Nonautonomous saddle-node bifurcations in the quasiperiodically forced logistic map

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Abstract

We provide a local saddle-node bifurcation result for quasiperiodically forced interval maps. As an application, we give a rigorous description of saddle-node bifurcations of 3-periodic graphs in the quasiperiodically forced logistic map with small forcing amplitude.

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1 Introduction

Recently, many papers have been devoted to bifurcation problems of non-autonomous dynamical systems. See e.g. Fabbri et al. [4], Johnson et al. [9], Kloeden [12], Kloeden and Siegmund [13], Novo et al. [18], Núñez and Obaya [19], Rasmussen [21, 22]. Kloeden and Siegmund [13] give a survey and introduce several definitions of attractor of nonautonomous dynamical systems. Johnson et al. [9] and Fabbri et al. [4] study an alternative local theory applicable in the context of differential equations depending on some small parameter. Langa et al. [14, 15] base their investigation on the notion of pullback attractor and observe bifurcations for certain nonautonomous differential equations. Rasmussen [21, 22] gives some new notions of attractivity and repulsivity which are different from those given by Langa et al. [14, 15] and uses them to develop a bifurcation theory of nonautonomous difference equation as well as nonautonomous differential equation. Núñez et al. [19] present a bifurcation theory by considering attraction properties of minimal sets for skew product flows generated by deterministic scalar differential equation. Kloeden [12] investigates pitchfork and transcritical bifurcations of a vector valued differential equation with an autonomous linear part and a homogeneous nonlinearity multiplied by an almost periodic function.

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In this paper, we study saddle-node bifurcations in families of quasiperiodically driven monotone interval maps

\[ F_\beta : \Omega \times I \to \Omega \times I, \quad F_\beta(\omega, x) = (\theta \omega, f_\beta(\omega, x)), \quad \beta \in [a, b], \]  

(1)

where \( \Omega = \mathbb{R}^d / \mathbb{Z}^d \) is the \( d \)-dimensional torus and \( \theta : \Omega \to \Omega \) is the rotation by some totally irrational vector in \( \Omega \). The assumptions on \( f_\beta \) include its continuity, the continuity of its derivative with respect to \( x \), its convexity with respect to \( x \) and some monotonicity conditions with respect to \( x \) and \( \beta \). Our main focus is the application of a general bifurcation result to the quasiperiodically forced logistic map given by (1) with

\[ f_{\beta, \epsilon}(\omega, x) = \beta (1 + \epsilon \cos(2\pi \omega)^p) x(1 - x) \]  

(2)

for small forcing parameter \( \epsilon \). The bifurcating objects we study are invariant or periodic graphs. For a quasiperiodically driven monotone interval map, these play the same role as fixed points for unperturbed maps. We prove the existence of a critical parameter \( \beta_0 \) at which the number and properties of periodic graphs of \( f_\beta \) in a certain region of the phase space change: two continuous periodic graphs exist, one has positive Lyapunov exponent and the other has negative Lyapunov exponent for \( \beta > \beta_0 \); only one continuous or a pair of pinched semi-continuous periodic graphs exists for \( \beta = \beta_0 \); and no periodic graph exists for \( \beta < \beta_0 \).

Our general bifurcation result (Theorem 12) can be considered as a discrete-time analogue of results by Novo et al. [18] and Núñez and Obaya [19] on non-autonomous scalar convex differential equation. For the proof, we draw on methods from the proof of [8, Theorem 2.1], were a special case was treated. For our purposes, we need to generalise this result and obtain a flexible version which holds under \( C^1 \)-open conditions on the parameter family. Since these conditions are always met by parameter families of unforced interval maps undergoing a classical saddle-node bifurcation, the statement still applies for small non-autonomous perturbations. In particular, we obtain local saddle-node bifurcations of three-periodic graphs in the quasiperiodically forced logistic map (2). This map has recently been described as a prototypical example for the birth of strange non-chaotic attractors [6, 17, 20] and a rigorous proof for a special case has been given by Bjerklöv in [3]. Our results provide a general setting for the further analysis of such bifurcation phenomena.

This paper is organized as follows. The general setting is described in Section 2, in which we include the most basic concepts and results on topological dynamics and ergodic theory used throughout the paper. Section 3 is devoted to the proof of the local saddle-node bifurcation theorem. Section 4 contains the application of this local saddle-node bifurcation theorem to the quasiperiodically forced logistic maps.

2 Quasiperiodically driven \( C^k \)-interval maps

2.1 Invariant graphs, invariant measures and Lyapunov exponents

In all of the following, we denote by \( \Omega = \mathbb{R}^d / \mathbb{Z}^d \) the \( d \)-dimensional torus and suppose \( \theta : \Omega \to \Omega \) is the rotation by some totally irrational vector \( \alpha \in \Omega \). In particular, \( \theta \) is a uniquely ergodic and minimal isometry. The Lebesgue measure on \( \Omega \) is denoted by \( \mathcal{P} \).

**Definition 1** (Quasiperiodically driven \( C^k \)-interval maps). Let \( X = [a, b] \subseteq \mathbb{R} \) be a compact interval. Then \( F : \Omega \times X \to \Omega \times X \) is called a *quasiperiodically driven (or
forced) $C^k$-interval map if it is a skew product of the form $F: \Omega \times X \to \Omega \times X$, $(\omega, x) \mapsto (\theta \omega, f(\omega, x))$ such that

(i) for all $\omega \in \Omega$ the map $x \mapsto f(\omega, x)$ is $k$ times continuously differentiable in $x$,

(ii) all derivatives $D^i f(x, \omega)$ of $x \mapsto f(\omega, x)$ up to order $i = k$ depend continuously on $(\omega, x)$.

Further, we say $F$ is monotone if the maps $x \mapsto f(\omega, x)$ are strictly monotonically increasing for all $\omega \in \Omega$.

Let $F \in F^k(\Omega \times X)$ be the set of all quasiperiodically driven $C^k$-interval maps on $\Omega \times X$ and denote the usual $C^k$-distance on $\mathcal{F}^k$ by $d_{C^k}$. Given $F \in \mathcal{F}^k(\Omega \times X)$ we let $f^n(\omega, x) = \pi_2 \circ F^n(\omega, x)$, where $\pi_2$ denotes the projection to the second coordinate. Thus $F^n(\omega, x) = (\theta^n \omega, f^n(\omega, x))$.

A fundamental concept in this context is that of invariant graphs.

**Definition 2 (Invariant Graph)** Let $F \in \mathcal{F}^k(\Omega \times X)$. An invariant graph of $F$ is a measurable function $\varphi: \Omega \to X$ that satisfies

$$f(\omega, \varphi(\omega)) = \varphi(\theta \omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (3)$$

For convenience, we will use the term invariant graph both for $\varphi$ and for the associated point set $\Phi := \{(\omega, \varphi(\omega)) : \omega \in \Omega\}$. An invariant graph of $F^n$, where $q \in \mathbb{N}$, is called a $q$-periodic graph of $F$.

We will usually identify invariant graphs that are equal $\mathbb{P}$-almost everywhere and thus, implicitly, always speak of equivalence classes. However, there is one important exception we have to address. When an invariant graph has some topological properties like continuity or semi-continuity, then these can certainly be destroyed by modifications on a set of measure zero. Hence, some care has to be taken and we will therefore use the following convention.

Whenever we say that an invariant graph is continuous or semi-continuous, this means that there exists a representative in the respective equivalence class with this property. Further, we require that for this particular representative equation (3) holds for all $\omega \in \Omega$. There is only one case where this terminology might be somewhat ambiguous. It is possible that an equivalence class of invariant graphs contains both an upper and a lower semi-continuous representative, but no continuous one. We will always explicitly mention this possibility whenever it may occur. However, in most cases it can be excluded due to a non-zero Lyapunov of the invariant graph, see Corollary 7 below.

Let $\varphi$ be a given invariant graph of $F$. We define a probability measure $\mathbb{P}_\varphi$ on $(\Omega \times X, \mathcal{B}(\Omega) \otimes \mathcal{B}(X))$ by

$$\mathbb{P}_\varphi(A) := \mathbb{P}(\{(\omega, \varphi(\omega)) : \omega \in A\}) \quad \forall A \in \mathcal{B}(\Omega) \otimes \mathcal{B}(X). \quad (4)$$

is called the induced probability measure of $\varphi$. Using the invariance of $\varphi$ and the ergodicity of $\theta$, it is easy to see that $\mathbb{P}_\varphi$ is ergodic with respect to $F$. The following result shows that when $F$ is monotone this construction can be reversed, such that there is a one-to-one correspondence between invariant graphs and ergodic invariant measures.

**Lemma 3** (Invariant Graphs and Ergodic Measures, Theorem 1.8.4 in [1]). Let $F \in \mathcal{F}^0(\Omega \times X)$ be monotone and assume that $\mu$ is an $F$-invariant ergodic measure. Then there exists an invariant graph $\varphi$ such that $\mu = \mathbb{P}_\varphi$, where $\mathbb{P}_\varphi$ is defined as in (4).
The proof in [1] is given for continuous-time systems, but literally remains valid for discrete time.

In order to assign a well-defined point set to an equivalence class of almost everywhere defined invariant graphs, we introduce the essential closure of an invariant graph \( \varphi \) by
\[
\bar{\Phi}^{\text{ess}} := \text{supp}(\mathbb{P}_\varphi) = \{(\omega, x) : \mathbb{P}_\varphi(U) > 0 \ \forall \text{open neighborhoods } U \text{ of } (\omega, x)\}.
\] (5)

In the following lemma, we recall some basic properties of semi-continuous invariant graphs and their essential closure.

**Lemma 4 ([7]).** Let \( F \in \mathcal{F}^0(\Omega \times X) \) be monotone and suppose that \( \varphi \) is an invariant graph of \( F \). Then the following holds.

(i) \( \bar{\Phi}^{\text{ess}} \) is a compact invariant set of \( F \).
(ii) \( \mathbb{P}_\varphi(\bar{\Phi}^{\text{ess}}) = 1 \).
(iii) If \( \tilde{\varphi}(\omega) = \varphi(\omega) \) \( \mathbb{P}_\varphi\)-a.e., then then \( \bar{\Phi}^{\text{ess}} \equiv \bar{\Phi}^{\text{ess}} \).
(iv) \( \bar{\Phi}^{\text{ess}} \) is contained in every compact set which contains \( \mathbb{P}_\varphi\)-a.e. point of \( \Phi \).

The Lyapunov exponent of an invariant graph \( \varphi \) is given by
\[
\lambda(\varphi) := \int_{\Omega \times X} \log |Df(\omega, x)| \, d\mathbb{P}_\varphi(\omega, x) = \int_{\Omega} \log |Df(\omega, \varphi(\omega))| \, d\mathbb{P}(\omega),
\] where \( Df(\omega, x) \) denotes the derivative of \( f(\omega, x) \) with respect to \( x \). Due to Birkhoff’s Ergodic Theorem, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log |Df^n(\omega, x)| = \lambda(\varphi) \quad \text{for } \mathbb{P}_\varphi\text{-a.e. } (\omega, x) \in (\Omega \times X). \tag{6}
\]
If \( \varphi \) is continuous then the Uniform Ergodic Theorem (see, for example, [10]) implies that (6) holds for all \( \omega \in \Omega \) and the convergence is even uniform on \( \Omega \). This observation entails the following perturbation result.

**Lemma 5** (Persistence of Continuous Invariant Graphs, [23]). Let \( F \in \mathcal{F}^1(\Omega \times X) \) and suppose that \( F \) has a continuous invariant graph \( \varphi \). Assume that \( \lambda(\varphi) < 0 \). Then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( \tilde{F} \in \mathcal{F}^1(\Omega \times X) \) with \( d_{\mathcal{C}1}(F, \tilde{F}) \leq \delta \) there exists a continuous invariant graph \( \tilde{\varphi} \) of \( \tilde{F} \) satisfying that \( |\varphi(\omega) - \tilde{\varphi}(\omega)| \leq \varepsilon \) for all \( \omega \in \Omega \). When \( F \) is monotone, the same holds in \( \lambda(\varphi) > 0 \).

Furthermore, we have the following criterion for the continuity of invariant graphs.

**Theorem 6** (Theorem 1.14 and Corollary 1.15 in [25]). Let \( F \in \mathcal{F}^1(\Omega \times X) \) and suppose \( A \subseteq \Omega \times X \) is a compact \( F \)-invariant set such that all invariant graphs contained in \( A \) have strictly negative Lyapunov exponent. Then \( A \) is a finite union of continuous curves. In particular, if \( A_\omega = \{x \in X \mid (\omega, x) \in A\} \) is a single interval for all \( \omega \in \Omega \), then \( A \) is a continuous invariant graph.

This allows to show that the pathological case of two semi-continuous, but no continuous invariant graphs in the same equivalence class can only occur in combination with Lyapunov exponent zero.
Corollary 7 (Proposition 4.1 in [24]). Suppose $F \in \mathcal{F}^1(\Omega \times X)$ is monotone and $\varphi^-$ and $\varphi^+$ are lower, respectively upper semi-continuous invariant graphs with $\varphi^-(\omega) = \varphi^+(\omega)$ $\mathbb{P}$-a.e. Then $\lambda(\varphi^\pm) \neq 0$ implies that the two graphs coincide and are continuous.

Finally, there is also an intimate relation between invariant graphs and compact invariant sets of quasi-periodically driven monotone interval maps. Suppose $F \in \mathcal{F}^0(\Omega \times X)$ is monotone and $A$ is a compact $F$-invariant set. Then the upper bounding graph $\varphi_A^+$ of $A$ defined by
\[
\varphi_A^+(\omega) = \sup \{ x \in X \mid (\omega, x) \in A \}
\] is an upper-semi continuous invariant graph. (In particular, the equation $\varphi_A^+(\theta \omega) = f(\omega, \varphi_A^+ (\omega))$ holds for all $\omega \in \Omega$, as required in the discussion below 2.) Analogously, the lower bounding graph $\varphi_A^-$ of $A$ defined by
\[
\varphi_A^-(\omega) = \inf \{ x \in X \mid (\omega, x) \in K \}
\] is a lower-semi continuous invariant graph. The set $A$ is called pinched if $\varphi_A^+$ and $\varphi_A^-$ coincide on a residual subset of $\Omega$ (residual in the sense of Baire’s Theorem). Similarly, we call a lower semi-continuous invariant graph $\varphi^-$ and an upper semi-continuous invariant graph $\varphi^+$ with $\varphi^- \leq \varphi^+$ pinched, if the set $[\varphi^-, \varphi^+] := \{ (\omega, x) \mid x \in [\varphi^-(\omega), \varphi^+(\omega)] \}$ is pinched, that is, if $\{ \omega \in \Omega \mid \varphi^- (\omega) = \varphi^+ (\omega) \}$ is residual. The following basic observations related to these notions are taken from [24].

Lemma 8 ([24]). Suppose $F \in \mathcal{F}^0(\Omega \times X)$ is monotone and $\varphi^+, \varphi^- : \Omega \to X$ are upper semi-continuous, respectively lower semi-continuous invariant graphs of $F$. Then the following holds:

(i) If $\varphi^- (\omega) > \varphi^+ (\omega)$ for some $\omega \in \Omega$, then there exists $\varepsilon > 0$ such that $\varphi^- (\omega) > \varphi^+(\omega) + \varepsilon \forall \omega \in \Omega$.

(ii) If $\varphi^- (\omega) \leq \varphi^+(\omega)$ for some $\omega \in \Omega$, then $\varphi^- (\omega) \leq \varphi^+(\omega) \forall \omega \in \Omega$.

(iii) If $\varphi^- (\omega) \leq \varphi^+(\omega)$ for all $\omega \in \Omega$ and there exists $\omega_0 \in \Omega$ such that $\varphi^+(\omega_0) = \varphi^- (\omega_0)$, then $\varphi^-$ and $\varphi^+$ are pinched.

(iv) The sets $\{ (\omega, x) \in \Omega \times X : x \leq \varphi^+ (\omega) \}$ and $\{ (\omega, x) \in \Omega \times X : x \geq \varphi^-(\omega) \}$ are closed.

(v) A compact invariant set $A \subseteq K$ of $F$ is pinched provided that either $\Phi_A^+ \subset \Phi_A^-$ or $\Phi_A^- \subset \Phi_A^+$, where $\Phi$ denotes the topological closure of $\Phi$.

(vi) $A$ is minimal if and only if $A = \Phi_A^- = \Phi_A^+$. In particular, any minimal set is pinched.

2.2 Preliminary results

In order to describe bifurcations of invariant graphs, it will be crucial to have an upper bound on the number of these objects that can coexist in a given system. This is achieved by the following result, which is originally due to Keller. We concentrate on the invariant graphs in a certain region $K = \{ (\omega, x) \in \Omega \times X \mid x \in [\gamma^- (\omega), \gamma^+ (\omega)] \}$, where $\gamma^+ : \Omega \to X$ are measurable (and usually continuous) functions. We let $K_\omega = [\gamma^- (\omega), \gamma^+ (\omega)]$.

Theorem 9 (Theorem 3.3 in [8]). Let $F \in \mathcal{F}^2(\Omega \times X)$ be monotone and suppose that for all $\omega \in \Omega$ the maps $x \mapsto f(\omega, x)$ are strictly convex on $K_\omega$. Then the following holds:

(i) $F$ has at most two invariant graphs in $K$ (in the sense of equivalence classes).
(ii) If $F$ has exactly two invariant graphs $\varphi^-$ and $\varphi^+$ in $K$ and $\varphi^-(\omega) < \varphi^+(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, then $\lambda(\varphi^-) < 0 < \lambda(\varphi^+)$. 

If in addition the curves $\gamma^\pm$ are both mapped above themselves and there exist two invariant graphs in $K$, then it turns out that these invariant graphs can be obtained as the pointwise limits of the forwards iterates of $\gamma^-$ and the backwards iterates of $\gamma^+$, respectively. This characterization will play an important role in the proof of the bifurcation result (Theorem 12) below. We define two sequences of curves $(\varphi_n^-)_{n=0}^\infty$ and $(\varphi_n^+)_{n=0}^\infty$ by

$$
\varphi_n^-(\omega) := f^n(\theta^{-n}\omega, \gamma^- (\theta^{-n}\omega)) \quad \text{and} \quad \varphi_n^+(\omega) := f^{-n}(\theta^n\omega, \gamma^+ (\theta^n\omega)). \quad (9)
$$

**Lemma 10.** Let $F \in \mathcal{F}^0(\Omega \times X)$ be monotone and suppose $\gamma^-$ and $\gamma^+$ are both continuous and there holds $f(\omega, \gamma^\pm(\omega)) \geq \gamma^\pm(\theta \omega)$ for all $\omega \in \Omega$. Then we have:

(i) The map $F$ has an invariant graph in $K$ if and only if $\varphi_n^-(\omega), \varphi_n^+(\omega) \in K_\omega$, for all $n \in \mathbb{N}$ and $\omega \in \Omega$.

(ii) Suppose that $F$ has an invariant graph in $K$. Then the limits

$$
\varphi^-(\omega) := \lim_{n \to \infty} \varphi_n^-(\omega) \quad \text{and} \quad \varphi^+(\omega) := \lim_{n \to \infty} \varphi_n^+(\omega)
$$

exist and are invariant graphs in $K$. Furthermore, $\varphi^-$ is lower semi-continuous and $\varphi^+$ is upper semi-continuous and $\varphi^-(\omega) \leq \varphi(\omega) \leq \varphi^+(\omega)$ for all invariant graphs $\varphi$ of $F$ in $K$.

**Proof.** (i) An elementary computation yields that

$$
\varphi_n^-(\omega) = f^{n-k}(\theta^{-(n-k)}\omega, \varphi_{k-n}^- (\theta^{-(n-k)}\omega)) \quad \text{for all } 0 \leq k \leq n, \omega \in \Omega. \quad (10)
$$

Suppose that $\varphi_n^-(\omega) \in K_\omega$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. In particular, $\varphi_n^-(\omega) \geq \gamma^-(\omega) = \varphi_0^-(\omega)$, which together with (10) and the monotonicity of $F$ implies that the sequence $(\varphi_n^- (\omega))_{n \in \mathbb{N}}$ is increasing. At the same time it is bounded above by $\gamma^+$, such that the limit $\varphi^-(\omega) := \lim_{n \to \infty} \varphi_n^-(\omega)$ exists. Setting $k = n - 1$ in (10) and using the continuity of $f$ gives that $\varphi^-$ is an invariant graph of $F$.

Conversely, let $\varphi$ be an invariant graph of $F$ in $K$. In this case $A = \overline{\Phi}^{ss} \subseteq K$ and hence $\varphi_A^+$ are invariant graphs in $K$ that satisfy (3) for all $\omega \in \Omega$. We may thus assume that $\varphi$ is defined and invariant everywhere.

By Definition 2,

$$
\varphi(\omega) = f^n(\theta^{-n}\omega, \varphi(\theta^{-n}\omega)).
$$

This together with the monotonicity of $F$ implies that $\varphi^-(\omega) \leq \varphi(\omega) \leq \gamma^+(\omega) \forall n \in \mathbb{N}$. Similarly, we obtain $\varphi^-(\omega) \geq \gamma^-(\omega) \forall n \in \mathbb{N}$, and by replacing $f$ with $f^{-1}$ we can repeat the same argument to show that $\varphi_n^+(\omega) \in K_\omega \forall n \in \mathbb{N}$ as well. This completes the proof of part (i).

(ii) Let $\varphi$ be an invariant graph of $F$ in $K$. As in part (i), we obtain that for each fixed $\omega \in \Omega$ the sequence $(\varphi_n^-(\omega))_{n \in \mathbb{N}}$ is contained in $K$ and its limit $\varphi^-(\omega) := \lim_{n \to \infty} \varphi_n^-(\omega)$ is an invariant graph. Further, as the limit of an increasing sequence of continuous curves $\varphi^-$ is lower semi-continuous.

Similarly, we see that $\varphi^+$ is an upper semi-continuous invariant graph of $F$ in $K$. Finally, using the invariance of $\varphi$ and the monotonicity of $F$ on $K$ we get $\varphi_n^-(\omega) \leq \varphi(\omega) \leq \varphi_n^+(\omega) \forall n \in \mathbb{N}$. Letting $n \to \infty$ gives that $\varphi^-(\omega) \leq \varphi(\omega) \leq \varphi^+(\omega)$. \qed
If the fibre maps $x \mapsto f(\omega, x)$ are convex on $K_\omega$ in the preceding statement, then Theorem 9 implies that $\varphi^-$ and $\varphi^+$ are the only invariant graphs in $K$. In this situation there only exist two alternatives concerning the structure of these graphs: Either they are pinched, or they are both continuous. This is the content of the following statement, which is formulated in a slightly more general way.

**Proposition 11.** Let $F \in \mathcal{F}^1(\Omega \times X)$ be monotone and suppose that $F$ has exactly two invariant graphs $\varphi^- \leq \varphi^+$ in $K$ which are lower, respectively upper semi-continuous and have non-zero Lyapunov exponents. Then either $\varphi^-$ and $\varphi^+$ are pinched, or they are both continuous invariant graphs.

**Proof.** Suppose that $\varphi^-$ and $\varphi^+$ are not pinched, that is, $\varphi^-(\omega) < \varphi^+(\omega)$ for all $\omega \in \Omega$. We denote by $\psi^+$ the upper bounding graph of $\Phi^-$, that is, $\psi^+ = \varphi^+_{\Phi^-}$ in the sense of (7). Similarly, we let $\psi^- = \varphi^-_{\Phi^+}$. Then due to Lemma 8(v) the pairs $\varphi^-, \psi^+$ and $\psi^-, \varphi^+$ are pinched, which means that there exists a residual set $R \subseteq \Omega$ such that

$$\varphi^-(\omega) = \psi^+(\omega) \quad \text{and} \quad \psi^-(\omega) = \varphi^+(\omega) \quad \forall \omega \in R. \quad (11)$$

We distinguish two cases. First, suppose that $\psi^+ \geq \psi^-$. Then we obtain that

$$\varphi^-(\omega) = \psi^+(\omega) \geq \psi^-(\omega) = \varphi^+(\omega) \quad \forall \omega \in R,$$

such that $\varphi^-$ and $\varphi^+$ are pinched.

Secondly, assume that there exists some $\omega \in \Omega$ with $\psi^+(\omega) < \psi^-(\omega)$. Then due to Lemma 8(i) we have $\psi^+(\omega) < \psi^-(\omega) \forall \omega \in \Omega$. Consequently, the compact $F$-invariant sets $A^- = [\varphi^-, \psi^+]$ and $A^+ = [\psi^-, \varphi^+]$ are disjoint, and each of them contains exactly one invariant graph $\varphi^-$, respectively $\varphi^+$. Since both these invariant graphs have non-zero Lyapunov exponent, Theorem 11 implies that both sets are just continuous curves, and must therefore coincide with the graphs $\Phi^-$ and $\Phi^+$. \hfill \Box

### 3 Saddle-node bifurcations in quasiperiodically forced interval maps

We now consider parameter families of quasiperiodically forced $C^2$-interval maps $(F_\beta)_{\beta \in [0,1]}$ and write $F_\beta(\omega, x) = (\theta \omega, f_\beta(\omega, x))$. As in the previous section, we concentrate on a region $K = \{(\omega, x) \in \Omega \times X \mid x \in [\gamma^-(\omega), \gamma^+(\omega)]\}$, where $\gamma^\pm : \Omega \to X$ are continuous curves.

**Theorem 12** (Local Saddle-Node Bifurcation Theorem). Suppose that the parameter family of quasiperiodically forced $C^2$-interval maps $(F_\beta)_{\beta \in [0,1]}$ satisfies the following conditions:

1. **(H1)** $Df_\beta(\omega, x) > 0$ for all $(\omega, x) \in K$ and $\beta \in [0,1]$ (monotonicity);
2. **(H2)** for all $\omega \in \Omega$ and $\beta \in [0,1]$, the map $x \mapsto f_\beta(\omega, x)$ is strictly convex on $K_\omega$ (convexity);
3. **(H3)** for all $\omega \in \Omega$ and $\beta \in [0,1]$

$$f_\beta(\omega, \gamma^-(\omega)) > \gamma^-(\theta \omega) \quad \text{and} \quad f_\beta(\omega, \gamma^+(\omega)) > \gamma^+(\theta \omega) \quad (12)$$

(Dynamical behaviour at the boundaries of $K$);
(H4) the maps \((\beta, \omega, x) \mapsto f_\beta(\omega, x)\) and \((\beta, \omega, x) \mapsto Df_\beta(\omega, x)\) are continuous (continuity in the parameter);

(H5) for all \((\omega, x) \in K\), the map \([0, 1] \to X, \beta \mapsto f_\beta(\omega, x)\) is strictly decreasing (monotonicity in the parameter);

(H6) \(F_1\) has two continuous invariant graphs in \(K\), and \(F_0\) has no invariant graphs in \(K\) (behaviour at the extremal parameters).

Then there exists a critical parameter \(\beta_c \in (0, 1)\) such that the following statements hold

(i) If \(\beta < \beta_c\), then \(F_\beta\) has no invariant graphs in \(K\).

(ii) If \(\beta = \beta_c\), then \(F_\beta\) has either one or two invariant graphs in \(K\). We denote them by \(\varphi^-\) and \(\varphi^+\) (allowing \(\varphi^- = \varphi^+\)), where \(\varphi^- \leq \varphi^+\). Then \(\varphi^-\) is lower semi-continuous and \(\varphi^+\) is upper semi-continuous. Further, one of the two following holds:

\[- \varphi^- \text{ equals } \varphi^+ \text{ } \mathbb{P}\text{-a.e. and then } \lambda(\varphi^-) = \lambda(\varphi^+) = 0 \text{ (Smooth Bifurcation).}\]

\[- \varphi^- \neq \varphi^+ \text{ } \mathbb{P}\text{-a.e., } \lambda(\varphi^-) < 0, \lambda(\varphi^+) > 0 \text{ and both invariant graphs are non-continuous (Non-smooth Bifurcation).}\]

In any case, the set \(B := \{(\omega, x) \in \Omega \times X \mid x \in [\varphi^-(\omega), \varphi^+(\omega)]\}\) is pinched.

(iii) If \(\beta > \beta_c\), then \(F_\beta\) has exactly two invariant graphs in \(K\), both of which are continuous. We denote them by \(\varphi^-_\beta\) and \(\varphi^+\), with \(\varphi^-_\beta < \varphi^+_\beta\). There holds \(\lambda(\varphi^-_\beta) < 0, \lambda(\varphi^+_\beta) > 0\), and the dependence of the graphs on \(\beta\) is continuous and monotone: If \(\beta\) is decreased, then \(\varphi^-_\beta\) moves upward and \(\varphi^+_\beta\) moves downward uniformly on all fibers.

Remark 13. Using the reflection \((\omega, x) \mapsto (\omega, -x)\), one can formulate a dual version of Theorem 12 for concave fiber maps. Replacing assumptions (H2), (H3) and (H5) by

(H2') for all \(\omega \in \Omega\) and \(\beta \in [0, 1]\), the map \(x \mapsto f_\beta(\omega, x)\) is strictly concave on \(K_\omega\);

(H3') for all \(\omega \in \Omega\) and \(\beta \in [0, 1]\)

\[f_\beta(\omega, \gamma^-(\omega)) < \gamma^-(\theta \omega) \quad \text{and} \quad f_\beta(\omega, \gamma^+(\omega)) < \gamma^+(\theta \omega);\]

(H5') for all \((\omega, x) \in K\), the map \([0, 1] \to X, \beta \mapsto f_\beta(\omega, x)\) is strictly increasing;

the statements of Theorem 12 remain true, with the only difference that the signs of the Lyapunov exponents are reversed, that is, \(\lambda(\varphi^-) < 0, \lambda(\varphi^+) > 0\) and \(\lambda(\varphi^-_\beta) < 0, \lambda(\varphi^+_\beta) > 0\) in Theorem 12 is replaced by \(\lambda(\varphi^-) > 0, \lambda(\varphi^+) < 0\) and \(\lambda(\varphi^-_\beta) > 0, \lambda(\varphi^+_\beta) < 0\).

Proof of Theorem 12. First of all, note that we may assume without loss of generality that all maps \(F_\beta\) are monotone on all of \(X\). Since we are only interested in the dynamics and the invariant graphs inside \(K\), we may just modify the \(F_\beta\) outside of \(K\) otherwise. Inside of \(K\) monotonicity is guaranteed by (H1).

\(^1\)This includes the case of a single neutral continuous invariant graph, but due to the zero Lyapunov exponent the possibility of having only semi-continuous, but no continuous representatives in the equivalence class cannot be excluded (see the discussion after (3) and Corollary 7).
Due to (H6), the map $F_1$ has an invariant graph. We define the critical parameter by

$$\beta_c := \inf \{ \beta \in [0, 1] : F_\beta \text{ has an invariant graph in } K \}.$$ 

(i) By definition of $\beta_c$, $F_\beta$ has no invariant graphs in $K$ for all $\beta < \beta_c$. 

(ii) Fix $\omega \in \Omega$ and $n \in \mathbb{N}$. By the definition of $\beta_c$, there exists a sequence $(\beta_k)_{k=1}^\infty$ with $\beta_k > \beta_c$ and $\lim_{k \to \infty} \beta_k = \beta_c$ such that the map $F_{\beta_k}$ has an invariant graph in $K$ for all $k \in \mathbb{N}$. Consequently, in view of Lemma 10(i) we get

$$\varphi_{\beta_k, n}^-(\omega) := f_{\beta_k}^n(\theta^{-n}\omega, \gamma^{-n}(\theta^{-n}\omega)) \in K_\omega.$$ 

Letting $k \to \infty$ yields that $\varphi_{\beta, n}^- = f_{\beta_k}^n(\theta^{-n}\omega, \gamma^{-n}(\theta^{-n}\omega)) \in K_\omega$. As $\omega$ and $n$ are arbitrary, Lemma 10(i) implies that $F_{\beta_k}$ has an invariant graph in $K$. Therefore, according to Lemma 10(ii), $\varphi_{\beta_c}^-$ and $\varphi_{\beta_c}^+$ are invariant graphs of $F_{\beta_c}$ in $K$ (allowing $\varphi_{\beta_c}^- = \varphi_{\beta_c}^+$).

We now show that $B := \left[ \varphi_{\beta_c}^-, \varphi_{\beta_c}^+ \right]$ is a pinched set. Suppose the opposite. Then in view of Lemma 8(iii) we have $\varphi_{\beta_c}^- (\omega) < \varphi_{\beta_c}^+ (\omega)$ for all $\omega \in \Omega$. Consequently, using Theorem 9(ii), we obtain that $\lambda(\varphi_{\beta_c}^-) < 0 < \lambda(\varphi_{\beta_c}^+)$. Together with Theorem 11, this implies that $\varphi_{\beta_c}^-$ and $\varphi_{\beta_c}^+$ are continuous invariant graphs. Due to Theorem 5 and (H4), these invariant graphs will persist for all parameters sufficiently close to $\beta_c$, contradicting the definition of $\beta_c$. Hence, the set $B$ must be pinched. To conclude the proof of this part we consider the following cases:

- $\varphi_{\beta_c}^- = \varphi_{\beta_c}^+$ $\mathbb{P}$-a.e.. If $\lambda(\varphi_{\beta_c}^+) \neq 0$, then by Corollary 7 the two graphs coincide and are continuous. Then for all $\beta$ in a small neighbourhood of $\beta_c$, contradicting the definition of $\beta_c$. Hence, we must have $\lambda(\varphi_{\beta_c}^+) = 0$.

- $\varphi_{\beta_c}^- \neq \varphi_{\beta_c}^+$ $\mathbb{P}$-a.e. Then according to Lemma 9(ii), we get $\lambda(\varphi_{\beta_c}^-) < 0 < \lambda(\varphi_{\beta_c}^+)$. Consequently, using the definition of $\beta_c$ and Theorem 5 again, the invariant graphs $\varphi_{\beta_c}^-$ and $\varphi_{\beta_c}^+$ cannot be continuous and we are in the case of a non-smooth bifurcation.

(iii) According to part (ii), $F_{\beta_k}$ has at least one invariant graph $\varphi$ in $K$. Fix $\beta > \beta_c$. Due to (H5) we obtain $\varphi_{\beta_k, n}^+(\omega) < \varphi_{\beta_k, n}^-(\omega) \leq \varphi(\omega)$ for all $n \in \mathbb{N}$. Therefore, by Lemma 10(i) the limit $\varphi_{\beta}^-(\omega) := \lim_{n \to \infty} \varphi_{\beta_k, n}^-(\omega)$ exists and is a lower semi-continuous invariant graph in $K$ of $F_{\beta}$. Lemma 10(ii) therefore implies that the limit $\varphi_{\beta}^+(\omega) := \lim_{n \to \infty} \varphi_{\beta_k, n}^+(\omega)$ exists and is an invariant graph of $F_{\beta}$ in $K$ as well. Moreover, $\varphi_{\beta}^+$ is upper semi-continuous and $\varphi_{\beta}^-(\omega) \geq \varphi(\omega)$. By conditions (H1) and (H5) we get

$$\varphi_{\beta}^-(\omega) = f_{\beta}(\theta^{-1}\omega, \varphi_{\beta}^-(\theta^{-1}\omega)) < f_{\beta}(\theta^{-1}\omega, \varphi(\theta^{-1}\omega)) = \varphi(\omega) \quad \forall \omega \in \Omega.$$ 

Similarly, $\varphi_{\beta}^+(\omega) > \varphi(\omega)$ and hence $\varphi^+(\omega) > \varphi^-(\omega)$ for all $\omega \in \Omega$, in particular the two graphs are not pinched. Further, Lemma 9(i) implies that $\varphi_{\beta}^-$ and $\varphi_{\beta}^+$ are the only invariant graphs of $F_{\beta}$ in $K$. We can therefore apply Proposition 11 to see that the invariant graphs $\varphi_{\beta}^-$ and $\varphi_{\beta}^+$ are continuous as required.

An important point concerning the application of this result to the quasi periodically forced logistic map in the next section is the fact that conditions (H1)–(H6) are satisfied on open sets in the space of parameter families equipped with a suitable topology. In order to make this precise, we let

$$\mathcal{P}[0, 1] = \{ F_1 : [0, 1] \to \mathcal{F}(\Omega \times X), \ \beta \mapsto F_\beta \mid (\beta, \omega, x) \mapsto f_{\beta}(\omega, x), \ (\beta, \omega, x) \mapsto \partial_\beta f_{\beta}(\omega, x), \ (\beta, \omega, x) \mapsto D f_{\beta}(\omega, x) \text{ and } (\beta, \omega, x) \mapsto D^2 f_{\beta}(\omega, x) \text{ are continuous} \}.$$
Suppose $K$ and Prasad et al. [20]. Note that (p) where by (H6) is an open condition as well and hence $U$ This together with Lemma 10(i) implies that $\tilde{\omega}$ $V$ such that (H6) holds on the open set $V$. Then according to Theorem 12. Let $\tilde{\omega}$ $\varepsilon$ $\Omega$ $\gamma$ $\gamma^+$ for all $\tilde{\omega}$ $\gamma$ $\Omega$ and $n_0 \in \mathbb{N}$. Hence, there exists a neighbourhood $V_2 \subseteq V_1$ of $F$ such that $\tilde{F}$ has two continuous invariant graphs in $K$ for all $\tilde{F}$ $V_2$. Further, in view of Lemma 10(i) and the fact that $F_0$ has no invariant graphs in $K$, we have $f_0^{n_0}(\theta^{-n_0}\omega_0, \gamma^{-n_0}\omega_0) > \gamma^+(\omega_0)$ for some $\omega_0 \in \Omega$ and $n_0 \in \mathbb{N}$. This together with Lemma 10(i) implies that $\tilde{F}$ has no invariant graphs in $K$ for all $\tilde{F} \in V_2$, such that (H6) holds on the open set $V_2$. Since $F$ was arbitrary, this shows that (H6) is an open condition as well and hence $\mathcal{U}_K[0,1]$ and $\tilde{\mathcal{U}}_K[0,1]$ are open. 

4 A local saddle-node bifurcation theorem for a family of quasiperiodically forced logistic maps

Let $T^1 = \mathbb{R}/\mathbb{Z}$ be the one dimensional torus and $\theta_\alpha : T^1 \to T^1$ the rotation by some irrational number $\alpha \in T^1$. We consider a family of quasiperiodically forced logistic maps $F_{\beta,\varepsilon} : T^1 \times [0,1] \to T^1 \times [0,1]$ defined by $F_{\beta,\varepsilon}(\omega, x) = (\theta_\alpha \omega, f_{\beta,\varepsilon}(\omega, x)) \quad \forall (\omega, x) \in T^1 \times [0,1], \quad (14)$ where $\varepsilon \geq 0, \beta \in \left[0, \frac{4}{1+\varepsilon}\right]$ and for each $\beta, \varepsilon$ the function $f_{\beta,\varepsilon} : T^1 \times [0,1] \to [0,1]$ is given by $f_{\beta,\varepsilon}(\omega, x) = \beta \left(1 + \varepsilon \cos(2\pi\omega)^p\right) x(1-x) \quad \forall (\omega, x) \in T^1 \times [0,1],$ where $p \in \mathbb{N}$. For more information about this family of maps, we refer to Negi et al. [17] and Prasad et al. [20]. Note that $(f_{\beta,0})_{\beta \in [0,1]}$ is the family of unforced logistic maps, i.e. $f_{\beta,0}(\cdot, x) = \beta x(1-x) := f_{\beta}(x) \quad \forall x \in [0,1].$ (15) As depicted in Figure 1, the map $f_{\beta}$, where $\beta = \beta_c := 1+\sqrt{8}$, has a unique 3-periodic orbit
in \([0, 1]\) which is denoted by \(\{x_1, x_2, x_3\}\) with \(x_1 < x_2 < x_3\). Furthermore, \(Df^3_{\beta_c}(x_1) = Df^3_{\beta_c}(x_3) = 1\) and the following inequalities hold

\[
\begin{align*}
D^2f^3_{\beta_c}(x_1) &> 0, & D^2f^3_{\beta_c}(x_2) &> 0, & D^2f^3_{\beta_c}(x_3) &< 0, \\
\partial_\beta f^3_{\beta_c}(x_1)|_{\beta = \beta_c} &< 0, & \partial_\beta f^3_{\beta_c}(x_2)|_{\beta = \beta_c} &< 0, & \partial_\beta f^3_{\beta_c}(x_3)|_{\beta = \beta_c} &> 0,
\end{align*}
\]

(16)

see e.g. [2, 16]. A bifurcation result illustrated in Figure 1 is formulated theoretically as follows (see [2, 16] for more details):

**Lemma 15.** Consider the family of unforced logistic maps \((f_\beta)_{\beta \in [0,4]}\) defined as in (15). Then there exist \(\eta > 0\) and intervals \(I_1, I_2, I_3 \subset [0,1]\) containing \(x_1, x_2, x_3\), respectively, such that

(i) If \(\beta \in (\beta_c - \eta, \beta_c)\), then \(f_\beta\) has no \(3\)-periodic points in \(I_1 \cup I_2 \cup I_3\).

(ii) If \(\beta = \beta_c\), then each of the intervals \(I_1, I_2, I_3\) contains a unique \(3\)-periodic point \(x_1, x_2, x_3\) of \(f_\beta\), respectively.

(iii) If \(\beta \in (\beta_c, \beta_c + \eta)\), then \(f_\beta\) has exactly two \(3\)-periodic points in \(I_i\) denoted by \(x_i(\beta)^- < x_i(\beta)^+\). Furthermore, \(x_1(\beta), x_2(\beta), x_3(\beta)\) and \(x_1^+(\beta), x_2^+(\beta), x_3^-(\beta)\) are \(3\)-periodic orbits of \(f_\beta\) and satisfy

\[
\begin{align*}
Df^3_{\beta_c}(x_1(\beta)) &= Df^3_{\beta_c}(x_2^-(\beta)) = Df^3_{\beta_c}(x_3^+(\beta)) < 1, \\
Df^3_{\beta_c}(x_1^+(\beta)) &= Df^3_{\beta_c}(x_2^+(\beta)) = Df^3_{\beta_c}(x_3^-(\beta)) > 1.
\end{align*}
\]

Moreover, for \(i = 1,2,3\) the functions \(x_i^-(\cdot)\) are monotonically decreasing and \(x_i^+(\cdot)\) are monotonically increasing and \(\lim_{\beta \to \beta_c} x_i^+(\beta) = x_i\).

So far we have introduced a bifurcation result of \(3\)-periodic orbits for the family of unforced logistic maps. Now we consider \(3\)-periodic graphs of \(F_{\beta, \epsilon}\). Relations between periodic points of \(f_\beta\) and periodic graphs of \(F_{\beta, \epsilon}\) are our next step to get bifurcation results of the family of quasiperiodically forced logistic maps \(F_{\beta, \epsilon}\) defined as in (14). One of these relations is provided in the following remark.

**Remark 16.** (i) If \(x \in [0,1]\) is a \(q\)-periodic point of \(f_\beta\) then the constant map \(\varphi : T^1 \to [0,1], \omega \mapsto x\) is a \(q\)-periodic graph of \(F_{\beta, \epsilon}\).
(ii) Let $I$ be a compact interval of $[0,1]$ and suppose that $\varphi : \mathbb{T}^1 \to I$ is a $q$-periodic graph of $F_{\beta,0}$. Then $f_{\beta}$ has a $q$-periodic point in $I$. To see this, we suppose the opposite, i.e. $f_{\beta}$ has no $q$-periodic points in $I$. By continuity, we assume w.l.o.g. that $f_{\beta}^3(x) > x$ for all $x \in I$. This together with the invariance of $\varphi$ implies that $\varphi(\theta_{q_{a}} \omega) = f_{\beta}^3(\varphi(\omega)) > \varphi(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \mathbb{T}^1$. Therefore, $f_{\beta}^3(\varphi(\theta_{q_{a}} \omega)) d\mathbb{P}(\omega) > f_{\beta}^3(\varphi(\omega)) d\mathbb{P}(\omega)$ which contradicts the fact that $\theta_{q_{a}}$ preserves the probability measure $\mathbb{P}$. Hence, $f_{\beta}$ has a $q$-periodic point in $I$.

Let $\eta$ be given as in Lemma 15. By (16) and the fact that $\lim_{\beta \to \beta_c} x_i^+ (\beta) = x_i, i = 1, 2, 3$, we can choose $\eta$ smaller, if necessary, to get that for all $\beta \in [\beta_c - \eta, \beta_c + \eta]$ the following statements hold:

\[
\begin{align*}
Df_{\beta}^3(x) > 0, & \quad D^2f_{\beta}^3(x) > 0, & \quad \partial_{\beta}f_{\beta}^3(x) < 0 & \forall x \in [x_1^- (\beta_c + \eta), x_1^+ (\beta_c + \eta)], \\
Df_{\beta}^3(x) > 0, & \quad D^2f_{\beta}^3(x) > 0, & \quad \partial_{\beta}f_{\beta}^3(x) < 0 & \forall x \in [x_2^- (\beta_c + \eta), x_2^+ (\beta_c + \eta)], \\
Df_{\beta}^3(x) > 0, & \quad D^2f_{\beta}^3(x) < 0, & \quad \partial_{\beta}f_{\beta}^3(x) > 0 & \forall x \in [x_3^- (\beta_c + \eta), x_3^+ (\beta_c + \eta)].
\end{align*}
\]

(17)

For an arbitrary $\zeta \in [0,\eta]$, we define

\[
K_1(\zeta) := \mathbb{T}^1 \times [x_1^- (\beta_c + \zeta), x_1^+ (\beta_c + \zeta)] \quad \text{for } i = 1, 2, 3.
\]

(18)

Since $\lim_{\beta \to \beta_c} x_i^+ (\beta) = x_i$ and the fact that $\{x_1, x_2, x_3\}$ is a periodic orbit of $f_{\beta_1}$, it follows that there exist $\delta^* \in (0,\eta)$ and $\varepsilon_1 > 0$ such that the following inclusions

\[
F_{\beta_c}(K_1(\delta^*)) \subset K_2(\eta), \quad F_{\beta_c}(K_2(\delta^*)) \subset K_3(\eta), \quad F_{\beta_c}(K_3(\delta^*)) \subset K_1(\eta),
\]

(19)

hold for all $\beta \in [\beta_c - \delta^*, \beta_c + \delta^*] \times [0, \varepsilon_1]$. Now we choose and fix $\zeta \in [\delta^*, \eta]$. By (17) the family of maps $F_{\beta_0}^{3} : \mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1]$, where $\beta \in [\beta_c - \delta^*, \beta_c + \delta^*]$, fulfils the local properties (H1), (H2), (H4) and (H5) on the set $K_1(\zeta)$ and

\[
x_i^+ (\beta_c + \zeta) = f_{\beta_c+\zeta}^3(x_i^+ (\beta_c + \zeta)) < f_{\beta_c}^3(x_i^+ (\beta_c + \zeta)) \quad \forall \beta \in [\beta_c - \delta^*, \beta_c + \delta^*].
\]

(20)

It means that property (H3) is also satisfied, which describes the dynamical behaviour of $F_{\beta_0}^{3}$, where $\beta \in [\beta_c - \delta^*, \beta_c + \delta^*]$, at the boundary of $K_1(\zeta)$. Lemma 15 and Remark 16 imply (H6) and hence $F_{\beta_0}^{3}(\cdot,0) \in \mathcal{U}_{K_1(\zeta)}[\beta_c - \delta^*, \beta_c + \delta^*]$, where the set $\mathcal{U}_{K_1(\zeta)}[\beta_c - \delta^*, \beta_c + \delta^*]$ is defined as in (13) with the parameter interval $[0,1]$ replaced by $[\beta_c - \delta^*, \beta_c + \delta^*]$. Similarly, we get that $F_{\beta_0}^{3}(\cdot,0) \in \mathcal{U}_{K_2(\zeta)}[\beta_c - \delta^*, \beta_c + \delta^*] \cap \mathcal{U}_{K_3(\zeta)}[\beta_c - \delta^*, \beta_c + \delta^*]$. Then, using Proposition 14 and the fact that

\[
\lim_{\varepsilon \to 0} d_P \left( F_{\beta_0}^{3}(\cdot,\varepsilon) - F_{\beta_0}^{3}(\cdot,0) \right) = 0,
\]

there exists $\varepsilon^* \in (0, \varepsilon_1)$ such that for all $\varepsilon \in [0, \varepsilon^*)$

\[
F_{\beta_0}^{3}(\cdot,\varepsilon) \in \mathcal{U}_{K_1(\varepsilon)}[\beta_c - \delta^*, \beta_c + \delta^*] \cap \mathcal{U}_{K_2(\varepsilon)}[\beta_c - \delta^*, \beta_c + \delta^*] \cap \mathcal{U}_{K_3(\varepsilon)}[\beta_c - \delta^*, \beta_c + \delta^*]
\]

(21)

and

\[
F_{\beta_0}^{3}(\cdot,\varepsilon) \in \mathcal{U}_{K_1(\delta^*)}[\beta_c - \delta^*, \beta_c + \delta^*] \cap \mathcal{U}_{K_2(\delta^*)}[\beta_c - \delta^*, \beta_c + \delta^*] \cap \mathcal{U}_{K_3(\delta^*)}[\beta_c - \delta^*, \beta_c + \delta^*].
\]

(20)

We are now in a position to state and prove a local saddle-node bifurcation theorem for 3-periodic graphs of the family of quasi-periodically forced logistic maps (14).
Theorem 17 (Local Saddle-node Bifurcation of 3-Periodic Graphs in Quasiperiodically Forced Logistic Map). Consider the family of maps $F_{\beta,\xi} : T^3 \times [0,1] \to T^3 \times [0,1]$, where $\xi > 0, \beta \in \left[0, \frac{4}{1+\xi}\right]$, defined as in (14). Let $\delta^*, \varepsilon^* > 0$ and $\eta > 0$ such that (19), (20) and (21) hold. Then there exists a function $\beta_c : [0, \varepsilon^*) \to \left[0, \frac{4}{1+\xi}\right]$ such that the following statements hold for each $\xi \in [0, \varepsilon^*)$:

(i) If $\beta \in (\beta_c(\xi) - \delta^*, \beta_c(\xi))$, then $F_{\beta,\xi}$ has no 3-periodic graphs in $K_1(\eta), K_2(\eta)$ or $K_3(\eta)$.

(ii) If $\beta = \beta_c(\xi)$, then each of sets $K_1(\eta), K_2(\eta), K_3(\eta)$ contains exactly one continuous 3-periodic graph or a pair of pinched semi-continuous 3-periodic graphs of $F_{\beta,\xi}$.

(iii) If $\beta \in (\beta_c(\xi), \beta_c(\xi) + \delta^*)$, then $F_{\beta,\xi}$ has two continuous 3-periodic graphs in $K_i(\eta)$, denoted by $\varphi_{-\beta,c,i}$ and $\varphi_{+\beta,c,i}$ with $\varphi_{-\beta,c,i}(\omega) < \varphi_{+\beta,c,i}(\omega)$ for all $\omega \in T^1$ and $i = 1, 2, 3$. Furthermore,

$$\Phi_{\beta,\xi,2}^+ = F_{\beta,\xi}(\Phi_{\beta,\xi,1}^+), \Phi_{\beta,\xi,3}^-(\omega) = F_{\beta,\xi}(\Phi_{\beta,\xi,2}^+), \Phi_{\beta,\xi,1}^- = F_{\beta,\xi}(\Phi_{\beta,\xi,3}^-),$$

and their Lyapunov exponents satisfy

$$\lambda(\varphi_{\beta,c,1}^-) = \lambda(\varphi_{\beta,c,2}^-) = \lambda(\varphi_{\beta,c,3}^+) < 0 < \lambda(\varphi_{\beta,c,2}^+) = \lambda(\varphi_{\beta,c,3}^-).$$

Moreover, $\varphi_{\beta,c,1}^-, \varphi_{\beta,c,2}^-$ and $\varphi_{\beta,c,3}^+$ move upward, whereas $\varphi_{\beta,c,1}^+, \varphi_{\beta,c,2}^+, \varphi_{\beta,c,3}^+$ move downward as $\beta$ is decreased.

Proof. Choose and fix $\xi \in [0, \varepsilon^*)$. Using (20) and Theorem 12, for $i = 1, 2, 3$ there exists a critical point $\beta_{c,i}(\xi)$ for $F_{(\xi)}$ on the set $K_i(\delta^*)$. It is easy to see that for $i = 1, 2, 3$ an application of Theorem 12 and (21) yields the same critical point $\beta_{c,i}(\xi)$ for $F_{(\xi)}$ on the larger set $K_i(\eta) \supset K_i(\delta^*)$. According to Theorem 12(iii), for all $\beta \in (\beta_{c,i}(\xi), \beta_c + \delta^*)$, $i = 1, 2, 3$, the map $F_{\beta,\xi}$ has two 3-periodic continuous graphs in $K_i(\delta^*) \subset K_i(\eta)$ denoted by $\varphi_{-\beta,c,i}$ and $\varphi_{+\beta,c,i}$ with $\varphi_{-\beta,c,i}(\omega) < \varphi_{+\beta,c,i}(\omega)$ for all $\omega \in T^1$. To prove that $\beta_{c,i}$ is independent of $i$, let $\beta \in (\beta_{c,1}(\xi), \beta_c + \delta^*)$ be chosen arbitrarily. Since $\Phi_{\beta,c,1}^*$ are 3-periodic graphs of $F_{\beta,\xi}$, it follows that $F_{\beta,\xi}(\Phi_{\beta,c,1}^*)$ are also 3-periodic graphs of $F_{\beta,\xi}$. By (19), we get

$$F_{\beta,\xi}(\Phi_{\beta,c,1}^*) \subset F_{\beta,\xi}(K_1(\delta^*)) \subset K_2(\eta),$$

which implies that $\beta_{c,2}(\xi) \leq \beta_{c,1}(\xi)$. Analogously, we have $\beta_{c,3}(\xi) \leq \beta_{c,2}(\xi)$ and $\beta_{c,1}(\xi) \leq \beta_{c,3}(\xi)$. Thus, $\beta_{c,1}(\xi) = \beta_{c,2}(\xi) = \beta_{c,3}(\xi) =: \beta_c(\xi)$ and (22) is proved. The rest of this theorem is obtained by applying Theorem 12 to the map $F_{\beta,\xi}^3$ on the sets $K_1(\eta), K_2(\eta)$ and $K_3(\eta)$. The proof is complete.

Now we set $p = 9, \varepsilon = 0.0006, \alpha = (\sqrt{5} - 1)/2$ and consider the family of quasiperiodically forced logistic maps $F_{\beta,\xi}$ defined as in (14). A numerical investigation of the dynamics of $F_{\beta,\xi}$ yields a bifurcation at $\beta_c \approx 3.828529$. Figure 2 shows a forward orbit and some close-ups for $\beta < \beta_c$ before the bifurcation and Figure 3 displays the attracting and repelling three-periodic invariant graphs which exist after the bifurcation for $\beta > \beta_c$.

References

Figure 2: Before bifurcation, one forward orbit for $\beta = 3.828526 < \beta_c$. 


Figure 3: After bifurcation, repelling (red) and attracting (blue) three-periodic invariant graphs for $\beta = 3.828580 > \beta_c$.


