Linear sparse differential resultant formulas

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Abstract

Let \( \mathcal{P} \) be a system of \( n \) linear nonhomogeneous generic sparse ordinary differential polynomials in \( n - 1 \) differential indeterminates. In this paper, differential resultant formulas are presented to compute, whenever it exists, the sparse differential resultant \( \partial \text{Res}(\mathcal{P}) \) introduced by Li, Gao and Yuan in [12], as the determinant of the coefficient matrix of an appropriate set of derivatives of differential polynomials in \( \mathcal{P} \).

Keywords: differential elimination, linear differential polynomials, sparse differential resultant

1. Introduction

Elimination theory has proven to be a relevant tool in (differential) algebraic geometry (see [5],[6] and [1]). Elimination techniques have been developed using Gröbner bases, characteristic sets and (differential) resultants. The algebraic resultant has been broadly studied, regarding theory and computation, some significant references are [9], [2], [18] and [7]. Meanwhile, its counterpart the differential resultant is at an initial state of development, a survey on this development can be found in the introductions of [10] and [15]. Until very recently, the existing definitions of differential resultants for differential polynomials depended on the computation method [3]. In the recent paper [10], a rigorous definition of the differential resultant \( \partial \text{Res}(\mathcal{P}) \), of a set \( \mathcal{P} \) of \( n \) nonhomogeneous generic ordinary differential polynomials in \( n - 1 \) differential variables, has been presented: If the elimination ideal, of

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the differential ideal generated by \( \mathcal{P} \), is \( n - 1 \) dimensional then it equals the saturation ideal of a differential polynomial \( \partial \text{Res}(\mathcal{P}) \), the differential resultant of \( \mathcal{P} \). As in the algebraic case, the object that is naturally necessary for applications is the sparse differential resultant, and this was defined in [12], for a set of nonhomogeneous generic sparse ordinary differential polynomials.

It would be useful to represent the sparse differential resultant as the quotient of two determinants, as done for the algebraic case in [7]. In the differential case, these so called Macaulay style formulas do not exist, even in the simplest case. The matrices used in the algebraic case to define the Macaulay style formulas [7], are coefficient matrices of sets of polynomials obtained by multiplying the original ones by appropriate sets of monomials, [2]. In the differential case, in addition, derivatives of the original polynomials should be considered. The differential resultant formula defined by Carrà-Ferro in [3], is the algebraic resultant of Macaulay [13], of a set of derivatives of the ordinary differential polynomials in \( \mathcal{P} \). Already in the linear sparse generic case, these formulas vanish often, giving no information about the differential resultant \( \partial \text{Res}(\mathcal{P}) \).

In this paper, given a system of \( n \) linear nonhomogeneous generic sparse differential polynomials \( \mathcal{P} \), in \( n - 1 \) differential indeterminates, determinantal formulas to compute the linear sparse differential resultant \( \partial \text{Res}(\mathcal{P}) \) are provided. The linear case can be seen as a previous study to get ready to approach the nonlinear case. One can consider only the problem of taking the appropriate set of derivatives of the element in \( \mathcal{P} \) and forget about the multiplication by sets of monomials for the moment.

In [17], the linear complete differential resultant \( \partial \text{CRes}(\mathcal{P}) \) of a set of linear differential polynomials \( \mathcal{P} \) (non necessarily generic) was defined, as an improvement, in the linear case, of the differential resultant formula given by Carrà-Ferro. Still, \( \partial \text{CRes}(\mathcal{P}) \) is the determinant of a matrix having zero columns in many cases. The linear differential polynomials in \( \mathcal{P} \) can be described via differential operators. We use appropriate bounds of the supports of those differential operators to decide on a convenient set \( \mathcal{P}^\ast \) of derivatives of \( \mathcal{P} \), such that its coefficient matrix \( \mathcal{M}(\mathcal{P}^\ast) \) is squared and has no zero columns. In the generic case, we can guarantee that the linear sparse differential resultant \( \partial \text{Res}(\mathcal{P}) \) can always be computed as the determinant of a matrix \( \mathcal{M}(\mathcal{P}^\ast) \), for a convenient set \( \mathcal{P}^\ast \) of derivatives of polynomials in \( \mathcal{P} \).

Given a system of linear nonhomogeneous ordinary differential polynomials \( \mathcal{P} \), in Section 2, we describe appropriate sets bounding the supports of the differential operators describing the polynomials in \( \mathcal{P} \). Differential
resultant formulas for $\mathcal{P}$ are given in Section 3. In particular, the formula $\partial \text{FRes}(\mathcal{P})$ is defined, for the so-called super essential (irredundant) systems, as the determinant of a matrix $\mathcal{M}(\mathcal{P}\mathcal{S})$ with no zero columns. In Section 4, it is shown that every system $\mathcal{P}$ contains a super essential subsystem $\mathcal{P}^*$. Some results on linear differential polynomial parametric equations (linear DPPEs) are given in Section 5, they will be used in Section 6 to prove the main result of this paper. Namely, given a linear nonhomogeneous generic sparse system $\mathfrak{P}$ of ordinary differential polynomials, the linear sparse differential resultant $\partial \text{Res}(\mathfrak{P})$ equals $\partial \text{FRes}(\mathfrak{P}^*)$ up to a constant.

2. Preliminary notions

Let $\mathbb{D}$ be an ordinary differential domain with derivation $\partial$. Let us consider the set $U = \{u_1, \ldots, u_{n-1}\}$ of differential indeterminates over $\mathbb{D}$. By $\mathbb{N}_0$ we mean the natural numbers including 0. For $k \in \mathbb{N}_0$, we denote by $u_{j,k}$ the $k$-th derivative of $u_j$ and for $u_{j,0}$ we simply write $u_j$. We denote by $\{U\}$ the set of derivatives of the elements of $U$, $\{U\} = \{\partial^k u \mid u \in U, k \in \mathbb{N}_0\}$, and by $\mathbb{D}\{U\}$ the ring of differential polynomials in the differential indeterminates $U$, which is a differential ring with derivation $\partial$,

$$\mathbb{D}\{U\} = \mathbb{D}[u_{j,k} \mid j = 1, \ldots, n, k \in \mathbb{N}_0].$$

Given a subset $U \subset \{U\}$, we denote by $\mathbb{D}\{U\}$ the ring of polynomials in the indeterminates $U$. Given $f \in \mathbb{D}\{U\}$ and $y \in U$, we denote by $\text{ord}(f, y)$ the order of $f$ in the variable $y$. If $f$ does not have a term in $y$ then we define $\text{ord}(f, y) = -1$. The order of $f$ equals $\max\{\text{ord}(f, y) \mid y \in U\}$. We refer to [11] and [14] for concepts and results on differential algebra.

Let $\mathcal{P} := \{f_1, \ldots, f_n\}$ be a system of linear differential polynomials in $\mathbb{D}\{U\}$. We assume that:

1. The order of $f_i$ is $o_i \geq 0$, $i = 1, \ldots, n$. So that no $f_i$ belongs to $\mathbb{D}$.
2. $\mathcal{P}$ contains $n$ distinct polynomials.
3. $\mathcal{P}$ is a nonhomogeneous system. There exist $a_i \in \mathbb{D}$ and $h_i$ homogeneous differential polynomials in $\mathbb{D}\{U\}$, such that $f_i(U) = a_i - h_i(U)$ and, for some $i \in \{1, \ldots, n\}$, $a_i \neq 0$.

We denote by $\mathbb{D}[\partial]$ the ring of differential operators with coefficients in $\mathbb{D}$. There exist differential operators $\mathcal{L}_{i,j} \in \mathbb{D}[\partial]$ such that

$$f_i = a_i - \sum_{j=1}^{n-1} \mathcal{L}_{i,j}(u_j).$$
We denote by $|S|$ the number of elements of a set $S$. We call the indeterminates $U$ a set of parameters. The number of parameters of $P$ equals

$$\nu(P) := \{|j \in \{1, \ldots, n-1\} \mid L_{i,j} \neq 0 \text{ for some } i \in \{1, \ldots, n\}\}|.$$  

We assume that $\nu(P) = n - 1$.

Given a nonzero differential operator $L = \sum_{k \in \mathbb{N}_0} a_k \partial^k \in \mathbb{D}[\partial]$, let us denote the support of $L$ by $\mathcal{S}(L) = \{k \in \mathbb{N}_0 \mid a_k \neq 0\}$, and define

$$\text{ldeg}(L) := \min \mathcal{S}(L), \text{deg}(L) := \max \mathcal{S}(L).$$

For $j = 1, \ldots, n - 1$, we define the next positive integers, to construct convenient intervals bounding the supports of the differential operators $L_{i,j}$,

$$\gamma_j(P) := \min \{o_i - \text{deg}(L_{i,j}) \mid L_{i,j} \neq 0, i = 1, \ldots, n\},$$

$$\tau_j(P) := \min \{\text{ldeg}(L_{i,j}) \mid L_{i,j} \neq 0, i = 1, \ldots, n\},$$

$$\nu(P) := \tau_j(P) + \gamma_j(P).$$

Therefore, for $L_{i,j} \neq 0$ the next set of lattice points contains $\mathcal{S}(L_{i,j})$,

$$I_{i,j}(P) := [\gamma_j(P), o_i - \tau_j(P)] \cap \mathbb{Z}.$$

Finally, to explain the construction of Section 3, we will use the integer

$$\gamma(P) := \sum_{j=1}^{n-1} \gamma_j(P).$$

Let $\mathcal{K}$ be a differential field of characteristic zero with derivation $\partial$ and $C = \{c_1, \ldots, c_n\}$ a set of differential indeterminates over $\mathcal{K}$. By $\mathcal{K}\langle C \rangle$ we denote the differential field extension of $\mathcal{K}$ by $C$, the quotient field of $\mathcal{K}\{C\}$.

Let us consider the following rankings (see [11], page 75):

- The order $u_1 < \cdots < u_{n-1}$ induces an orderly ranking on $U$ (i.e. an order on $\{U\}$) as follows: $u_{i,j} < u_{k,l} \iff (j, i) <_{\text{lex}} (l, k)$. We set $1 < u_1$.

- Let $(i, j), (k, l) \in \mathbb{N}_0^2$ be distinct. We write $(i, j) < (k, l)$ if $i > k$, or $i = k$ and $j < l$. The order $c_n < \cdots < c_1$, induces a ranking on $C$, using the monomial order $\prec$: $c_{i,j} < c_{k,l} \iff (i, j) < (k, l)$.

We call $r$ the ranking on $C \cup U$ that eliminates $U$ with respect to $C$, that is $\partial^k x < \partial^k u$, for all $x \in C$, $u \in U$ and $k, k^* \in \mathbb{N}_0$. 

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3. Differential resultant formulas

Let us consider a subset \( PS \) of \( \partial P := \{ \partial^k f_i \mid i = 1, \ldots, n, k \in \mathbb{N}_0 \} \) and a set of differential indeterminates \( U \subset \{ U \} \) verifying:

- \((ps1)\) \( PS = \{ \partial^k f_i \mid k \in [0, L_i] \cap \mathbb{Z}, L_i \in \mathbb{N}_0, i = 1, \ldots, n \} \),
- \((ps2)\) \( PS \subset \mathbb{D}[U] \) and \( |U| = |PS| - 1 \).

Let \( N := \sum_{i=1}^n o_i \).

**Remark 3.1.** Sets \( PS \) and \( U \) verifying \((ps1)\) and \((ps2)\) were given in [3] and [16] (see also [17]).

1. In [3], \( L_i = N - o_i \) and \( U = \{ u_{j,k} \mid k \in [0, N] \cap \mathbb{Z}, j = 1, \ldots, n - 1 \} \).
2. In [16], Section 3, \( L_i = N - o_i - \hat{\gamma} \), where \( \hat{\gamma} := \sum_{j=1}^{n-1} \hat{\gamma}_j \),

\[
\hat{\gamma}_j := \min\{ \overline{\gamma}_j(P), \min\{ o_i \mid L_{i,j} = 0, i = 1, \ldots, n \} \},
\]

and \( U = \{ u_{j,k} \mid k \in [0, N - \hat{\gamma}_j - \hat{\gamma}] \cap \mathbb{Z}, j = 1, \ldots, n - 1 \} \).

Observe that both choices coincide if \( \hat{\gamma} = 0 \).

The coefficient matrix \( M(PS, U) \) of the differential polynomials in \( PS \) as polynomials in \( \mathbb{D}[U] \) is a \( |PS| \times |PS| \) matrix.

**Definition 3.2.** Given \( PS \) and \( U \) verifying \((ps1)\) and \((ps2)\), we call

\[
\det(M(PS, U))
\]

a differential resultant formula for \( P \).

The differential resultant formulas for \( P \) given in [3] and [17] are determinants of matrices with zero columns in many cases. Let \( PS^h := \{ \partial^k h_i \mid \partial^k f_i \in PS \} \), the set containing the homogeneous part of the polynomials in \( PS \). The coefficient matrix \( L(PS^h, U) \) of \( PS^h \), as a set of polynomials in \( \mathbb{D}[U] \), is a submatrix of \( M(PS, U) \) of size \( |PS| \times (|PS| - 1) \). We assumed that \( P \) is a nonhomogeneous system, thus if \( M(PS, U) \) has zero columns, those are columns of \( L(PS^h, U) \).
Remark 3.3. The differential resultant formula for $\mathcal{P}$ given in [16] is called the linear complete differential resultant of $\mathcal{P}$ and denoted $\partial\text{CRes}(\mathcal{P})$. With $\mathcal{P}$ and $\mathcal{U}$ as in Remark 3.1 (2), $\partial\text{CRes}(\mathcal{P}) = \det(M(\mathcal{P}, \mathcal{U}))$. Observe that, if $\gamma_j(\mathcal{P}) \neq 0$ for some $j \in \{1, \ldots, n-1\}$, then the columns of $\mathcal{L}(\mathcal{P}, \mathcal{U})$ indexed by $u_j, \ldots, u_j\gamma_j(\mathcal{P})-1$ are zero. If $\gamma_j(\mathcal{P}) > \hat{\gamma}_j$ for some $j \in \{1, \ldots, n-1\}$, then the columns of $\mathcal{L}(\mathcal{P}, \mathcal{U})$ indexed by $u_j, u_j\gamma_j(\mathcal{P})-\hat{\gamma}_j+1, \ldots, u_j, N-\gamma_j-\hat{\gamma}_j$ are zero.

If $N-o_i-\gamma(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$, the sets of lattice points $\mathbb{I}_i := [0, N-o_i-\gamma(\mathcal{P})] \cap \mathbb{Z}$ are nonempty. We define the set of differential polynomials

$$\text{ps}(\mathcal{P}) := \{\partial^k f_i \mid k \in \mathbb{I}_i, i = 1, \ldots, n\},$$

containing $L := \sum_{i=1}^{n} (N-o_i-\gamma(\mathcal{P})+1)$ differential polynomials, in the set $\mathcal{V}$ of $L-1$ differential indeterminates

$$\mathcal{V} := \{u_{j,k} \mid k \in \mathbb{I}_j, N-\gamma_j(\mathcal{P})-\gamma(\mathcal{P}) \cap \mathbb{Z}, j = 1, \ldots, n-1\}.$$

Let us assume that $\text{ps}(\mathcal{P}) = \{P_1, \ldots, P_L\}$. For $i = 1, \ldots, n$ and $k \in \mathbb{I}_i$,

$$P_{l(i,k)} := \partial^{N-o_i-\gamma(\mathcal{P})-k} f_i,$$

$$l(i,k) := \sum_{h=1}^{i-1} (N-o_h-\gamma(\mathcal{P})+1) + N-o_i-\gamma(\mathcal{P}) - k \in \{1, \ldots, L\}.$$

The matrix $M(\mathcal{P}) := M(\text{ps}(\mathcal{P}), \mathcal{V})$ is an $L \times L$ matrix. We assume that the $l$th row of $M(\mathcal{P})$, $l = 1, \ldots, L$ contains the coefficients of $P_l$ as a polynomial in $\mathbb{D}[\mathcal{V}]$, and that the coefficients are written in decreasing order with respect to the orderly ranking on $U$.

Thus, if $N-o_i-\gamma(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$, we can define a linear differential resultant formula for $\mathcal{P}$, denoted by $\partial\text{FRes}(\mathcal{P})$, and equal to:

$$\partial\text{FRes}(\mathcal{P}) := \det(M(\mathcal{P})). \quad (5)$$

In general, we cannot guarantee that the columns of $M(\mathcal{P})$ are nonzero, as the next example shows. In the next section, sufficient conditions on $\mathcal{P}$ are given for $M(\mathcal{P})$ to have no zero columns.

Example 3.4. Let $\mathcal{P} = \{f_1, f_2, f_3\}$, with $o_1 = 5$, $o_2 = 1$, and $o_3 = 1$. Let $f_1 = \mathcal{L}_{1,1}(u_1)$, $f_2 = \mathcal{L}_{2,2}(u_2)$, $f_3 = \mathcal{L}_{3,2}(u_2)$, with $\mathcal{G}(\mathcal{L}_{1,1}) = \{1, 5\}$ and $\mathcal{G}(\mathcal{L}_{2,2}) = \mathcal{G}(\mathcal{L}_{3,2}) = \{0, 1\}$. Then $\gamma(\mathcal{P}) = \gamma_3(\mathcal{P}) = 1$ and $N-o_1-\gamma(\mathcal{P}) = 1$, $N-o_2-\gamma(\mathcal{P}) = N-o_3-\gamma(\mathcal{P}) = 5$. Therefore $M(\mathcal{P})$ can be defined but columns indexed by $u_{1,3}$ and $u_{1,4}$ are zero.
3.1. Linear super essential systems

Let $S_{n-1}$ be the permutation group of $\{1, \ldots, n-1\}$. A linear differential system $P$ is called \textit{differentially essential} if, there exist $i \in \{1, \ldots, n\}$ and $\tau_i \in S_{n-1}$ such that
\[ \begin{cases} L_{j,\tau_i(n-j)} \neq 0, & j = 1, \ldots, i-1, \\ L_{j,\tau_i(n-j+1)} \neq 0, & j = i+1, \ldots, n. \end{cases} \] (6)

Observe that, if $P$ is differentially essential then $\nu(P) = n-1$ but the converse is false. Differentially essential systems of generic, non necessarily linear, differential polynomials were defined in [12], Definition 3.3 and for linear differential polynomials this requirement can be stated by (6).

\textbf{Definition 3.5.} A linear differential system $P$ is called \textit{super essential} if, for every $i \in \{1, \ldots, n\}$, there exists $\tau_i \in S_{n-1}$ verifying (6).

Given a super essential system $P$, it will be proved that $\partial \text{FRes}(P)$ can be defined and that the matrix $M(P)$ has no zero columns. For this purpose, let $\tau_i$, $i = 1, \ldots, n$ be as in Definition 3.5, to define the bijections $\mu_i : \{1, \ldots, n\} \setminus \{i\} \rightarrow \{1, \ldots, n-1\}$, by
\[ \mu_i(j) = \begin{cases} \tau_i(n-j), & j = 1, \ldots, i-1, \\ \tau_i(n-j+1), & j = i+1, \ldots, n. \end{cases} \] (7)

\textbf{Lemma 3.6.} Given a super essential system $P$, $N - o_i - \gamma(P) \geq 0$, $i = 1, \ldots, n$.

\textit{Proof.} Given $i \in \{1, \ldots, n\}$,

\[ N - o_i - \gamma(P) = \sum_{j \in \{1, \ldots, n\} \setminus \{i\}} (o_j - \gamma_{\mu_i(j)}(P)). \]

By Definition 3.5 and (7), $L_{j,\mu_i(j)} \neq 0$, $j \in \{1, \ldots, n\} \setminus \{i\}$ and, by (3), $o_j - \gamma_{\mu_i(j)}(P) \geq 0$. Which proves the result. \hfill \Box

By Lemma 3.6, the differential resultant formula $\partial \text{FRes}(P)$ can be defined. It will be proved in Theorem 6.6 that $M(P)$ has no zero columns, as a consequence of some results for generic differential polynomials.
4. Irredundant systems of linear differential polynomials

The linear differential system \( P \) is an overdetermined system, in the differential variables \( U \). Recall that we assumed \( \nu(P) = n - 1 = |P| - 1 \). In this section, it will be proven that the super essential condition on \( P \) is equivalent with every proper subsystem \( P' \) of \( P \) not being overdetermined, in the differential variables \( U \).

**Definition 4.1.** A system of linear differential polynomials \( P \) is called irredundant (for differential elimination purposes), if every proper subsystem \( P' \) of \( P \) verifies \( |P'| \leq \nu(P') \). Otherwise, \( P \) is called redundant.

It will be shown in this section that super essential and irredundant are equivalent conditions on \( P \). Furthermore, it will be proven that every linear differential system \( P \) contains a super essential subsystem \( P^* \).

**Proposition 4.2.** If \( P \) is super essential then \( P \) is irredundant.

Proof. For every proper subset \( P' = \{f_{h_1}, \ldots, f_{h_m}\} \) of \( P \), there exists \( i \in \{1, \ldots, n\} \) such that \( P' \subseteq P_i \). Therefore \( h_1, \ldots, h_m \in \{1, \ldots, n\}\{i\} \) and given \( \mu_i \) as in (7),

\[
L_{h_t, \mu_i(h_t)} \neq 0, \quad t = 1, \ldots, m.
\]

Since \( \mu_i \) is a bijection, \( \nu(P') \geq m = |P'| \).

Let \( x_{i,j}, i = 1, \ldots, n, j = 1, \ldots, n-1 \) be algebraic indeterminates over \( \mathbb{Q} \), the field of rational numbers. Let \( X(P) = (X_{i,j}) \) be the \( n \times (n-1) \) matrix, such that

\[
X_{i,j} := \begin{cases} 
  x_{i,j}, & L_{i,j} \neq 0, \\
  0, & L_{i,j} = 0.
\end{cases}
\]

We denote by \( X_i(P), i = 1, \ldots, n, \) the submatrix of \( X(P) \) obtained by removing its \( i \)th row. Thus \( X(P) \) is an \( n \times (n-1) \) matrix with entries in the field \( \mathbb{K} := \mathbb{Q}(X_{i,j} | X_{i,j} \neq 0) \).

**Lemma 4.3.** Given \( i \in \{1, \ldots, n\}, \det(X_i(P)) \neq 0 \) if and only if there exists \( \tau_i \in S_{n-1} \) verifying (6).

Proof. Given \( \tau \in S_{n-1} \), define the bijection \( \mu_\tau : \{1, \ldots, n\}\{i\} \longrightarrow \{1, \ldots, n-1\} \) by

\[
\mu_\tau(j) = \begin{cases} 
  \tau(n-j), & j = 1, \ldots, i-1, \\
  \tau(n-j+1), & j = i+1, \ldots, n.
\end{cases}
\]
We can write
\[
\det(X_i(\mathcal{P})) = \sum_{\tau \in S_{n-1}} \prod_{j \in \{1, \ldots, n\} \setminus \{i\}} X_{j, \mu_\tau(j)}.
\] (8)

The entries of \(X_i(\mathcal{P})\) are either algebraic indeterminates or zero. Thus \(\det(X_i(\mathcal{P})) = 0\) if and only if every summand of (8) is zero, it contains a zero entry. That is, for every \(\tau \in S_{n-1}\), there exists \(j \in \{1, \ldots, n\} \setminus \{i\}\) such that \(X_{j, \mu_\tau(j)} = 0\), thus \(\mathcal{L}_{j, \mu_\tau(j)} = 0\). This proves that, \(\det(X_i(\mathcal{P})) = 0\) if and only if there is no \(\tau \in S_{n-1}\) verifying (6).

Given the set \(\mathcal{P} := \{p_1, \ldots, p_n\}\) of algebraic polynomials in \(\mathbb{K}[C, U]\),
\[
p_i := c_i + \sum_{j=1}^{n-1} X_{i,j} u_j, \ i = 1, \ldots, n.
\]

A coefficient matrix \(M(\mathcal{P})\) of \(\mathcal{P}\) is an \(n \times (2n - 1)\) matrix and it can be obtained by concatenating \(X(\mathcal{P})\) with the identity matrix of size \(n\),
\[
M(\mathcal{P}) = \begin{bmatrix}
1 & \cdots & 0 \\
X(\mathcal{P}) & \ddots \\
0 & \cdots & 1
\end{bmatrix}.
\]

The reduced echelon form of \(M(\mathcal{P})\) is the coefficient matrix of the reduced Gröbner basis \(\mathcal{B} = \{e_0, e_1, \ldots, e_{n-1}\}\) of the algebraic ideal \((\mathcal{P})\) generated by \(p_1, \ldots, p_n\) in \(\mathbb{K}[C, U]\), with respect to lex monomial order with \(u_1 > \cdots > u_{n-1} > c_1 > \cdots > c_n\) ([5], p. 95, Exercise 10). We assume that \(e_0 < e_1 < \cdots < e_{n-1}\).

Observe that the elements of \(\mathcal{B}\) are linear homogeneous polynomials in \(\mathbb{K}[C, U]\) and at least
\[
e_0 \in \mathcal{B}_0 := \mathcal{B} \cap \mathbb{K}[C].
\]

Given a linear homogeneous polynomial \(e \in \mathbb{K}[C]\), \(e = \sum_{h=1}^{n} a_h c_h, \ a_h \in \mathbb{K}\), let \(I(e) := \{h \in \{1, \ldots, n\} \mid a_h \neq 0\}\). Let us consider the system
\[
\mathcal{P}^* := \{f_h \mid h \in I(e_0)\}.
\] (9)

From Definition 3.5 and Lemma 4.3 it follows that
\[
\mathcal{P} \text{ is super essential } \iff I := \{i \in \{1, \ldots, n\} \mid \det(X_i(\mathcal{P})) = 0\} = \emptyset. \quad (10)
\]
If \( \mathcal{P} \) is differentially essential then \( I \neq \{1, \ldots, n\} \), and up to a constant

\[
e_0 = \sum_{i \in I(e_0)} \det(X_i(\mathcal{P}))c_i, \text{ with } I(e_0) = \{1, \ldots, n\} \setminus I,
\]

the determinant of the matrix obtained by concatenating \( X(\mathcal{P}) \) with the column vector containing \( c_1, \ldots, c_n \).

**Lemma 4.4.** If \( \mathcal{P} \) is super essential then \( \mathcal{P} = \mathcal{P}^* \), otherwise \( \mathcal{P}^* \not\subset \mathcal{P} \).

**Proof.** If \( \mathcal{P} \) is super essential, by (10) and (11), \( I(e_0) = \{1, \ldots, n\} \). Otherwise, \( I \neq \emptyset \). If \( I \neq \{1, \ldots, n\} \) then \( I(e_0) \subsetneq \{1, \ldots, n\} \). If \( I = \{1, \ldots, n\} \) then rank(\( X(\mathcal{P}) \)) < \( n - 1 \) and \( e_1 \in \mathcal{B}_0 \) with \( e_0 < e_1 \). Therefore, \( I(e_0) \subsetneq \{1, \ldots, n\} \).

We will prove next that \( \mathcal{P}^* \) is a super essential subsystem of \( \mathcal{P} \).

**Lemma 4.5.** For every \( \mathcal{P}' \not\subset \mathcal{P}^* \), rank(\( X(\mathcal{P}') \)) = \( |\mathcal{P}'| \) and rank(\( X(\mathcal{P}^*) \)) = \( |\mathcal{P}^*| - 1 \).

**Proof.** Given a proper subsystem \( \mathcal{P}' \) of \( \mathcal{P} \), the matrix \( X(\mathcal{P}') \) has \( |\mathcal{P}'| \) rows. Thus rank(\( X(\mathcal{P}') \)) \leq |\mathcal{P}'|. If \( \mathcal{P}' \not\subset \mathcal{P}^* \) then

\[
\mathcal{P}' := \{p_h \mid f_h \in \mathcal{P}'\} \not\subset \mathcal{P}^* := \{p_h \mid h \in I(e_0)\}.
\]

If rank(\( X(\mathcal{P}') \)) < |\( \mathcal{P}' \) then there exists \( e \in (\mathcal{P}') \cap \mathbb{K}[C] \) such that \( I(e) \not\subset I(e_0) \). This contradicts that \( e_0 \) is the smallest element in the reduced Gröbner basis of \( (\mathcal{P}) \cap \mathbb{K}[C] \). Therefore rank(\( X(\mathcal{P}') \)) = |\( \mathcal{P}' \) |

Let \( m = |\mathcal{P}^*| \). We have shown that rank(\( X(\mathcal{P}^*) \)) \geq m - 1. Since \( e_0 \in (\mathcal{P}^*) \cap \mathbb{K}[C] \), rank(\( X(\mathcal{P}^*) \)) < \( m \). Therefore rank(\( X(\mathcal{P}^*) \)) = m - 1.

**Theorem 4.6.** If \( \mathcal{P} \) is not super essential then, the system \( \mathcal{P}^* \) given by (9) is a proper super essential subsystem of \( \mathcal{P} \), with \( \nu(\mathcal{P}^*) = |\mathcal{P}^*| - 1 \).

**Proof.** We can write \( \mathcal{P}^* = \{g_1 := f_{i_1}, \ldots, g_m := f_{i_m}\} \). Let \( \mathcal{P}_{m}^* := \mathcal{P}^* \setminus \{g_m\} \). By Lemma 4.5, there exists \( J = \{j_1, \ldots, j_{m-1}\} \subsetneq \{1, \ldots, n-1\} \), such that the matrix \( Y_m(\mathcal{P}^*) = (Y_{h,k}) \), with

\[
Y_{h,k} := X_{i_h,j_k}, \quad h = 1, \ldots, m - 1, \quad k = 1, \ldots, m - 1,
\]

is a squared submatrix of \( X(\mathcal{P}_{m}^*) \) of size \( m - 1 \) and

\[
\det(Y_m(\mathcal{P}^*)) \neq 0.
\]
Let
\[ Y_{m,k} := X_{i_m,j_k}, \quad k = 1, \ldots, m-1, \]
and define the matrix \( Y(P^*) = (Y_{h,k}), \quad h = 1, \ldots, m, \quad k = 1, \ldots, m-1 \). We will prove that, the only nonzero entries of \( X(P^*) \) are the ones in the submatrix \( Y(P^*) \), that is
\[ g_h = c_i + \sum_{k=1}^{m-1} Y_{h,k} u_{j_k}, \quad h = 1, \ldots, m, \quad (13) \]
and that
\[ \text{det}(Y_h(P^*)) \neq 0, \quad h = 1, \ldots, m, \quad (14) \]
where \( Y_h(P^*) \) is the submatrix of \( Y(P^*) \) obtained by removing the \( h \)th row.

By (10), this shows that \( P^* \) is a super essential subsystem of \( P \), with \( \nu(P^*) = m - 1 = |P^*| - 1 \). For this purpose, we will prove the following claims. For \( l \in \{1, \ldots, m\} \), if \( \text{det}(Y_l(P^*)) \neq 0 \) then
\[ g_l = c_i + \sum_{k=1}^{m-1} Y_{l,k} u_{j_k} \quad \text{and} \quad (15) \]
there exists a bijection \( \eta_l : \{1, \ldots, m\} \setminus \{l\} \rightarrow \{1, \ldots, m-1\} \) such that
\[ \text{det}(Y_t(P^*)) \neq 0, \quad \forall t \in T_l := \{ t \in \{1, \ldots, m\} \setminus \{l\} \mid Y_{l,\eta_l(t)} \neq 0 \}. \quad (16) \]

1. Proof of (15). Otherwise, there exists \( j \in \{1, \ldots, n-1\} \setminus J \) such that \( X_{i_l,j} \neq 0 \). This means that the matrix
\[
\begin{bmatrix}
X_{i_l,j} \\
\vdots \\
Y(P^*) \\
X_{i_m,j}
\end{bmatrix},
\]
is nonsingular, which contradicts \( \text{rank}(X(P^*)) = m - 1 \), see Lemma 4.5.

2. Proof of (16). Since \( \text{det}(Y_l(P^*)) \neq 0 \), by Lemma 4.3, there exists \( \tau_l \in S_{m-1} \) and a bijection
\[ \eta_l : \{1, \ldots, m\} \setminus \{l\} \rightarrow \{1, \ldots, m-1\}, \]
\[ \eta_l(j) := \begin{cases} \tau_l(m-h), & h = 1, \ldots, l-1, \\ \tau_l(m-h+1), & h = l+1, \ldots, m, \end{cases} \]
such that
\[ Y_{h,\eta(h)} \neq 0, h \in \{1, \ldots, m\}\backslash\{l\}. \]  
(17)

Given \( t \in T_l \) and the permutation \( \rho(l, t) : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\} \), such that
\[ \rho(l, t)(h) = \begin{cases} 
  t, & h = l, \\
  l, & h = t, \\
  h, & h \in \{1, \ldots, m\}\backslash\{t, l\},
\end{cases} \]
we define the bijection \( \eta_t : \{1, \ldots, m\}\backslash\{t\} \rightarrow \{1, \ldots, m\} \). Thus, by (17) and the definition of \( T_l \), \( Y_{h,\eta_t(h)} \neq 0, h \in \{1, \ldots, m\}\backslash\{t\} \), which proves that \( \det(Y_t(P^*)) \neq 0 \).

Finally, to prove the result we follow the next algorithm.

1. Set \( T := T_m \) and \( P' := \{g_h \mid h \in T \cup \{m\}\} \).
2. If \( T = \{1, \ldots, m-1\} \) then \( P^* = P' \), which proves (13) and (14), by (12), (16) and (15).
3. If \( T \neq \{1, \ldots, m-1\} \) then, there exists \( l \in T \) such that \( T_l \backslash T \neq \emptyset \).
   Otherwise, \( \nu(P') \leq |T| \), thus \( \text{rank}(X(P')) \leq \nu(P') \leq |T| \), contradicting Lemma 4.5 since \( |P'| = |T| + 1 \). Set \( T := T \cup (T_l \backslash T) \), \( P' := \{g_h \mid h \in T \cup \{m\}\} \) and go to step 2.

Observe that the loop finishes because each time we go to step 3, at least one new element is added to \( T \).

In particular, Theorem 4.6 shows that if \( P \) is not super essential then \( P \) is redundant, which together with Proposition 4.2 proves the next result.

**Corollary 4.7.** A linear differential system \( P \) is irredundant if and only if it is super essential.

5. Systems of linear DPPEs

In this section, we set \( D = K\{C\} \) and consider a system of linear differential polynomials in \( D\{U\} = K\{C, U\} \),

\[ P = \{F_i := c_i - H_i(U), i = 1, \ldots, n\}, \]
(18)
with \( H_i(U) = \sum_{j=1}^{n-1} L_{i,j}(u_j) \), \( L_{i,j} \in K[\delta] \). Let \( [P]_{D(U)} \) be the differential ideal generated by \( P \) in \( D\{U\} \). By [8], Lemma 3.2 and Theorem 3.1, the elimination ideal of \([P]_{D(U)} \) in \( D \) equals

\[ \text{ID}(P) := [P]_{D(U)} \cap D = \{f \in D \mid f(H_1(U), \ldots, H_n(U)) = 0\}. \]
It is called the implicit ideal of the system of linear differential polynomial parametric equations (linear DPPEs)

\[
\begin{aligned}
c_1 &= H_1(U), \\
&\vdots \\
c_n &= H_n(U).
\end{aligned}
\]

The implicitization of linear DPPEs by differential resultant formulas was studied in [17] and [16]. In this section, some of the results presented in [16] are extended, to be used in Section 6.

Let \( P' \) be a subset of \( P \). If \(|P'| = m\) then \( P' = \{F_{h_1}, \ldots, F_{h_m}\} \) and the implicit ideal of \( P' \) equals

\[
\text{ID}(P') = \{f \in \mathcal{K}\{C'\} \mid f(H_{h_1}(U), \ldots, H_{h_m}(U)) = 0\}, \tag{19}
\]

where \( C' = \{c_h \mid F_h \in P'\} \). If \(|P'| \leq \nu(P')\) then it may happen that \( \text{ID}(P') = \{0\} \). If \(|P'| > \nu(P')\), by [8], Lemma 3.1, \( \text{ID}(P') \) is a differential prime ideal with generic zero \((H_{h_1}(U), \ldots, H_{h_m}(U))\). Let \( \mathcal{A} \) be a characteristic set of \( \text{ID}(P') \) (with respect to any ranking, see [4], Section 4.2), the differential dimension of \( \text{ID}(P') \) is \( \text{dim}(\text{ID}(P')) = m - |\mathcal{A}| = m - 1 \) and coincides with the differential transcendence degree of \( \mathcal{K}(H_{h_1}(U), \ldots, H_{h_m}(U)) \) over \( \mathcal{K} \),

\[
\text{dim}(\text{ID}(P')) = d.tr.deg \frac{\mathcal{K}(H_{h_1}(U), \ldots, H_{h_m}(U))}{\mathcal{K}}.
\]

If \( P \) is redundant then, there exists \( P' \subsetneq P \), with \( \nu(P') < |P'| \) and, by the previous observation

\[
\{0\} \neq \text{ID}(P') \subset \text{ID}(P). \tag{20}
\]

Let \( PS \subset \partial P \) and \( U \subset \{U\} \) be sets verifying (ps1) and (ps2) (as in Section 3 but with \( P \) as in (18)). The set \( PS \) belongs to the polynomial ring \( \mathcal{K}[C_{PS}, U] \), with

\[
C_{PS} := \{c_{i,k} \mid k \in [0, L_i] \cap \mathbb{Z}, i = 1, \ldots, n\}.
\]

Let \((PS)\) be the algebraic ideal generated by \( PS \) in \( \mathcal{K}[C_{PS}, U] \). Let \( \mathbb{D}' = \mathcal{K}\{C'\} \).

**Proposition 5.1.** Given \( P' \subsetneq P \) with \( \nu(P') < |P'| \), if \( \text{dim}(\text{ID}(P)) = n - 1 \) then \( \text{ID}(P) = [A]_{\mathbb{D}} \), where \( [A]_{\mathbb{D}} \) is the differential ideal generated in \( \mathbb{D} \), by a nonzero linear differential polynomial \( A \) such that \( \text{ID}(P') = [A]_{\mathbb{D}'} \).
Proof. Let \( \mathcal{G} \) be the Gröbner basis of (PS) with respect to lex monomial order induced by the ranking \( \mathbf{r} \). Observe that \( \mathcal{G} \) can be obtained from PS by Gaussian elimination, thus \( \mathcal{G} \) is a set of linear differential polynomials in (PS) and \( \mathcal{G}_0 = \mathcal{G} \cap \mathbb{D} \neq \emptyset \) (the proof is analogous to \cite{17}, Lemma 11 (1)). We have \( \text{ID}(\mathcal{P}) = [A]_\mathbb{D} \) and given \( B_0 \in \mathcal{G}_0 \), \( B_0 = \mathcal{L}(A) \), with \( \mathcal{L} \in \mathbb{D}[\partial] \). Since \( B \) is linear then \( \mathcal{L} \in \mathcal{K}[\partial] \) and \( A \) is also linear.

We are assuming \( \nu(\mathcal{P}') < |\mathcal{P}'| = m \), thus there exists a proper subset \( U' \) of \( U \) with \( |U'| = |\mathcal{P}'| - 1 \) such that \( \mathcal{P}' \subset \partial \mathcal{P}' \) and \( \mathcal{U}' \subset \{U'\} \) verifying (ps1) and (ps2). By (20), \( \dim(\text{ID}(\mathcal{P}')) = m - 1 \) and, by the previous paragraph, \( \text{ID}(\mathcal{P}') = [B]_{\mathcal{B}'} \), with \( B \) a linear differential polynomial. Thus \( B = \mathcal{L}(A) \), with \( 0 \neq \mathcal{L} \in \mathcal{K}[\partial] \). Since \( B \) is linear then \( \mathcal{L} \in \mathcal{K}[\partial] \) and \( A \in \text{ID}(\mathcal{P}') \) and \( \mathcal{L} \in \mathcal{K} \).

Given a nonzero linear differential polynomial \( B \) in \( \text{ID}(\mathcal{P}) \), by \cite{16}, Lemma 4.4 there exist unique \( \mathcal{F}_i \in \mathcal{K}[\partial] \) such that
\[
B = \sum_{i=1}^{n} \mathcal{F}_i(c_i) \quad \text{and} \quad \sum_{i=1}^{n} \mathcal{F}_i(H_i(U)) = 0. \tag{21}
\]
We denote a greatest common left divisor of \( \mathcal{F}_1, \ldots, \mathcal{F}_n \) by \( \text{gcl}(\mathcal{F}_1, \ldots, \mathcal{F}_n) \).

As in \cite{16}, Definition 4.9:

1. The ID-content of \( B \) equals \( \text{IDcont}(B) := \text{gcl}(\mathcal{F}_1, \ldots, \mathcal{F}_n) \). We say that \( B \) is ID-primitive if \( \text{IDcont}(B) \in \mathcal{K} \).
2. There exist \( \mathcal{L}_i \in \mathcal{K}[\partial] \) such that \( \mathcal{F}_i = \text{IDcont}(B)\mathcal{L}_i \), \( i = 1, \ldots, n \), and \( \mathcal{L}_1, \ldots, \mathcal{L}_n \) are coprime. An ID-primitive part of \( B \) equals
\[
\text{IDprim}(B) := \sum_{i=1}^{n} \mathcal{L}_i(c_i).
\]

If \( B \) belongs to (PS) then \( \text{ord}(B, c_i) \leq L_i \), \( i = 1, \ldots, n \). Given a nonzero linear differential polynomial \( B \) in (PS), we define the co-order with respect to PS of \( B \) to be the highest positive integer \( c_{PS}(B) \) such that \( \partial^{PS}(B) B \in \) (PS). Observe that, this definition was given in \cite{16}, Definition 4.7, for a choice of PS.

**Theorem 5.2.** Let \( \mathcal{P} \) be a system of linear DPPEs as in (18). Let \( \text{PS} \subset \partial \mathcal{P} \) and \( \mathcal{U} \subset \{U\} \) be sets verifying (ps1) and (ps2). If \( \dim(\text{ID}(\mathcal{P})) = n - 1 \) then \( \text{ID}(\mathcal{P}) = [A]_\mathbb{D} \), where \( A \) is a linear differential polynomial verifying:
1. $A$ is ID-primitive and $A \in (\text{PS}) \cap \mathbb{D}$.
2. $c_{\text{PS}}(A) = |\text{PS}| - 1 - \text{rank}(\mathcal{L}(\text{PS}, \mathcal{U}))$.

**Proof.** We can adapt the proof of [16], Theorem 5.2. We can also adapt the proof of [17], Theorem 10 (1) $\iff$ (3) to show that

$$\det(\mathcal{M}(\text{PS}, \mathcal{U})) \neq 0 \iff \text{rank}(\mathcal{L}(\text{PS}, \mathcal{U})) = |\text{PS}| - 1.$$ (22)

6. Computation of the sparse linear differential resultant

The field $\mathbb{Q}$ of rational numbers is a field of constants of the derivation $\partial$. For $i = 1, \ldots, n$ and $j = 1, \ldots, n - 1$, let us consider subsets $\mathcal{S}_{i,j}$ of $\mathbb{Z}$ to be the supports of differential operators

$$G_{i,j} := \begin{cases} \sum_{k \in \mathcal{S}_{i,j}} c_{i,j,k} \partial^k & \mathcal{S}_{i,j} \neq \emptyset, \\ 0 & \mathcal{S}_{i,j} = \emptyset, \end{cases}$$

whose coefficients are differential indeterminates over $\mathbb{Q}$ in the set

$$\overline{C} := \bigcup_{i=1}^n \bigcup_{j=1}^{n-1} \{ c_{i,j,k} \mid k \in \mathcal{S}_{i,j} \}.$$  

Let $F_i$, $i = 1, \ldots, n$ be a generic sparse linear differential polynomial as follows,

$$F_i := c_i - \sum_{j=1}^{n-1} G_{i,j}(u_j) = c_i - \sum_{j=1}^{n-1} \sum_{k \in \mathcal{S}_{i,j}} c_{i,j,k} u_{j,k}.$$  

In this section, $\mathcal{K} = \mathbb{Q}(\overline{C})$, a differential field extension of $\mathbb{Q}$ with derivation $\partial$, and $\mathbb{D} = \mathcal{K}\{C\}$. Consider the system of linear DPPEs in $\mathbb{D}\{U\}$

$$\mathfrak{P} := \{ F_i = c_i - \mathbb{H}_i(U) \mid i = 1, \ldots, n \}.$$  

Let us assume that the order of $F_i$ is $o_i \geq 0$, $i = 1, \ldots, n$ so that, if $G_{i,j} \neq 0$,

$$\mathcal{S}_{i,j} \subset I_{i,j}(\mathfrak{P}) = [\gamma_j(\mathfrak{P}), o_i - \tau_j(\mathfrak{P})] \cap \mathbb{Z}.$$  

By [12], Corollary 3.4, the dimension of $\text{ID}(\mathfrak{P}) = [\mathfrak{P}]_{\mathbb{D}(U)} \cap \mathbb{D}$ is $n - 1$ if and only if $\mathfrak{P}$ is a differentially essential system. In such case, $\text{ID}(\mathfrak{P}) = \text{sat}(R)$, the saturation ideal of a differential polynomial $R$ in $\mathbb{D}$. By clearing denominators when necessary, we can assume that $R \in \mathbb{Q}(\overline{C}, C)$ is irreducible. By [12], Definition 3.5, $R$ is the sparse differential resultant of $\mathfrak{P}$. We will denote it by $\partial \text{Res}(\mathfrak{P})$ and call it the **sparse linear differential resultant** of $\mathfrak{P}$.  

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Remark 6.1. Given a differentially essential system \( \mathfrak{P} \), by Theorem 5.2, \( \text{ID}(\mathfrak{P}) = [\partial \text{Res}(\mathfrak{P})]_{\mathbb{D}} \) and \( \partial \text{Res}(\mathfrak{P}) \) is a linear ID-primitive differential polynomial in \( \text{ID}(\mathfrak{P}) \). Furthermore, given \( \mathfrak{P} \subset \partial \mathfrak{P} \) and \( U \subset \{U\} \) verifying (ps1) and (ps2), it holds that:

1. \( \partial \text{Res}(\mathfrak{P}) \) belongs to \( (\mathfrak{P}) \cap \mathbb{D} \) and,
2. \( c_{\mathfrak{P}}(\partial \text{Res}(\mathfrak{P})) = |\mathfrak{P}| - 1 - \text{rank}(\mathcal{L}(\mathfrak{P}, U)) \).

Theorem 6.2. Let \( \mathfrak{P} \) be a differentially essential system. Given \( \mathfrak{P} \subset \partial \mathfrak{P} \) and \( U \subset \{U\} \) verifying (ps1) and (ps2), the following statements are equivalent:

1. \( \det(\mathcal{M}(\mathfrak{P}, U)) \neq 0 \).
2. \( \text{ord}(\partial \text{Res}(\mathfrak{P}), c_{i}) \leq L_{i}, \ i = 1, \ldots, n \) and there exists \( k \in \{1, \ldots, n\} \) such that \( \text{ord}(\partial \text{Res}(\mathfrak{P}), c_{k}) = L_{k} \).

Furthermore, if \( \det(\mathcal{M}(\mathfrak{P}, U)) \neq 0 \) then \( \det(\mathcal{M}(\mathfrak{P}, U)) = \alpha \partial \text{Res}(\mathfrak{P}) \) for some \( \alpha \in \mathcal{K} \).

Proof. By (22), 1 is equivalent to \( \text{rank}(\mathcal{L}(\mathfrak{P}, U)) = |\mathfrak{P}|-1 \). Furthermore, by Remark 6.1(2), it is equivalent to \( c_{\mathfrak{P}}(\partial \text{Res}(\mathfrak{P})) = 0 \) and, since \( \partial \text{Res}(\mathfrak{P}) \in (\mathfrak{P}) \), this is equivalent to 2. Finally, if \( D = \det(\mathcal{M}(\mathfrak{P}, U)) \neq 0 \) then \( D \in (\mathfrak{P}) \cap \mathbb{D} \) and \( c_{\mathfrak{P}}(D) = 0 \) as well. Since \( \partial \text{Res}(\mathfrak{P}) \) is ID-primitive, there exists \( \alpha \in \mathcal{K} \) such that \( D = \alpha \partial \text{Res}(\mathfrak{P}) \). \( \square \)

We will prove that, if \( \mathfrak{P} \) is super essential then \( \partial \text{FRes}(\mathfrak{P}) \neq 0 \) and therefore \( \partial \text{Res}(\mathfrak{P}) = \alpha \partial \text{FRes}(\mathfrak{P}) \), for some \( \alpha \in \mathcal{K} \).

Lemma 6.3. Given \( i \in \{1, \ldots, n\} \), if \( \det(\mathfrak{C}_{i}(\mathfrak{P})) \neq 0 \) then, for every subset \( \mathfrak{P}' \) of \( \mathfrak{P}_{i} \), the differential ideal \( \text{ID}(\mathfrak{P}') \) contains no linear differential polynomial.

Proof. Let \( \mathfrak{P}' = \{\mathbb{F}_{h_{1}}, \ldots, \mathbb{F}_{h_{m}}\} \). Therefore \( h_{1}, \ldots, h_{m} \in \{1, \ldots, n\} \backslash \{i\} \) and, by Lemma 4.3, we can define \( \mu_{i} \) as in (7), such that \( \mathcal{G}_{h_{t}, \mu_{i}(h_{t})} \neq 0, t = 1, \ldots, m \). By (19),

\[
\text{ID}(\mathfrak{P}') = \{ f \in \mathcal{K}\{C'\} \mid f(\mathbb{H}_{h_{1}}(U), \ldots, \mathbb{H}_{h_{m}}(U)) = 0 \}.
\]

Given a linear differential polynomial \( B \in \text{ID}(\mathfrak{P}') \), by (21), there exist \( \mathcal{F}_{h_{1}}, \ldots, \mathcal{F}_{h_{m}} \in \mathcal{K}[\partial] \) such that

\[
\mathcal{F}_{h_{1}}(\mathbb{H}_{h_{1}}(U)) + \cdots + \mathcal{F}_{h_{m}}(\mathbb{H}_{h_{m}}(U)) = 0.
\]
Replace by zero, the coefficients of $G_{ht,j}$, for $t = 1, \ldots, m$ and $j \neq \mu_i(h_t)$, in (23) to obtain

$$F_{h_1}(G_{h_1,\mu_i(h_1)}(u_{\mu_i(h_1)})) + \cdots + F_{h_m}(G_{h_m,\mu_i(h_m)}(u_{\mu_i(h_m)})) = 0.$$ 

Which is not possible, since $u_{\mu_i(h_1)}, \ldots, u_{\mu_i(h_m)}$ are differentially independent. This proves that $B$ does not exist. \hfill \Box

**Theorem 6.4.** If $\mathfrak{P}$ is super essential then $\partial \text{Res}(\mathfrak{P}) \neq 0$.

**Proof.** Observe that $\mathfrak{P}$ is differentially essential and thus $\text{ID}(\mathfrak{P}) = [R]_D$, where $R = \partial \text{Res}(\mathfrak{P})$ is a linear differential polynomial by Remark 6.1. Let $ps = ps(\mathfrak{P})$ and $\gamma = \gamma(\mathfrak{P})$. Let us assume that $\partial \text{Res}(\mathfrak{P}) = 0$ to reach a contradiction.

By (22), $\text{rank}(L(ps, V)) < L - 1$ and, by Theorem 5.2, $c_{ps}(R) \geq 1$. We denote $c_{ps}(R)$ simply by $c(R)$ in the remaining parts of the proof. Let $P_R := \left\{ F_i \in P \mid N - o_i - \gamma - c(R) \geq 0 \right\}$. If $P_R \subset P$, then, by Proposition 5.1, $R \in \text{ID}(P_R)$. On the other hand, $P_R \subset P_i$ for some $i \in \{1, \ldots, n\}$. Since $\mathfrak{P}$ is super essential, $\det(X_i(\mathfrak{P})) \neq 0$ and by Lemma 6.3, there is no linear differential polynomial in $\text{ID}(P_R)$, therefore $P = P_R$.

The set $\mathfrak{P} := \{ F_i \in \mathfrak{P} \mid N - o_i - \gamma - c(R) \geq 0 \}$.

If $P_R \subset P$, then, by Proposition 5.1, $R \in \text{ID}(P_R)$. On the other hand, $P_R \subset P_i$ for some $i \in \{1, \ldots, n\}$. Since $\mathfrak{P}$ is super essential, $\det(X_i(\mathfrak{P})) \neq 0$ and by Lemma 6.3, there is no linear differential polynomial in $\text{ID}(P_R)$, therefore $P = P_R$.

The set

$$\mathfrak{P} := \{ \partial^k F_i \mid k \in [0, N - o_i - \gamma - c(R)] \cap \mathbb{Z}, \ i = 1, \ldots, n \},$$

contains $L - n c(R)$ polynomials in $\mathcal{K}[C_{\mathfrak{P}}, W]$, where

$$C_{\mathfrak{P}} := \{ \partial^k c_i \mid k \in [0, N - o_i - \gamma - c(R)] \cap \mathbb{Z}, \ i = 1, \ldots, n \}, \text{and}$$

$$W := \{ u_{j,k} \mid k \in [0, N - \gamma_j - \gamma - c(R)] \cap \mathbb{Z}, \ j = 1, \ldots, n - 1 \},$$

with $|C_{\mathfrak{P}}| = L - n c(R)$ and $|W| = L - (n - 1) c(R) - 1$. Let $(\mathfrak{P})$ be the algebraic ideal generated by $\mathfrak{P}$ in $\mathcal{K}[C_{\mathfrak{P}}, W]$. Let $\mathcal{M}$ be the $|C_{\mathfrak{P}}| \times (|W| + |C_{\mathfrak{P}}|)$ coefficient matrix of $\mathfrak{P}$ as polynomials in $\mathcal{K}[C_{\mathfrak{P}}, W]$, with coefficients in decreasing order w.r.t. $\tau$. The submatrix $\mathcal{N}$, of the first $|W|$ columns of $\mathcal{M}$ is the coefficient matrix of

$$\mathfrak{P}^h := \{ \partial^k H_i \mid k \in [0, N - o_i - \gamma - c(R)] \cap \mathbb{Z}, \ i = 1, \ldots, n \}.$$

The submatrix of $\mathcal{M}$ of the columns indexed by $C_{\mathfrak{P}}$, is the identity matrix of size $|C_{\mathfrak{P}}|$, thus $\text{rank}(\mathcal{M}) = |C_{\mathfrak{P}}|$. Since $\text{rank}(\mathcal{N}) \leq |W|$, there exists a matrix
\[ \mathcal{E}, \text{ row equivalent to } \mathcal{M} \text{ and whose last } c(R) + 1 = |C_{\text{ps}}| - |W| \text{ rows, are the coefficients of } c(R) + 1 \text{ differential polynomials } A < A_1 < \cdots < A_{c(R)} \text{ in } (\overline{\mathcal{P}} \cap \mathbb{D}), \text{ ordered w.r.t. the ranking on } C. \]

Since \( \text{ID}(\mathcal{P}) = [R]_{\mathbb{D}} \) then \( A_t = F_t(A) \), with \( F_t \in \mathcal{K}[\partial], t = 1, \ldots, c(R) \) and \( \deg(F_1) < \cdots < \deg(F_{c(R)}). \) By construction of \( \mathcal{M}, c_{\text{ps}}(A_{c(R)}) \geq c(R). \) This implies that \( c_{\text{ps}}(A) \geq 2c(R) > c(R). \) Which is a contradiction since \( A \in [R]_{\mathbb{D}}, \) so \( A = L(R), L \in \mathcal{K}[\partial] \) and \( c_{\text{ps}}(A) \leq c(R). \) This proves that \( \partial \text{FRes}(\mathcal{P}) \neq 0. \)

There exists a super essential subsystem \( \mathcal{P}^* \) of \( \mathcal{P}. \) If \( \mathcal{P} \) is super essential then \( \mathcal{P}^* = \mathcal{P}, \) otherwise, by Theorem 4.6, \( \mathcal{P}^* \) can be obtained by (9).

**Theorem 6.5.** Let us consider a differentially essential system \( \mathcal{P}, \) of generic sparse linear differential polynomials, and a super essential subsystem \( \mathcal{P}^* \) of \( \mathcal{P}. \) There exists \( \alpha \in \mathcal{K} \) such that \( \partial \text{Res}(\mathcal{P}) = \alpha \partial \text{FRes}(\mathcal{P}^*). \)

**Proof.** By hypothesis \( \text{ID}(\mathcal{P}) = [\partial \text{Res}(\mathcal{P})]_{\mathbb{D}}. \)

1. If \( \mathcal{P} \) is super essential then \( \mathcal{P}^* = \mathcal{P}. \) By Theorems 6.2 and 6.4, \( \partial \text{Res}(\mathcal{P}) = \alpha \partial \text{FRes}(\mathcal{P}), \alpha \in \mathcal{K}. \)

2. If \( \mathcal{P} \) is not super essential then, by Theorem 4.6, \( \mathcal{P}^* \) given by (9) is a proper super essential subsystem of \( \mathcal{P}, \) with \( \nu(\mathcal{P}^*) = |\mathcal{P}^*| - 1. \) Thus \( \mathcal{P}^* \) is differentially essential and by (19), \( \text{ID}(\mathcal{P}^*) = [\partial \text{Res}(\mathcal{P}^*)]_{\mathbb{D}^\prime}, \) with \( \mathbb{D}^\prime = \mathcal{K}\{C^\prime\}. \) By 1, \( \partial \text{Res}(\mathcal{P}^*) = \beta \partial \text{FRes}(\mathcal{P}^*), \beta \in \mathcal{K}. \) By Proposition 5.1, \( \partial \text{Res}(\mathcal{P}) = \alpha \partial \text{FRes}(\mathcal{P}^*), \alpha \in \mathcal{K}. \)

\( \square \)

The previous result, together with Theorem 6.4, allows us to give a bound of the order of \( \partial \text{Res}(\mathcal{P}) \) in the differential indeterminates \( C, \) namely

\[ \text{ord}(\partial \text{Res}(\mathcal{P}), c_i) \leq N - o_i - \gamma(\mathcal{P}^*), \quad i = 1, \ldots, n, \]

and equality holds for some \( k \in \{1, \ldots, n\}. \)

To finish, we remark that, if \( \mathcal{P} \) is a system of linear differential polynomials, which are not generic, \( \partial \text{FRes}(\mathcal{P}) = 0 \) in many cases. If \( \mathcal{P} \) is super essential, we can guarantee that \( \mathcal{M}(\mathcal{P}) \) has no zero columns but we cannot guarantee that \( \partial \text{FRes}(\mathcal{P}) \neq 0, \) as Example 6.7 shows.

**Theorem 6.6.** Given a super essential system \( \mathcal{P} \) (as in Section 2), the matrix \( \mathcal{M}(\mathcal{P}) \) has no zero columns.
Proof. Let $\mathfrak{P}$ be the linear differential generic system with $S_{i,j} = S(L_{i,j})$. The system $\mathfrak{P}$ is also super essential. If $\mathcal{M}(\mathfrak{P})$ has a zero column, so does $\mathcal{M}(\mathfrak{P})$. But this contradicts Theorem 6.4 and proves the result. \qed

Example 6.7. Given the differentially essential system of generic differential polynomials $\mathfrak{P} = \{F_1, F_2, F_3, F_4\}$,

\[
F_1 = c_1 + c_{1,1,0}u_1 + c_{1,1,1}u_{1,1} + c_{1,3,0}u_3 + c_{1,3,1}u_{3,1}, \\
F_2 = c_2 + c_{2,2,0}u_2 + c_{2,2,1}u_{2,1}, \\
F_3 = c_3 + c_{3,1,0}u_1 + c_{3,3,0}u_3, \\
F_4 = c_4 + c_{4,1,0}u_1 + c_{4,2,0}u_2 + c_{4,3,0}u_3,
\]

let us consider the specialization $\mathcal{P}$ of $\mathfrak{P}$

$\mathcal{P} = \{c_1 + u_1 + 2u_{1,1} + u_3 + 2u_{3,1}, c_2 + u_2 + u_{2,1}, c_3 + u_1 + u_3, c_4 + u_1 + u_2 + u_3\}$.

It holds that $\partial \text{FRes}(\mathfrak{P}) \neq 0$ but $\partial \text{FRes}(\mathcal{P}) = 0$, even though $\mathcal{P}$ is super essential and $\mathcal{M}(\mathcal{P})$ has no zero columns. We can check, applying [16], Algorithm 7.1, that $\dim \text{ID}(\mathcal{P}) < 3$.

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