A note on a paper by Su Ke and He Zhen

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1. Introduction and preliminaries

The purpose of this note is to point out that the class of multivalued operators considered in [7] in connection with the study of evolution inclusion is a subclass of pseudomonotone operators. This means that the hypotheses made on the operator in [7] are strong enough so that the result on existence of solutions to the evolution inclusion in [7] is a consequence of a theorem of Bian and Webb [3].

Let $H$ be a separable Hilbert space and let $V$ be a dense subspace of $H$ carrying the structure of a separable reflexive Banach space with continuous embedding $V \subset H$. Identifying $H$ with its dual, the triple of spaces $(V, H, V^*)$ is called an evolution triple (cf. [4]). We denote by $(\cdot, \cdot)$ the duality pairing of $V$ and its dual $V^*$, by $\| \cdot \|$, $\| \cdot \|$ and $\| \cdot \|_{V^*}$ the norms in $V, H$ and $V^*$, respectively. Let $p \geq 2, 1/p + 1/q = 1$ and $(0, T)$ is a finite time interval. Given a Banach space $(X, \| \cdot \|_X)$ and $U \subset X$, we write $\|U\|_X = \sup\{\|x\|_X : x \in U\}$.

We recall the following definitions (cf. e.g. [1–4]).

Definition 1. Let $V$ be a reflexive Banach space and $A : V \to 2^{V^*}$ be an operator.

We say that $A$ is “pseudomonotone”, if

1. $A$ has values which are nonempty, bounded, closed and convex;
2. $A$ is usc from each finite dimensional subspace of $V$ into $V^*$ furnished with the weak topology;
3. if $v_n \to v$ weakly in $V$ and $v_n^* \in A(v_n)$ is such that $\lim \sup (v_n^*, v_n - v) \leq 0$, then for every $y \in V$, there exists $u(y) \in A(v)$ such that $(u(y), v - y) \leq \lim \inf (v_n^*, v_n - y)$.

We say that $A$ is “quasimonotone”, if for any sequences $v_n \to v$ weakly in $V$, $v_n^* \in A(v_n)$, we have $0 \leq \lim \sup (v_n^*, v_n - v)$.
We say that \( A \) is “of type \((M)\)”, if

1. \( A \) has values which are nonempty, bounded, closed and convex;
2. \( A \) is usc from each finite dimensional subspace of \( V \) into \( V^* \) furnished with the weak topology;
3. if \( v_n \to v \) weakly in \( V \), \( v_n^* \in A(v_n) \), \( v_n^* \to v^* \) weakly in \( V^* \) and \( \limsup (v_n^*, v_n - v) \leq 0 \), then \( v^* \in A(v) \).

We say that \( A \) is “generalized pseudomonotone”, if for any sequences \( v_n \to v \) weakly in \( V \), \( v_n^* \to v^* \) weakly in \( V^* \), \( v_n^* \in A(v_n) \) and \( \limsup (v_n^*, v_n - v) \leq 0 \), we have \( v^* \in A(v) \) and \( \langle v_n^*, v_n \rangle \to \langle v^*, v \rangle \).

**Remark 2.** It is easy to observe (cf. [2]) that \( A \) is quasimonotone if and only if for any sequences \( v_n \to v \) weakly in \( V \), \( v_n^* \in A(v_n) \), we have \( 0 \leq \liminf (v_n^*, v_n - v) \).

### 2. Problem statement

In this section we recall the Cauchy problem for an evolution inclusion of first order

\[
\begin{align*}
    u'(t) + B(t, u(t)) & \ni f(t) \quad \text{for a.e. } t \in (0, T) \\
    u(0) & = u_0,
\end{align*}
\]

where \( B: (0, T) \times V \to 2^{V^*} \) is a multivalued operator between a reflexive Banach space \( V \) and its dual \( V^* \) and \( f, u_0 \) are given.

This problem has been studied in many papers, cf. [1,2,5,6] and the references therein. Recently the problem (1) has been treated by Su Ke and He Zhen in [7] who generalized the results on existence of solutions to (1). They assumed that \( B(t, \cdot) \) is not pseudomonotone but of type \((M)\) and they employ a surjectivity theorem for an abstract operator inclusion in a reflexive Banach space. More precisely, the following hypotheses on the multivalued term are used in [7]:

- the operator \( B: (0, T) \times V \to 2^{V^*} \) satisfies
  - \((B1)\) \( B(\cdot, v) \) is measurable with nonempty closed convex values and \( B(t, \cdot) \) is of type \((M)\) for \( t \in (0, T) \);
  - \((B2)\) \( \|B(t, v)\|_{V^*} \leq b_1 \|v\| + b_2(t) \) for \( t \in (0, T) \), \( v \in V \) with \( b_1 \geq 0 \) and \( b_2(t) \) is \( L^1(0, T) \);
  - \((B3)\) \( \inf_{v \in B(\cdot, v)} \|v\|_{V^*} \geq b_3 \|v\| - b_4(t) \) for \( t \in (0, T) \), \( v \in V \) with \( b_3 \geq 0 \) and \( b_4 \in L^1(0, T) \);
  - \((B4)\) if \( v_n \to v \) weakly in \( V \) and \( v_n^* \in B(t, v_n) \), then \( 0 \leq \liminf (v_n^*, v_n - v) \).

However, in our opinion the result of [7] is not a new contribution in the field since under the hypotheses \((B1)\), \((B2)\) and \((B4)\), the operator \( B(t, \cdot) \) is pseudomonotone for \( t \in (0, T) \) and in this case the existence of solutions for (1) is well known (cf. Theorem 4.4 in [3]).

**Proposition 3.** Let \( V \) be a reflexive Banach space and \( A: V \to 2^{V^*} \) be an operator. If \( A \) is bounded, quasimonotone and of type \((M)\), then it is pseudomonotone.

**Proof.** Let \( A: V \to 2^{V^*} \) be bounded, quasimonotone and of type \((M)\). Recall (cf. e.g. Proposition 6.3.66 in [4]) that if \( A \) is a generalized pseudomonotone operator which is bounded (i.e. maps bounded sets to bounded ones) and for every \( v \in V \), \( A(v) \) is nonempty, closed and convex, then \( A \) is pseudomonotone. Therefore, it is enough to show that \( A \) is generalized pseudomonotone and apply the above result. Let us assume that \( v_n \to v \) weakly in \( V \), \( v_n^* \to v^* \) weakly in \( V^* \), \( v_n^* \in A(v_n) \) and \( \limsup (v_n^*, v_n - v) \leq 0 \). Since \( A \) is of type \((M)\), we obtain \( v^* \in A(v) \). By using the quasimonotonicity (cf. Remark 2), we have

\[
0 \leq \liminf (v_n^*, v_n - v) \leq \limsup (v_n^*, v_n - v) \leq 0
\]

which means that \( \langle v_n^*, v_n \rangle \to 0 \). From the equality

\[
\langle v_n^*, v_n \rangle = \langle v_n^*, v_n - v \rangle + \langle v_n^*, v \rangle,
\]

we get \( \langle v_n^*, v_n \rangle \to \langle v^*, v \rangle \). \( \square \)

On the other hand, it is still an open problem to prove the existence of solutions to (1) in the case \( B(t, \cdot) \) is of type \((M)\) and it is not pseudomonotone. It is desirable for the applicability to give an example of an operator which is not pseudomonotone but it is of type \((M)\).

### References