Dynamics analysis of the Star parallel manipulator

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Matrix relations in kinematics and dynamics of the Star parallel manipulator are established in this paper. The prototype of the manipulator is a three-degree-of-freedom mechanism, which consists of a system of parallel kinematical chains connecting to a moving platform. Knowing the translation motion of the platform, we develop first the inverse kinematics problem and determine the position, velocity and acceleration of each robot’s link. Further, the inverse dynamics problem is solved using an approach based on the principle of virtual work, but it has been verified the results in the framework of the Lagrange equations with their multipliers. Recursive formulae offer expressions and graphs for the power requirement comparison of each of three actuators in two computational complexities: complete dynamic model and simplified dynamic model.

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1. Introduction

Parallel manipulators are closed-loop mechanical structures presenting very good performance in terms of accuracy, rigidity and ability to manipulate large loads.

Generally, the mechanism has two platforms: one of them is attached to the fixed reference frame and the other one can perform arbitrary motions in its workspace, some moving legs made up as serial robots connecting the moving platform to the fixed platform. Spherical joints, revolute joints or prismatic joints can connect the elements of the robot to one another.

Typically, a parallel mechanism is said to be symmetrical if it satisfies the following conditions: the number of legs is equal to the number of degrees of freedom of the moving platform, every limb is controlled by a single actuator and the location and number of actuated joints in all limbs is equal (Tsai [1]).

During last two decades, parallel manipulators had increasingly focused attention of many researchers that consider their design as a valuable alternative for robotic mechanisms [2,3]. Parallel robots can be equipped with hydraulic or pneumatic actuators. They have a robust construction and move bodies of considerable mass and size with high speed. These complex architectures can be found in many technical applications where a high-speed displacement or orientation of a rigid body in space is desired.

Compared with serial robots, parallel robots have specific characteristics: higher structural rigidity, better orientation accuracy, stable functioning, larger dynamic charge capacity and suitable possibility to mount all actuators on or near to the fixed base. However, most existing parallel manipulators have limited and complicated workspace volume with singularities [4,5].

Recently, much effort has been devoted to the kinematic and dynamic analysis of fully parallel manipulators. Important companies such as Giddings & Lewis, Ingersoll, Hexel and others have developed them as high precision machining tools. The most known class of manipulators is the flight simulator with six degrees of freedom, which is in fact the Gough-Stewart platform (Stewart [6]; Merlet [7]; Parenti Castelli and Di Gregorio [8]; Baron and Angeles [9]).

The Star parallel manipulator (Hervé and Sparacino [10]; Tremblay and Baron [11]) and the Delta parallel robot (Clavel [12]; Tsai and Stamper [13]; Staicu [14]) are all equipped with three motors, which have a parallel setting and push the end-effector in a three degrees of freedom general translation. Angeles, Gosselin, Gagné and Wang [15–17] developed the direct kinematics and dynamics of an Agile Wrist prototype that presents three concurrent rotations.

The dynamics analysis of parallel manipulators is usually implemented through analytical methods in classical mechanics [18,19] in which projection and resolution of equations on the reference axes are written in a considerable number of cumbersome, scalar relations and the solutions are rendered by large scale computation together with time consuming computer codes.

Kane and Levinson [20] obtained some vector recursive relations concerning the equilibrium of generalised forces that are applied to a serial manipulator. Sorli et al. [21] conducted the dynamics modelling for a Turin parallel manipulator, though the mechanism has three identical legs, it has 6-DOFs. Geng et al. [22] developed Lagrange’s equations of motion under some simplifying assumptions regarding the geometry and inertia distribution of the manipulator. Dasgupta and Mruthyunjaya [23] used the Newton–Euler approach to develop closed-form dynamics
2. Inverse kinematics

The mechanism input of the robot is structured into three active revolute joints while the output body is connected to the fixed base through a set of three variable length legs with identical topology. The topology of one of the three kinematical chains of the manipulator is made of an electrical motor, a helical joint and an intermediary mechanism with four revolute links that connect four bars, which are parallel two by two, and finally a passive revolute joint of the moving platform (Fig. 1).

In the following we develop by inverse kinematics a means to determine the velocities and accelerations of all moving components of the robot, given the characteristics of the translation motion of the moving platform.

Let us locate a fixed Cartesian frame OxOyOz(T0) at the centre O of the triangular base of side l1√3 about which the Star parallel manipulator of known geometry moves. The elements of its legs have known size and given mass. To simplify the graphical image of the kinematical scheme of the mechanism, in what follows we will represent the intermediate reference systems by only two axes, as is used in most of the robotics papers [1,4,7]. It is noted that the relative rotation with angle ϕu.k−1 or the relative translation of each component body T0 with the displacement λu.k−1 must always be pointed along the direction of the z2 axis.

One of the three active elements of the robot is the first body of the leg A (Fig. 2). This is a homogenous screw of length OA1 = l1, mass m1 and tensor of inertia J1, which rotates around the A1z1 axis with an angular velocity ωu.1 = ωu.1 and angular acceleration ωu.1. It is noted that the cylindrical part 2 of length l2, mass m2 and tensor of inertia J2 is connected to the A2x2y2z2(T2) frame and it has a relative rotation–translation motion. The axial displacement λu.3 of this helical motion is proportional with the rotation angle ϕu.3, so that λu.3 = k31 ϕu.3.

Further on, two parallel identical bars A2A4 (3) and A6A7 (6) of same length l1, mass m1 = m6 and the same tensor of inertia J1 = J6 rotate about two parallel axes with the angle ϕu.4. The four-bar parallelogram is closed by an element T4(T4) identical with T2 and which rotates by an angle ϕu.4 = ϕu.6.

The moving platform 5 of the manipulator can be a three arm star of length r = l1 − l2 or an equilateral triangle of side l = r√3 and mass m5. The platform is joined to A5 and rotates about the neighbouring body T4.

At the central configuration, we also consider that all legs are symmetrically extended and the following angles give the initial position of the robot

$$a_A = \frac{\pi}{3}, \quad a_B = \frac{\pi}{3}, \quad a_C = -\frac{\pi}{3}$$  \tag{1}

The angle $a_A$, for example, is measured from the fixed x0 axis to the line connecting the centre O to the first actuated joint A1.

In that following we apply the method of successive displacements to geometric analysis of closed-loop chains and we note that a joint variable is the displacement required to move a link from the initial location to the actual position. If every link is connected to at least two other links, the chain forms one or more independent closed-loops. The variable angles ϕu.k−1 of rotation about the joint axes zk are the parameters needed to bring the next link from a reference configuration to the next configuration. We call the matrix
by multiplying three transformation matrices from

\[ p_{10} = p_{10}^A \theta_1 a_{10}^A, \quad p_{21} = p_{21}^A \theta_2, \quad p_{32} = p_{32}^A \theta_2, \quad p_{43} = p_{43}^A \theta_3, \quad p_{62} = p_{32} \]

where is denoted [24]:

\[ \theta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \theta_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \theta_4 = \begin{bmatrix} \cos \alpha_i & \sin \alpha_i & 0 \\ -\sin \alpha_i & \cos \alpha_i & 0 \end{bmatrix}, \quad \theta_5 = \begin{bmatrix} \cos \psi_{i,k-1} & \sin \psi_{i,k-1} & 0 \\ -\sin \psi_{i,k-1} & \cos \psi_{i,k-1} & 0 \end{bmatrix} \]

\[ p_{0,k-1} = \prod_{\tau=1}^{k} p_{k-\tau+1,k-\tau}, \quad (k = 1, 2, \ldots, 6). \]

The three rotation angles \( \psi_{i0}, \psi_{i1}, \psi_{i2} \) of the actuators \( A, B, C \) are the joint variables that give the input vector \( \vec{\psi}_{10} = [\psi_{i0}^A, \psi_{i1}^A, \psi_{i2}^A]^T \) of the instantaneous position of the mechanism. But, in the inverse geometric problem, it can be considered that three absolute coordinates \( x_i, y_i, z_i \) of mass centre \( G \) of the moving platform give the position of the mechanism. Since all translations take place successively about the moving coordinate axes, the diagonal unit matrix \( R_{23}R_{23}R_{10} = I \) of the moving platform is obtained by multiplying three transformation matrices from \( O_{x0}y_{0}z_{0} \) to \( G_{x1}y_{1}z_{1} \) reference

\[ R_{10} = \theta_1, \quad R_{21} = \theta_2 \theta_1, \quad R_{32} = \theta_3 \theta_1 \theta_2. \]

Supposing, for example, that the motion of the mass centre \( G \) of the platform along a curvilinear trajectory is given by the analytical relations

\[ \vec{v}_{i0}^G = \begin{bmatrix} x_i^G & y_i^G & z_i^G \end{bmatrix}^T \]

\[ x_i^G(t) = \dot{x}_i^G \sin \left( \frac{\pi}{15} t \right) \]

\[ y_i^G(t) = \dot{y}_i^G \sin \left( \frac{\pi}{15} t \right) \]

\[ z_i^G(t) = \dot{z}_i^G \left[ 1 - \cos \left( \frac{\pi}{15} t \right) \right] \]

the input angles \( \psi_{i0} \) \( (i = A, B, C) \) of the robot and the variables \( \psi_{i1}, \psi_{i2} \) will be obtained from the following geometrical conditions of constraint

\[ \vec{v}_{i0}^A + \sum_{k=1}^{4} a_{k0}^A \vec{v}_{i,k-1,k} + \vec{v}_{i0}^A = \vec{v}_{i0}^G \]

\[ = \vec{v}_{i0}^C + \sum_{k=1}^{4} a_{k0}^C \vec{v}_{i,k-1,k} + \vec{v}_{i0}^C = \vec{v}_{i0}^G, \]

where the following is denoted:

\[ \vec{v}_{i0}^A = l_1 \alpha_{i0}^A \vec{u}_1, \quad \vec{v}_{i0}^B = -l_2 \alpha_{i0}^B \vec{u}_3, \quad \vec{v}_{i0}^C = l_2 \vec{u}_3, \quad \vec{v}_{i1}^B = -l_3 \alpha_{i1}^B \vec{u}_2, \quad \vec{v}_{i1}^C = 0, \quad \vec{v}_{i2}^B = (l_1 - l_2) \alpha_{i2}^B \vec{u}_1, \quad \vec{v}_{i2}^C = (l_1 0 0) \]

\[ \vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \]

Actually, these equations mean that there is only one inverse geometric solution for the manipulator:

\[ x_i^C \tan(\alpha_{i0}^C - \phi_{i1}^C) = x_i^G \sin \alpha_i - y_i^G \cos \alpha_i \]

\[ l_2 \cos(\psi_{i0}^A - \psi_{i1}^A) \cos \phi_{i2}^C = z_i^G \]

\[ k_2 \psi_{i2}^C = l_1 \sin \phi_{i2}^C - x_i^G \cos \alpha_i - y_i^G \sin \alpha_i \quad (i = A, B, C), \]

where the “zero” position vector \( \vec{r}_{i0}^G = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \) corresponds to the joints variables \( \alpha_{i0}^A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \).

Knowing the rotation motion of the platform by the relations (5), we develop the inverse kinematical problem and determine the velocities \( \vec{v}_{i0}^A, \vec{v}_{i0}^B, \) and accelerations \( \vec{a}_{i0}^A, \vec{a}_{i0}^B, \) of each of the moving links in terms of the translation velocity \( \vec{v}_p = \vec{v}_{i0}^G \) of the moving platform.

The motions of the component elements of each leg (for example the first leg \( A \)) are characterised by the following skew-symmetric matrices [25]:

\[ \vec{a}_{i0}^A = a_{k-1,k-1,0} \vec{a}_{k-1,0}^A + a_{k,k-1,0} \vec{a}_{k,k-1}^A - \vec{a}_{k,k-1,0} \vec{a}_{k,k-1}^A, \]

which are associated to the absolute angular velocities given by the recursive relations

\[ a_{i0}^A = a_{i0}^A(l_1 k_{i0} \vec{a}_{i0}^A) \vec{a}_{i0}^A \vec{a}_{i0}^A + a_{i0}^A l_{i0} \vec{a}_{i0}^A \vec{a}_{i0}^A + a_{i0}^A l_{i0} \vec{a}_{i0}^A \vec{a}_{i0}^A + a_{i0}^A l_{i0} \vec{a}_{i0}^A \vec{a}_{i0}^A \]

\[ + a_{i0}^A l_{i0} \vec{a}_{i0}^A \vec{a}_{i0}^A + a_{i0}^A l_{i0} \vec{a}_{i0}^A \vec{a}_{i0}^A + a_{i0}^A l_{i0} \vec{a}_{i0}^A \vec{a}_{i0}^A \]

Equations of geometrical constraints (5) can be derived with respect to time to obtain the following matrix conditions of connectivity for the relative angular velocities [26,27]:

\[ \omega_{j0}^{A} \vec{a}_{j0}^A \vec{a}_{j0}^A \vec{a}_{j0}^A + \omega_{j0}^{A} \vec{a}_{j0}^A \vec{a}_{j0}^A \vec{a}_{j0}^A + \omega_{j0}^{A} \vec{a}_{j0}^A \vec{a}_{j0}^A \vec{a}_{j0}^A + \omega_{j0}^{A} \vec{a}_{j0}^A \vec{a}_{j0}^A \vec{a}_{j0}^A \]

\[ + \omega_{j0}^{A} \vec{a}_{j0}^A \vec{a}_{j0}^A \vec{a}_{j0}^A = -\vec{u}_j \vec{r}_j, \quad (j = 1, 2, 3), \]

where \( \vec{u}_j \) is a skew-symmetric matrix associated to the unit vector \( \vec{u}_j \). If the other two kinematical chains of the robot are pursued analogous relations can be easily obtained.

Relative angular velocities \( \omega_{j0}^{A}, \omega_{j0}^{B}, \omega_{j0}^{C} \) result from the above Eq. (12) as functions of the platform’s translation velocity. The same relations (12) give immediately the complete Jacobian matrix of the manipulator. This matrix is a fundamental element for the analysis of the robot workspace and the particular configurations of singularities where the manipulator becomes uncontrollable. The kinematical relations (12) will be further required in the computation of virtual velocity distribution of the elements of the manipulator.

The derivative with respect to time of conditions (8) leads to the matrix equation

\[ J \dot{\vec{\psi}}_{10} = J \vec{v}_{10}^G \]

Matrices \( J_1 \) and \( J_2 \) are, respectively, the inverse and forward Jacobian of the manipulator and can be expressed

\[ J_1 = \text{diag} \left( \vec{a}_{j0}^A, \vec{a}_{j0}^B, \vec{a}_{j0}^C \right) \]

\[ J_2 = \left[ \begin{array}{ccc} \vec{a}_{j0}^A & \vec{a}_{j0}^B & \vec{a}_{j0}^C \\ \vec{a}_{j0}^A & \vec{a}_{j0}^B & \vec{a}_{j0}^C \\ \vec{a}_{j0}^A & \vec{a}_{j0}^B & \vec{a}_{j0}^C \end{array} \right]. \]
with

\[ \delta_i = \sin \varphi_{32} \cos \varphi_{32} \]

\[ \beta_1 = -(1/k_{11}) \cos \varphi_{32} (\sin \alpha_i \sin \varphi_{54} \cos \varphi_{32} + \cos \alpha_i \sin \varphi_{32}) + (1/l_3) \sin \alpha_i \sin \varphi_{34} \sin \varphi_{32} \]

\[ \beta_2 = -(1/k_{31}) \cos \varphi_{32} (\cos \alpha_i \sin \varphi_{54} \cos \varphi_{32} - \sin \alpha_i \sin \varphi_{32}) - (1/l_3) \cos \alpha_i \sin \varphi_{34} \sin \varphi_{32} \]

\[ \beta_3 = -(1/k_{21}) \cos \varphi_{32} \cos \varphi_{32} - (1/l_3) \sin \alpha_i \sin \varphi_{34} \sin \varphi_{32} \]

(15)

The three kinds of singularity of the three closed-loop kinematical chains can be determined through the analysis of two Jacobian matrices \( J_1 \) and \( J_2 \).

Now, let us assume that the manipulator has a first virtual motion determined by the angular velocities \( \omega_{10}^{A} = 1, \omega_{0b}^{A} = 0, \omega_{0c}^{A} = 0 \). The characteristic virtual velocities are expressed as functions of the position of the mechanism by the kinematical constraints Eq. (12). Some other relations of connectivity can be obtained if one considers successively that \( \omega_{0b}^{A} = 1, \omega_{10}^{A} = 0, \omega_{10}^{A} = 0 \), and \( \omega_{0c}^{A} = 1, \omega_{10}^{A} = 0, \omega_{0b}^{A} = 0 \). These virtual velocities are required in the computation of the virtual work of all the forces applied to the component elements of the mechanism.

As for the relative characteristic angular accelerations \( \varepsilon_{10}^{A}, \varepsilon_{21}^{A}, \varepsilon_{32}^{A} \) of the bodies of first leg A, the derivatives with respect to time of the Eq. (12) give other conditions of connectivity

\[ \varepsilon_{10}^{A} = \dot{\omega}_{10}^{A} + \dot{\varphi}_{10}^{A} \dot{\varphi}_{10}^{A} \dot{\varphi}_{10}^{A} \]

and the acceleration \( \varepsilon_{10}^{A} \) of joints \( A \)

\[ \ddot{\varphi}_{10}^{A} + \ddot{\varphi}_{10}^{A} + \ddot{\varphi}_{10}^{A} = a_{k,k-1} (\ddot{\varphi}_{10}^{A} - 6 \dot{\varphi}_{10}^{A} + 12 \varphi_{10}^{A}) - a_{k-1,k} (\ddot{\varphi}_{10}^{A} - 6 \dot{\varphi}_{10}^{A} + 12 \varphi_{10}^{A}) + a_{k,k+1} (\ddot{\varphi}_{10}^{A} - 6 \dot{\varphi}_{10}^{A} + 12 \varphi_{10}^{A}) \]

(17)

The matrix relations (9)–(11) and (17) will be further used for the computation of the wrench of the inertia forces for every rigid component of the robot. The dynamic model of the mechanism would only be established in regard with the complete geometrical analysis and kinematics of the mechanical system.

The relations (12) and (16) represent the inverse kinematics model of the Star parallel manipulator.

3. Dynamics equations

The dynamics of parallel manipulators is complicated by the existence of multiple closed-loop chains. Difficulties commonly encountered in dynamics modelling of parallel robots include problematic issues such as: a complex spatial kinematical structure which possesses a large number of passive degrees of freedom, the dominance of inertial forces over the frictional and gravitational components and the problem linked to the solution of the inverse dynamics.

In the context of the real-time control, neglecting the friction forces and considering the gravitational effects, the relevant objective of the dynamics is to determine the input torques, which must be exerted by the actuators in order to produce a given trajectory of the effector. There are three methods, which can provide the same result concerning these actuating moments. The first one is using the Newton–Euler classical procedure [23,28], the second one uses the Lagrange's equations and multipliers formalism [22,29] and the third one is based on the principle of virtual work \([1,4,26,30,31]\).

3.1. Principle of virtual work

Knowing the position and kinematics state of each link as well as the external forces acting on the robot, in the present paper we apply first the principle of virtual work for the inverse dynamic problem in order to establish some matrix relations. The three input torques and powers required in a given translation motion of the platform will be computed using a recursive procedure.

Three independent electric motors \( A_1, B_1, C_1 \) that generate three moments \( m_{10}^{A} = m_{01}^{A}, m_{10}^{B} = m_{01}^{B}, \) and \( m_{10}^{C} = m_{01}^{C} \), which have fixed directions of the axes \( z_{1}^{A}, z_{1}^{B}, z_{1}^{C} \), control the motion of the platform \( B_{0}C_{0} \).

The parallel robot can artificially be transformed in a set of three open chains \( C_i \) \((i = A, B, C)\) subject to the constraints. This is possible by cutting each joint for the moving platform, and takes its effect into account by introducing the corresponding constraint conditions. The last and more simplified open tree system could comprise the moving platform only.

The force of inertia of an arbitrary rigid body \( T_k^{A} \), for example,

\[ \overrightarrow{\tau}_{k0}^{A} = -m_{k0}^{A} \left( \dot{\varphi}_{y0}^{A} + \left( \omega_{0b}^{A} - \omega_{0c}^{A} + \psi_{0b}^{A} \right) \dot{\varphi}_{b0}^{A} \right) \]

(18)

and the resulting moment of the forces of inertia

\[ \overrightarrow{m}_{k0}^{A} = -m_{k0}^{A} \left( \dot{\varphi}_{b0}^{A} \dot{\varphi}_{y0}^{A} + \dot{\varphi}_{b0}^{A} \right) \]

(19)

are determined with respect to the centre of joint \( A \). On the other hand, the wrench of two vectors \( \hat{f}_{k0}^{A} \) and \( \hat{m}_{k0}^{A} \) evaluates the influence of the action of the weight \( m_{k0}^{A} \) and of other external and internal forces applied to the same element \( T_k^{A} \) of the manipulator, for example:

\[ \hat{f}_{k0}^{A} = 9.81 m_{k0}^{A} a_{k0}^{A} \]

\[ \hat{m}_{k}^{A} = 9.81 m_{k0}^{A} a_{k0}^{A} \]

(20)

Considering three successive independent virtual motions of the robot, virtual displacements and velocities should be compatible with the virtual motions imposed by all kinematical constraints and joints at a given instant in time. By intermediate of the complete Jacobian matrix expressed by the conditions of connectivity (12), the absolute virtual velocities \( \overrightarrow{v}_{k_{1,1}}^{A}, \overrightarrow{v}_{k_{1,1}}^{A} \) associated with all moving links are related to a set of independent relative virtual velocities \( \overrightarrow{v}_{k_{1,1}}^{A}, \overrightarrow{v}_{k_{1,1}}^{A} \) and \( \overrightarrow{v}_{k_{1,1}}^{A} \).

The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism.

Assuming that frictional forces at the joints are negligible, the virtual work produced by the forces of constraint at the joints is zero.

Applying the fundamental equations of the parallel robots dynamics established by Stefan Staica [32], the following compact
and the input forces are compared to the values obtained in the first work of friction forces in joints. It is added. The new active torques will supply all the relative velocities and relative accelerations, through successive recasting the connectivity conditions that will constitute the virtual work. Now, we can calculate the friction forces and the friction torques in the joints, based on the friction coefficients and the maximum of the connecting forces in the joints. We apply again the explicit Eqs. (21) and (22), where the contribution of the virtual work of friction forces in joints is added. The new active torques and input forces are compared to the values obtained in the first calculus.

Compared with Tsai’s analytical method based on the principle of virtual work [1, 13], the advantages of the present approach are the followings:

- Geometrical constraint relations, under matrix form, generate through successive recasting the connectivity conditions that will supply all the relative velocities and relative accelerations, which characterise the independent kinematical chains.
- The accelerations of the mass centres, the angular accelerations and the twists of the inertia forces are expressed through matrix formulae, which contain the kinematical characteristics of the relative motion of the building elements of the manipulator.
- A single matrix relation supplies all the virtual velocities.
- The explicit dynamics equation represents a definitive formula, obtained by the transformation of the general expression of the virtual work where the relative virtual velocities only appear, generated by recursive relations.
- All intermediate analytical calculations were eliminated and the numerical computation is achieved through the numerical code, for each active force or torque applied by the driving system.

3.2. Equations of Lagrange

A solution of the dynamics problem of the Star parallel mechanism can be developed based on the Lagrange equations of second kind for a mechanical system with constraints. The generalised coordinates of the robot are represented by 12 independent parameters:

\[ q_1 = \dot{x}_0^C, \quad q_2 = \dot{y}_0^C, \quad q_3 = \dot{z}_0^C, \]
\[ q_4 = \dot{x}_0^B, \quad q_5 = \dot{y}_0^B, \quad q_6 = \dot{z}_0^B, \]
\[ q_7 = \dot{x}_0^L, \quad q_8 = \dot{y}_0^L, \quad q_9 = \dot{z}_0^L, \]
\[ q_{10} = \dot{\psi}_{k-1}^L, \quad q_{11} = \dot{\psi}_{k-1}^L, \quad q_{12} = \dot{\psi}_{k-1}^L. \]

The Lagrange’s equations with their nine multipliers \( \lambda_1, \lambda_2, \ldots, \lambda_9 \) will be expressed by 12 differential relations

\[
\frac{d}{dt} \left\{ \lambda_k \sqrt{q_i} \right\} - \frac{\partial L}{\partial q_i} = 0 \quad (k = 1, 2, \ldots, 12),
\]

which contain following 12 generalised forces

\[ Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = m_{10}^A, \quad Q_5 = 0, \]
\[ Q_6 = 0, \quad Q_7 = m_{10}^B, \quad Q_8 = 0, \quad Q_9 = 0, \]
\[ Q_{10} = m_{10}^L, \quad Q_{11} = 0, \quad Q_{12} = 0. \]

A number of nine kinematical conditions of constraint are given by the relations (12):

\[
\sum_{k=1}^{12} C_k q_k = 0 \quad (s = 1, 2, \ldots, 9).
\]

The general expression of the Lagrange function

\[
L = L_p + \sum_{s=1}^{4} \left( L_s^A + L_s^B + L_s^C \right) + L_{10}^A + L_{10}^B + L_{10}^L
\]

is expressed as analytical functions of the generalised coordinates and their first derivatives with respect to time:

\[ L_p = \frac{1}{2} m_p v_p^T \dot{v}_p - m_g g z_0^G, \]
\[ L_1^A = \frac{1}{2} \omega_1^A q_{10}^A, \]
\[ L_2^A = \frac{1}{2} m_f^A v_{10}^A v_0^A + \frac{1}{2} m_{20}^A q_{20}^A, \]
\[ L_3^A = L_0^A = \frac{1}{2} m_p_0^A v_{30}^A v_0^A + \frac{1}{2} m_{30}^A \dot{q}_{30}^A q_{30}^A + m_s^A_0 v_{30}^A q_{30}^A \]
\[ - \dot{m}_s^A g^A u_3 T_{20}^C (21) \]
\[ L_4^A = \frac{1}{2} m_f^A v_{40}^A v_0^A + \frac{1}{2} m_{40}^A \dot{q}_{40}^A q_{40}^A + m_s^A v_{40}^A q_{40}^A \]
\[ - \dot{m}_s^A g^A u_3 T_{43}^C (25) \]
\[ i = (A, B, C), \quad p = (a, b, c). \]

The joint’s velocities, the angular velocities, the skew-symmetric matrices associated to the angular velocities and the first derivatives of orthogonal matrices \( p_{k-1, k-1} \) are computed as follows:

\[
v_{10}^i = 0, \quad \dot{v}_{10}^i = \dot{\psi}_{10}^i \dot{u}_3, \]
\[
v_{20}^i = k_{21} \dot{\psi}_{21}^i \dot{u}_3, \]
\[
\dot{\omega}_{10}^i = (\dot{\psi}_{21}^i - \dot{\psi}_{10}^i) \dot{u}_3, \]
\[
\dot{\omega}_{20}^i = (\dot{\psi}_{21}^i - \dot{\psi}_{10}^i) \dot{u}_3, \]
\[
\dot{\psi}_{10}^i = p_{k-1, k-1} \dot{v}_{10}^i - 0 + p_{k-1, -1} \dot{\omega}_{10}^i, \]
\[
\dot{\psi}_{20}^i = p_{k-1, k-1} \dot{v}_{20}^i - 0 + p_{k-1, -1} \dot{\omega}_{20}^i, \]
\[
\dot{\psi}_{k-1}^i = p_{k-1, k-1} \dot{v}_{k-1, 0}^i + \dot{\psi}_{k-1, -1}^i \dot{u}_3, \]
\[
\dot{\psi}_{k-1}^i = p_{k-1, k-1} \dot{v}_{k-1, 0}^i + \dot{\psi}_{k-1, -1}^i \dot{u}_3, \]
\[
\frac{\partial p_{k-1, k-1}}{\partial \psi_{k-1, k-1}} = \dot{p}_{k-1, k-1}^i \dot{u}_3. \]
A long and tedious calculus about the partial derivatives \( \frac{\partial}{\partial \theta_k} \frac{\partial}{\partial \dot{\theta}_k} \) of the functions (28) and the derivatives \( \frac{\partial}{\partial \theta_k} \phi_k (\dot{\theta}_1, \ldots, \dot{\theta}_{12}) \) with respect to time leads to an algebraic system of 12 relations. In the direct or inverse dynamics problem, after elimination of the nine multipliers, finally we obtain same expressions (21) for the three input torques \( T_{10}^i, m_{10}^i, c_{10}^i \) required by the three revolute actuators.

For simulation purposes let us consider a manipulator which has the following characteristics:

- \( \dot{x}_0^c = 0.05 \text{ m} \), \( \dot{y}_0^c = 0.10 \text{ m} \), \( \dot{z}_0^c = 0.15 \text{ m} \) \( \Delta t = 15 \text{ s} \)
- \( l_1 = 0.50 \text{ m} \), \( l_2 = 0.15 \text{ m} \), \( l_3 = 0.90 \text{ m} \) \( m_1 = 1.5 \text{ kg} \)
- \( m_2 = m_3 = 3 \text{ kg} \), \( m_4 = 5 \text{ kg} \), \( m_p = 15 \text{ kg} \)
- \( k_{12} = \frac{4}{\pi} \times 10^{-3} \text{ m} \)

\[
\begin{bmatrix}
0.1 & 0.001 \\
0.05 & 0.002 \\
1.5 & 0.02 & 1.5 \\
\end{bmatrix}
\]

4. Dynamics simulation procedure

The procedure for solving the inverse dynamics of the Star parallel manipulator by using the principle of virtual work can be summarised in several basic steps.

1°. For a period of \( \Delta t = 15 \text{ s} \), it is assumed that the time-history of the moving platform is specified in terms of its position about the fixed frame from Eq. (5). The relations (8) give the evolution of joint variables \( \psi_{10}^i, \psi_{21}^i, \psi_{32}^i \) (\( i = A, B, C \)).

2°. Using the relations (2) and (3) we compute the transformation matrices of three legs \( A : B, C : \psi_{10}^i, \psi_{21}^i, \psi_{32}^i \).

3°. Determine the velocities and accelerations of all links by performing the inverse kinematics analysis in terms of prescribed velocities \( \dot{x}_0^c, \dot{y}_0^c, \dot{z}_0^c \) and accelerations \( \ddot{x}_0^c, \ddot{y}_0^c, \ddot{z}_0^c \) of the mass centre \( G \). Specifically, for each leg, from the conditions of connectivity (12) and (16) we compute the relative angular velocities \( \dot{\omega}_{10}^i, \dot{\omega}_{21}^i, \dot{\omega}_{32}^i \), and the relative angular accelerations \( \ddot{\omega}_{10}^i, \ddot{\omega}_{21}^i, \ddot{\omega}_{32}^i \).

4°. Using same Eq. (12), where we introduce \( \omega_{10}^{A_1} = 1, \omega_{10}^{B_1} = 0, \omega_{10}^{C_1} = 0 \), we compute, for example, the virtual characteristic velocity of each element of the robot. Other sets of virtual velocities are obtained if we consider successively that \( \omega_{10}^{A_2} = 1, \omega_{10}^{B_2} = 0, \omega_{10}^{C_2} = 1, \omega_{10}^{A_3} = 0, \omega_{10}^{B_3} = 0, \omega_{10}^{C_3} = 0 \).

5°. We compute the velocity and the acceleration of the centre of mass as well as the angular velocity and the angular acceleration of each link using the Eqs. (9)–(11) and (17).

6°. Decompose artificially the robot in several open-loop chains by cutting open at the moving revolute joints \( A_5, B_5, C_5 \). More simplified system is just the moving platform.

7°. For each link and platform we determine the inertia force, the moment of inertia forces (18) and (19), the resulting force and the resulting moment (excluding the actuator torque) exerted to the rigid body \( T_c \), from recursive Eq. (20).

8°. Finally, we find the torque \( m_{10}^i, m_{10}^c \) and the power \( P_{10}^i = \omega_{10}^i m_{10}^i \) from the recursive Eq. (21). Analogous relations give the input torques \( m_{10}^i, m_{10}^c \) and the powers \( P_{10}^i, P_{10}^c \) of other two actuators.

Based on the algorithm derived from above equations, a computer program was developed to solve the inverse dynamics of the Star parallel mechanism, using the MATLAB software. To validate the dynamics modelling, it is assumed that the platform starts at rest from a central configuration and its centre moves along a rectilinear or curvilinear trajectory in a platform’s translation motion. Furthermore, at the initial location, the moving platform is assumed to be located 0.9 m lower than the fixed base, namely \( t = 0 : x_0^c = 0, y_0^c = 0, z_0^c = 0.9 \text{ m} \).

Assuming that there is no external force and moment acting on the moving platform, the time-history evolution of powers of the three actuators during the platform’s evolution are shown in two computational complexities: complete dynamic model and simplified dynamic model in which the mass \( m_1 \) and inertia tensor \( J_1 \) of the element \( T_1 \) are negligible. The computational accuracy is now compared.

The following examples are solved to illustrate the algorithm. For the first example, the moving platform moves along the vertical \( z_0 \text{ direction} \) with variable acceleration while all the other positional parameters are held equal to zero. As can be seen from Fig. 3, it is proved to be true that all actuating powers are permanently equal to one another. When the moving platform is going to the fixed base, the limbs become more horizontally oriented, therefore increasing the actuating torques.

If the platform’s centre \( G \) moves with constant velocity in the vertical \( x_0z_0 \) plane along a half-circle of radius \( x_0^c \), the powers required by the three actuators are calculated by the programme and plotted versus time as follows \( P_{10}^i, P_{10}^c \) (Fig. 4) and \( P_{10}^i, P_{10}^c \) (Fig. 5).

In a third example of curvilinear translation, the centre \( G \) of the platform starts from its initial position and describes in the vertical bisector plane an ellipse arc in agreement with the equations

\[
\begin{align*}
\dot{x}_0^c &= \dot{y}_0^c = \dot{y}_0^c \sin \left( \frac{\pi}{15} t \right) \\
\ddot{z}_0^c &= \ddot{x}_0^c = \ddot{y}_0^c = \ddot{z}_0^c = 1 - \cos \left( \frac{\pi}{15} t \right) \sin \left( \frac{\pi}{15} t \right) \end{align*}
\]

The powers developed by the actuators are graphically sketched in Figs. 6–8.

The simulation through the MATLAB programme certifies that one of the major advantages of the current matrix recursive formulation is a reduced number of additions or multiplications and consequently a smaller processing time of numerical computation in comparison with the approach based on the Lagrange equations. Also, the proposed method can be applied to various types of complex manipulators of higher degrees of freedom, when the number of components of the mechanism is increased.
5. Conclusions

Most of dynamical models based on the Lagrange formalism neglect the weight of intermediate bodies and take into consideration the active forces or moments only and the wrench of applied forces on the moving platform. The number of relations given by this approach is equal to the total number of the position variables and Lagrange multipliers inclusive. Also, the analytical calculations involved in these equations present a risk of errors.

The commonly known Newton–Euler method, which takes into account the free-body-diagrams of the mechanism, leads to a large number of equations with unknowns including also the connecting forces in the joints. Finally, the actuating torques can be obtained.

Within the inverse kinematics analysis some exact relations that give in real-time the position, velocity and acceleration of each element of the parallel robot have been established in the present paper. The dynamics model takes into consideration the mass, the tensor of inertia and the action of weight and inertial force introduced by each element of the manipulator.

Based on the principle of virtual work, the new approach is far more efficient, can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of torques and powers required by the three actuators.

The recursive matrix relations (21) and (22) represent a set of explicit equations of the dynamic simulation and, in a context of automatic command, can easily be transformed into a robust model for computerised control of the Star parallel manipulator.

It has been show that some simplified models are not appropriate. Also, the method described above is available in forward and inverse mechanics of serial and parallel mechanisms, the platform of which behaves in translation, spherical evolution or more general six-degrees-of-freedom motion.
References


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