>
<tr>
<td>$\ddot{\omega}_{k,-1}$</td>
<td>Relative angular acceleration of $T_k$</td>
</tr>
<tr>
<td>$\ddot{\omega}_{k,-1}$</td>
<td>Absolute angular acceleration of $T_k$</td>
</tr>
<tr>
<td>$\dot{\epsilon}_{k,-1}$</td>
<td>Skew-symmetric matrix associated with the angular acceleration $\dot{\epsilon}_{k,-1}$</td>
</tr>
<tr>
<td>$\bar{u}^A_{k,-1}$</td>
<td>Relative position vector of the centre of $A_k$ joint</td>
</tr>
<tr>
<td>$\bar{v}^A_{k,-1}$</td>
<td>Relative velocity of the centre $A_k$</td>
</tr>
<tr>
<td>$\ddot{v}^A_{k,-1}$</td>
<td>Relative acceleration of the centre $A_k$</td>
</tr>
<tr>
<td>$m_k$</td>
<td>Mass of $T_k$ rigid body</td>
</tr>
<tr>
<td>$\bar{J}_k$</td>
<td>Symmetric matrix of tensor of inertia of $T_k$ about the link-frame $A_kX_kY_kZ_k$</td>
</tr>
<tr>
<td>$J_1, J_2$</td>
<td>Two Jacobian matrices of the manipulator</td>
</tr>
<tr>
<td>$m^{a}<em>{10}, m^{a}</em>{10}, m^{c}<em>{10}, m^{0}</em>{10}$</td>
<td>Torques of four actuators</td>
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</table>