Even and Odd Pairs in Linegraphs of Bipartite Graphs

Stefan Hougardy

Two vertices in a graph are called an even pair (odd pair) if all induced paths between these two vertices have even (odd) length. Even and odd pairs have turned out to be of importance in conjunction with perfect graphs. We will characterize all linegraphs of bipartite graphs that contain an even resp. odd pair. In general, it is a co-NP-complete problem to decide whether a graph contains an even pair. For the class of linegraphs of bipartite graphs we will show that testing for even resp. odd pairs can be done in polynomial time.

1. Introduction

Two vertices in a graph are called an even pair if every induced path between these two vertices has even length. A graph is called strict quasi-parity if every induced subgraph contains an even pair or is a clique. The class of strict quasi-parity graphs was introduced by Meyniel [11] and is denoted SQP for short.

A graph is called minimal non strict quasi-parity if the graph does not belong to SQP but every proper induced subgraph does. Meyniel [11] has posed the problem to characterize all minimal non strict quasi-parity graphs.

To attack this problem we proposed after discussions with Chinh Hoàng the following conjecture [8] which—if true—shows the significance of linegraphs of bipartite graphs in conjunction with even pairs.

Conjecture 1. Every minimal non strict quasi-parity graph is:
(i) an odd cycle of length at least five, or
(ii) the complement of a cycle of length at least seven, or
(iii) the linegraph of a bipartite graph.

For the class of planar graphs this conjecture has been proved recently by Linhares et al. [10].

The main motivation for studying strict quasi-parity graphs is their connection to perfect graphs. A graph is called perfect if for every induced subgraph the chromatic number of the subgraph equals its clique number. A minimal imperfect graph is a graph that itself is not perfect but all its proper induced subgraphs are.

Meyniel [11] has shown that no minimal imperfect graph can contain an even pair and thereby proved that the graphs in SQP are perfect.

Many of the known classes of perfect graphs, such as bipartite graphs, comparability graphs, permutation graphs, interval graphs and triangulated graphs—to mention just the ‘classical’ ones—are subsets of SQP. On the other hand, there are several classes of perfect graphs, strongly perfect graphs [2], BIP* [4], alternately orientable graphs [7] and slim graphs [6], that are supposed to be in SQP, but so far no-one has been able to prove this. Here a better knowledge of minimal non strict quasi-parity graphs would be very helpful.

Another link between perfect graphs and strict quasi-parity graphs was established in [9]. Berge’s famous Strong Perfect Graph Conjecture [1] states that a graph is perfect iff the graph and its complement do not contain odd induced cycles of length at
least five. In [9] it was shown that this conjecture is equivalent to the statement that minimal imperfect graphs are minimal non strict quasi-parity.

2. Even Pairs

If Conjecture 1 turns out to be true then the next problem in characterizing minimal non strict quasi-parity graphs is to decide which linegraphs of bipartite graphs are minimal non strict quasi-parity. The main result of this section proceeds in this direction by giving a characterization of those linegraphs of bipartite graphs that contain an even pair.

**Theorem 1.** The linegraph $G$ of a bipartite graph $H$ contains an even pair iff there exist two non-incident edges $a_1a_2$ and $b_1b_2$ in $H$ such that $a_1$ and $b_1$ resp. $a_2$ and $b_2$ belong to the same colorclass of the bipartite graph $H$ and to different components of $H - a_2 - b_2$ resp. $H - a_1 - b_1$.

**Proof.** Let $G$ be the linegraph of a bipartite graph $H$ and let $(a, b)$ be an even pair in $G$. By $a_1a_2$ resp. $b_1b_2$ we denote the edges in $H$ which correspond to the vertices $a$ resp. $b$ in $G$. We may assume that $a_1$ and $b_1$ resp. $a_2$ and $b_2$ belong to the same colorclass of $H$. Since $a$ and $b$ cannot be adjacent, the edges $a_1a_2$ and $b_1b_2$ are non-incident. Now suppose that there exists a path in $H$ between $a_1$ and $b_1$ that uses neither the vertex $a_2$ nor the vertex $b_2$. Then the length of this path must be even. A shortest such path will result in an odd induced path between $a$ and $b$ in $G$ (see Figure 1).

This contradicts the fact that $(a, b)$ is an even pair in $G$. Thus we know that every path in $H$ connecting $a_1$ and $b_1$ must contain at least one of the two vertices $a_2$ and $b_2$. This means that $a_1$ and $b_1$ belong to different components of $H - a_2 - b_2$. Using symmetrical arguments we see that $a_2$ and $b_2$ belong to different components of $H - a_1 - b_1$.

We now assume that $H$ has the property stated in the theorem. We have to show that $G$ contains an even pair. Let $a$ and $b$ be the vertices of $G$ which correspond to the edges $a_1a_2$ and $b_1b_2$ in $H$. Since the edges $a_1a_2$ and $b_1b_2$ are non-incident the vertices $a$ and $b$ are non-adjacent. Suppose now that there exists an odd induced path between $a$ and $b$ in $G$. This path corresponds to an even path in $H$ which connects the two edges $a_1a_2$ and $b_1b_2$. Taking a shortest such path and using symmetry we may assume that

![Figure 1](image-url)
this path connects $a_1$ with $b_1$ and uses neither $a_2$ nor $b_2$. But such a path cannot exist since $a_1$ and $b_1$ belong to different components of $H - a_2 - b_2$. \square

**Remark.** Bienstock [3] has shown that deciding whether a graph contains an even pair is a co-NP-complete problem. Restricted to the class of linegraphs of bipartite graphs, the above characterization shows that this problem can be solved in polynomial time. To this end one has to make use of the well known fact that a graph can be derived from its linegraph in polynomial time [14].

We can also give a characterization of complements of linegraphs of bipartite graphs that contain an even pair. It is well known that a linegraph of a bipartite graph cannot contain a diamond [5] (a diamond is the graph that is obtained from the complete graph on four vertices by removing one edge). Thus the desired characterization can easily be obtained from the following observation.

**Lemma 1.** The complement of a diamond-free graph contains an even pair iff there exist two non-adjacent vertices which are not connected by an induced path of length 3.

**Proof.** The necessity of the above condition is obvious. The sufficiency follows from the simple fact that the complement of an odd path of length at least 5 contains a diamond. \square

**Corollary 1.** The complement of a linegraph of a bipartite graph contains an even pair iff there exist two non-adjacent vertices which are not connected by an induced path of length 3.

**Remark.** Again, this characterization yields a polynomial time algorithm to detect even pairs in complements of linegraphs of bipartite graphs.

Obviously, Theorem 1 together with Corollary 1 gives a characterization of linegraphs of bipartite graphs such that the graph and its complement are even pair free. Unfortunately, these conditions are rather technical. We therefore want to give a simpler condition guaranteeing that a linegraph of a bipartite graph and its complement do not contain an even pair. To this end, we first prove the following lemma.

**Lemma 2.** Let $H$ be a bipartite graph of minimum degree at least 3. Then the complement of its linegraph does not contain an even pair.

**Proof.** Let $G$ denote the linegraph of $H$. We have to prove that any two non-adjacent vertices $a$ and $b$ in $G$ are connected by an odd induced path. Let $a_1a_2$ and $b_1b_2$ be the edges in $H$ corresponding to the vertices $a$ and $b$ in $G$. Since $a$ and $b$ are non-adjacent in $G$ they are adjacent in $G$ and therefore the edges $a_1a_2$ and $b_1b_2$ in $H$ are incident. Thus we may assume that $a_1 = b_1$. Since $H$ is bipartite the vertices $a_2$ and $b_2$ cannot be adjacent. Since by assumption these vertices have degree at least 3 there must exist two non-incident edges $a_2v$ and $b_2w$ such that $v \neq w$. (Take as $v$ any neighbor of $a_2$ different from $a_1$ and as $w$ any neighbor of $b_2$ different from $a_1$ and $v$). Let $x$ resp. $y$ denote the vertices in $G$ corresponding to the edges $a_2v$ and $b_2w$ in $H$. Then $axyb$ is an induced $P_4$ in $G$. \square

Now we can easily give a sufficient condition for a linegraph of bipartite graphs such that the graph and its complement are even pair free.
FIGURE 2. A bipartite graph and its linegraph.

LEMMA 3. Let $H$ be a 3-connected bipartite graph. Then its linegraph and the complement of its linegraph do not contain an even pair.

PROOF. Since $H$ is 3-connected each vertex of $H$ has degree at least 3 and therefore by Lemma 2 the complement of the linegraph of $H$ is even pair free. From Theorem 1 it also follows that the linegraph of $H$ cannot contain an even pair. □

REMARK. From Lemma 2 it also follows that the complement of a linegraph of a 3-edge-connected bipartite graph is even pair free. In contrast to this, there exist arbitrarily high edge-connected bipartite graphs the linegraphs of which contain an even pair. This is shown by the example in Figure 2.

The bipartite graph consists of four copies of the $K_{3,3}$ that are identified as shown in the figure. It is easy to see that this graph is 3-edge-connected. However, two opposite vertices of the inner quadrilateral of its linegraph form an even pair.

By taking a $K_{p,p}$ instead of the $K_{3,3}$ one obtains in an analogous way arbitrarily high edge-connected bipartite graphs the linegraphs of which contain an even pair.

3. ODD PAIRS

In this section we will show that nearly the same results as shown in the last section are valid for odd pairs. First we give a counterpart to Theorem 1 by giving a characterization of linegraphs of bipartite graphs that contain an odd pair.

THEOREM 2. The linegraph $G$ of a bipartite graph $H$ contains an odd pair iff there exist two non-incident edges $a_1a_2$ and $b_1b_2$ in $H$ such that $a_1$ and $b_1$ resp. $a_2$ and $b_2$ belong to different colorclasses of the bipartite graph $H$ and to different components of $H - a_2 - b_2$ resp. $H - a_1 - b_1$.

PROOF. The proof is similar to that of Theorem 1. Instead of looking at paths between $a_1$ and $b_1$ resp. $a_2$ and $b_2$, one has to look at paths between $a_1$ and $b_2$ resp. $a_2$ and $b_1$. Then the proof is essentially the same as for Theorem 1 and is therefore omitted. □

We will now show that a counterpart of Lemma 3 also holds for odd pairs.
**Lemma 4.** Let $H$ be a 3-connected bipartite graph. Then its linegraph and the complement of its linegraph do not contain an odd pair.

**Proof.** From Theorem 2 it follows that the linegraph of $H$ cannot contain an odd pair.

We will now show that $\bar{G}$, the complement of the linegraph of $H$, also does not contain an odd pair. Let $a$ and $b$ be any two non-adjacent vertices of $\bar{G}$. These vertices are adjacent in $G$ and therefore two incident edges of $H$ correspond to these two vertices. Since $H$ is 3-connected and bipartite there must exist an edge in $H$ that is not incident with any of these two edges. But this means that $a$ and $b$ are connected by a path of length 2 in $\bar{G}$. Therefore $a$ and $b$ do not form an odd pair.

**Remark.** As mentioned in the introduction, Meyniel has shown that no minimal imperfect graph can contain an even pair. Whether an analogous statement holds for odd pairs is still an open question [12]. Nevertheless, one could think of defining a class of graphs similar to the class SQP, but in terms of odd pairs. But since a diamond does not contain an odd pair and diamond-free Berge graphs are known to be perfect [13] this would not give an interesting new class of perfect graphs.

**References**

13. K. R. Parthasarathy and G. Ravindra, The validity of the Strong Perfect-Graph Conjecture for $(K_4-e)$-free graphs, *J. Combin. Theory, Ser. B* 26 (1979), 98–100. (The proof contained an error that was corrected in [15].)

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Stefan Hougardy
Humboldt-Universität zu Berlin
Institut für Informatik,
Lehrstuhl Algorithmen und Komplexität,
Unter den Linden 6, 10099 Berlin, Germany
hougardy@informatik.hu-berlin.de