Attractors for Nonautonomous Reaction-Diffusion Systems with Symbols without Strong Translation Compactness

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Abstract

The long time behavior of the solutions of a general reaction-diffusion system (RDS) that covers many examples, such as the RDS with polynomial nonlinearity and Ginzburg-Landau equation, is discussed. First, the existence of a compact uniform attractor $A_0$ in $H$ is proved without additional assumptions on the interaction functions. Then the structure of the attractor is obtained for a certain class of interaction functions without strong translation compactness. For instance, the interaction functions are not required to be uniformly continuous. Moreover, an interesting problem arises naturally from this paper.

Keywords: uniform attractor, reaction-diffusion system, normal symbol
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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^1$. Consider the long time behavior of the solutions of the following nonautonomous reaction-diffusion system (RDS):

$$\begin{align*}
\partial_t u - a \Delta u + f(u, t) &= g(x, t), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega, \\
\left. u \right|_{t=\tau} &= u_\tau, \quad \tau \in \mathbb{R},
\end{align*}$$

(RDS)

where $a = \{a_{ij}\}_{i=1, \ldots, N}^{j=1, \ldots, N}$ is an $N \times N$ real matrix with positive symmetric part $\frac{1}{2}(a + a^*) \geq \beta I$, $\beta > 0$; $u = u(x, t) = (u^1, \ldots, u^N)$, $g = (g^1, \ldots, g^N)$, $f = (f^1, \ldots, f^N)$. Denote the spaces $H = (L^2(\Omega))^N$ and $V = (H^1_0(\Omega))^N$ with
of all complete trajectories of the process \( \{ \mathcal{K} \} \) exist and equal, and their structures are described by

\[
\tau \quad \text{(w.r.t.})
\]

The long time behavior of the solutions is described by the compact uniform \( \{ \mathcal{U} \} \) attractor \( A \). This system is quite general that covers many examples, such as the RDS with polynomial nonlinearity, Ginzburg-Landau equation, Chafee-Infante equation, Fitz-Hugh-Nagumo equations and Lotka-Volterra competition system. As a fundamental model in the theory of nonautonomous infinite dimensional dynamical systems, it has been extensively studied in [CV1, CV2, CV3, CV4, CV5].

It is known (cf. [CV5, Te1]) that for a given symbol \( \sigma_0(s) = (f_0(v, s), g_0(x, s)) \) and \( u_\tau \in H \), the system (RDS) has a unique weak solution \( u \in C(\{ \tau, \tau \}; H) \). Hence a process \( \{ U_{\sigma_0}(t, \tau) \} t \geq \tau, \tau \in \mathbb{R} \) is defined on \( H: U_{\sigma_0}(t, \tau)u_\tau = u(t) \).

The long time behavior of the solutions is described by the compact uniform (w.r.t. \( \tau \in \mathbb{R} \)) attractor \( \mathcal{A}_0 \), i.e., the minimal compact uniformly (w.r.t. \( \tau \in \mathbb{R} \)) attracting set. A is called an attracting set if for any bounded set \( B \subset H \),

\[
\lim_{t \to +\infty} \sup_{\tau \in \mathbb{R}} |U(t + \tau, \tau) B, A| = 0 \quad \text{(see [Ha1, Ha2]).}
\]

On the other hand, according to the idea of Chepyzhov-Vishik [CV1, CV2], one should consider the associated family of processes \( \{ U_{\sigma}(t, \tau) \} \), \( \sigma \in \Sigma \) with the symbol space \( \Sigma \) including \( \sigma_0 \). The corresponding notion of attractor is uniform (w.r.t. \( \sigma \in \Sigma \)) attractor \( \mathcal{A}_\Sigma \), i.e., a minimal compact uniformly (w.r.t. \( \sigma \in \Sigma \)) attracting set: for any bounded set \( B \subset H \),

\[
\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} |U_{\sigma}(t, \tau) B, A_\Sigma| = 0. \quad \text{Assuming some additional conditions on} \ \sigma_0, \ \text{they proved that both kinds of uniform attractors exist and equal, and their structures are described by}
\]

\[
\mathcal{A}_0 = \mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0). \tag{4}
\]

Here \( \mathcal{K}_\sigma(0) \) is the kernel section at time \( t = 0 \) of the kernel \( \mathcal{K}_\sigma \) which is the set of all complete trajectories of the process \( \{ U_{\sigma}(t, \tau) \} \):

\[
\mathcal{K}_\sigma = \{ u(t) | U_{\sigma}(t, \tau)u(\tau) = u(t), |u(t)| \leq C_u, \forall t \geq \tau, \tau \in \mathbb{R} \}.
\]
Relation (4) is a nonautonomous version of the leading concept of invariance of the global attractor for autonomous system and thus it is important for the further study of the properties of the uniform attractor, such as the spatial complexity. By another so-called trajectory attractor method [CV3, CV4] (see also [Se]), Chepyzhov and Vishik improved the results. That is, if $g_0 \in L^2_b(\mathbb{R}; H)$ and $f_0(v, s)$ is bounded and uniformly continuous in every cylinder $Q(R) = \{(v, s) \mid \|v\|_{\mathbb{R}^N} \leq R, s \in \mathbb{R}\}, R > 0$, then $A_\Sigma$ exists and the second equality of (4) is valid. Indeed, they proved $A_\Sigma$ is weakly compact in $V$ and then is compact in $H$. The assumption on $f_0$ is equivalent to that $f_0$ is strongly translation compact in $C(\mathbb{R}; \mathcal{M})$, where $\mathcal{M} = C(\mathbb{R}^N; \mathbb{R}^N)$ and both $\mathcal{M}$ and $C(\mathbb{R}; \mathcal{M})$ are endowed with the well-known local uniform convergence topology. However, there is no information about $A_0$.

In this paper, motivated by the recent work in [LWZ, Lu], we will consider (RDS) with more general symbol $\sigma_0$. Specifically, both $f_0$ and $g_0$ are not assumed to be strongly translation compact in the corresponding functional spaces endowed with natural strong topologies. In [Lu], the author obtained the equalities (4) for 2D Navier-Stokes equations with the so-called normal external forces. However, the situation with (RDS) is more complicated since the nonlinearity $f_0$ is time-dependent.

We first obtain the existence of uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $A_0$ only assuming that $g_0$ is normal in $L^2_{loc}(\mathbb{R}; V')$ (see definition below). It is worthwhile to point out the remarkable fact that there is no additional condition on the nonlinearity other than (1)-(3). Thus even weak compactness in $V$ is unavailable for the associated family of processes. Note that the class of normal functions in $L^2_{loc}(\mathbb{R}; V')$ is larger than the class of translation compact functions in $L^2_{loc}(\mathbb{R}; V')$. Then we study the structure of compact uniform attractor $A_0$. We describe a class of interaction functions $f_0$ for which both kinds of uniform attractors $A_0$ and $A_\Sigma$ for (RDS) exist and satisfy (4). This class, denoted by $C^p.u.(\mathbb{R}; \mathcal{M})$, consists of continuous functions in $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ that are bounded and uniformly continuous with respect to the first variable in every cylinder $Q(R)$. Since the boundedness arises naturally from (2), the nonlinearity $f_0 \in C^p.u.(\mathbb{R}; \mathcal{M})$ need not to be strongly translation compact in $C(\mathbb{R}; \mathcal{M})$. For $f_0 \notin C^p.u.(\mathbb{R}; \mathcal{M})$, we have obtained the existence of $A_0$ for (RDS) we do not know its structure yet. This interesting problem is worth being studied further in the future.

This paper is organized as follows. In section 2, we present the the main results and formulate two problems arising from our work. In section 3, after we recall some abstract results we prove the main theorem by obtaining compactness and weak continuity respectively. In section 4, we prove the properties of the class $C^p.u.(\mathbb{R}; \mathcal{M})$ and construct an example of the interaction function that is not uniform continuous in the cylinder $Q(R)$s.

2 Main results and open problems

We now introduce a class of functions that was defined first in [LWZ].
Definition 2.1. Let $E$ be a Banach space. A function $\varphi$ is said to be normal in $L^2_{\text{loc}}(\mathbb{R}; E)$ if $\varphi \in L^2_{\text{loc}}(\mathbb{R}; E)$ and for any $\varepsilon > 0$, there exists $\eta > 0$ such that
\[
\sup_{t \in \mathbb{R}} \int_{t}^{t+\eta} \| \varphi(s) \|^2_E \, ds \leq \varepsilon.
\]

We denote by $L^2_{\text{n}}(\mathbb{R}; E)$ the set of all normal functions in $L^2_{\text{loc}}(\mathbb{R}; E)$.

Remark 2.1. Obviously, $L^2_{\text{n}}(\mathbb{R}; E) \subset L^2_{\text{u}}(\mathbb{R}; E)$. Denote by $L^2_{\text{p}}(\mathbb{R}; E)$ the class of translation compact functions $\varphi(s), \ s \in \mathbb{R}$, i.e., the functions $\varphi(s)$ such that the closure of its translation family $\mathcal{H}_0(\varphi)$ in $L^2_{\text{loc}}(\mathbb{R}; E)$ is compact in $L^2_{\text{loc}}(\mathbb{R}; E)$. It is proved in [LWZ] that $L^2_{\text{n}}(\mathbb{R}; E)$ is closed subspaces of $L^2_{\text{p}}(\mathbb{R}; E)$, but the latter is a proper subset of the former. Note that if $\varphi \in L^2_{\text{loc}}(\mathbb{R}; E)$ such that $\{\varphi(s)\} s \in \mathbb{R}$ is bounded in $E$ then by the definition it must be normal, but it is easy to construct one that does not belong to $L^2_{\text{n}}(\mathbb{R}; E)$. For more detail and examples we refer to [LWZ, Lu].

Let $\mathcal{M} = C(\mathbb{R}^N, \mathbb{R}^N)$ be endowed with the topology of local uniform convergence. We denote by $C^{p.u.}(\mathbb{R}; \mathcal{M})$ the space $C(\mathbb{R}; \mathcal{M})$ endowed with the topology of following convergence: $\varphi_n(s) \to \varphi(s)$ as $n \to \infty$ in $C^{p.u.}(\mathbb{R}; \mathcal{M})$, if $\varphi_n(v, s)$ is uniformly bounded on any ball in $\mathbb{R}^N \times \mathbb{R}$ and
\[
\max_{\|v\|_{\mathbb{R}^N} \leq R} \| \varphi_n(v, s) - \varphi(v, s) \|_{\mathbb{R}^N} \to 0, \quad \text{as } n \to \infty,
\]
for every $s \in \mathbb{R}$, $R > 0$. And we denote by $C^{p.p.}(\mathbb{R}; \mathcal{M})$ the space $C(\mathbb{R}; \mathcal{M})$ endowed with the topology of following convergence: $\varphi_n(s) \to \varphi(s)$ as $n \to \infty$ in $C^{p.p.}(\mathbb{R}; \mathcal{M})$, if $\varphi_n(v, s)$ is uniformly bounded on any ball in $\mathbb{R}^N \times \mathbb{R}$ and
\[
\| \varphi_n(v, s) - \varphi(v, s) \|_{\mathbb{R}^N} \to 0, \quad \text{as } n \to \infty,
\]
for every $(v, s) \in \mathbb{R}^N \times \mathbb{R}$.

Denote by $C^{p.u.}_{tr.c.}(\mathbb{R}; \mathcal{M})$, $C^{p.p.}_{tr.c.}(\mathbb{R}; \mathcal{M})$ and $C_{tr.c.}(\mathbb{R}; \mathcal{M})$ the classes of the translation compact functions in $C^{p.u.}(\mathbb{R}; \mathcal{M})$, $C^{p.p.}(\mathbb{R}; \mathcal{M})$ and $C(\mathbb{R}; \mathcal{M})$ respectively.

The main result of this paper is

Theorem 2.1. For any given $\sigma_0 = (f_0(v, s), g_0(s))$ such that $g_0 \in L^2_{\text{n}}(\mathbb{R}; V')$ and $f_0$ satisfies conditions (1)-(3), the system (RDS) has a compact uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathcal{A}_0$. Furthermore, if $f_0 \in C^{p.u.}_{tr.c.}(\mathbb{R}; \mathcal{M})$ then the uniform (w.r.t. $\sigma \in \mathcal{H}_w(\sigma_0)$) attractor $\mathcal{A}_{\mathcal{H}_w(\sigma_0)}$ also exists and
\[
\mathcal{A}_0 = \mathcal{A}_{\mathcal{H}_w(\sigma_0)} = \bigcup_{\sigma \in \mathcal{H}_w(\sigma_0)} \mathcal{K}_{\sigma}(0).
\]
Here $\mathcal{H}_w(\sigma_0)$ is the compact closure of $\{\sigma_0(\cdot + h)| h \in \mathbb{R}\}$ in $C^{p.u.}(\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')$. $\mathcal{K}_{\sigma}(0)$ is nonempty for all $\sigma \in \mathcal{H}_w(\sigma_0)$.

We also have the following characterization of the functions in $C^{p.u.}_{tr.c.}(\mathbb{R}; \mathcal{M})$.
Theorem 2.2. \( \varphi(s) \in C^{p.u.}_{tr.c.}(\mathbb{R}; \mathcal{M}) \) if and only if \( \varphi(s) \in C^{p.p.}_{tr.c.}(\mathbb{R}; \mathcal{M}) \) and one of the following holds.

(i) \( \{ \varphi(s) \mid s \in \mathbb{R} \} \) is precompact in \( \mathcal{M} \).

(ii) For any \( R > 0 \), \( \varphi(v, s) \) is bounded in \( \mathcal{Q}(R) = \{(v, s) \mid \|v\|_R^N \leq R, s \in \mathbb{R}\} \)

\[ \|\varphi(v_1, s) - \varphi(v_2, s)\|_{\mathcal{M}} \leq \theta(\|v_1 - v_2\|_R^N, R), \quad \forall (v_1, s), (v_2, s) \in \mathcal{Q}(R), \quad (5) \]

where \( \theta(l, R) \) is positive function tending to 0 as \( l \to 0^+ \).

Let \( C_0(\mathbb{R}; \mathcal{M}) \) be the space of continuous functions with values in \( \mathcal{M} \) and endowed with the uniform convergence topology on every cylinder \( \mathcal{Q}(R) \). We have the following relationships.

Theorem 2.3. \( C_{tr.c.}(\mathbb{R}; \mathcal{M}) \subset C^{p.u.}_{tr.c.}(\mathbb{R}; \mathcal{M}) \subset C^{p.p.}_{tr.c.}(\mathbb{R}; \mathcal{M}) \subset C_0(\mathbb{R}; \mathcal{M}) \) with all inclusions being proper and the former three sets being closed in \( C_0(\mathbb{R}; \mathcal{M}) \).

Remark 2.2. The results of Theorem 2.1 is valid for the Neumann conditions or periodic boundary conditions as well. And when dealing with the Dirichlet boundary one can assume that \( p_k > 1 \). We shall consider the case \( p_k \geq 2 \) for simplicity (see [CV5]).

Remark 2.3. We refer also to [CV5] for more details for the concrete examples mentioned above.

In [Lu], the author studied the properties of the kernel section of a process \( \{U_\sigma(t, \tau)\} \) for every \( \sigma \in \Sigma \) by the properties of the family of processes \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \) with the symbols without strong translation compactness and estimated the fractal dimension of the kernel sections. By these abstract results, the estimate of the fractal dimension of the kernel sections of the uniform attractor in [CV5] is valid in our situation without any additional difficulty. For more detail, see [CV5, Lu].

Theorem 2.4. Under additional assumption that

\[ |f(v + w, s) - f(v, s) - f'_v(v, s)w| \leq C(1 + |v|^q + |w|^q)|w|^{1+\gamma}, \quad v, w \in \mathbb{R}^N, \]

where \( q < 4/(n-2) \), \( n > 2 \) and \( \gamma \) is positive and sufficiently small. The fractal dimension of the kernel sections of the uniform attractor obtained in Theorem 2.1 satisfies the estimate

\[ d_F K_\sigma(s) \leq N|\Omega| \left( \frac{D}{c_0\nu} \right)^{n/2}, \quad \forall s \in \mathbb{R}, \quad \sigma \in \mathcal{H}_w(\sigma_0), \]

where \( |\Omega| \) is the measure of \( \Omega \) and \( c_0 = \frac{4\pi n}{n+2}(\Gamma(1+n/2))^2/n \).

For more details about the constant \( c_0 \), see [LY, CV5].

We now formulate two open problems arising naturally from our work, which are interesting and worth being studied further in the future.

Open problems:
1. If \( f_0 \notin C^p_{tr,c}(\mathbb{R}; M) \), how to describe the structure of the uniform (w.r.t. \( \tau \in \mathbb{R} \)) attractor \( A_0 \) obtained in Theorem 2.1?

2. If \( f_0 \in C^p_{tr,c}(\mathbb{R}; M) \), how to estimate the Kolmogorov \( \varepsilon \)-entropy of the uniform (w.r.t. \( \tau \in \mathbb{R} \)) attractor \( A_0 \) obtained in Theorem 2.1?

Finally, we formulate the main properties of functions in \( C^p_{tr,c}(\mathbb{R}; M) \). They will be used in the proof of Theorem 2.1.

**Proposition 2.1.** Let \( \varphi \in C^p_{tr,c}(\mathbb{R}; M) \) and \( H_w(\varphi) \) be the closure of \( \{ \varphi(\cdot + h) | h \in \mathbb{R} \} \) in \( \varphi \in C^p_{tr,c}(\mathbb{R}; M) \). Then

(i) \( H_w(\varphi) \subset C^p_{tr,c}(\mathbb{R}; M) \), moreover, \( H_w(\varphi_1) \subset H_w(\varphi) \), for any \( \varphi_1 \in H_w(\varphi) \);

(ii) any \( \varphi_1 \in H_w(\varphi) \) satisfies (5) for the same \( \theta(l, R) \);

(iii) The translation group \( \{ T(t) \} \) is invariant and continuous on \( H_w(\varphi) \) in the topology of \( C^p_{tr,c}(\mathbb{R}; M) \).

## 3 Proof of Theorem 2.1

We first recall the abstract results that will be sufficient for this paper. We consider a family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) on a Banach space \( \mathcal{E} \) with symbol space \( \Sigma \). The following assumption is natural from applications,

**Assumption I.** Let \( \{ T(h) \}_{h \geq 0} \) be the translation semigroup acting on \( \Sigma \) and satisfying

i) \( T(h)\Sigma = \Sigma, \forall h \geq 0 \);

ii) translation identity

\[
U_\sigma(t + h, \tau + h) = U_{T(h)\sigma}(t, \tau), \forall \sigma \in \Sigma, t \geq \tau, \tau \in \mathbb{R}, h \geq 0.
\]

We have the following existence theorem (see [LWZ, Lu]).

**Theorem 3.1.** If the family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) satisfies Assumption I and has a compact uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing set, then it possesses a compact uniform (w.r.t. \( \sigma \in \Sigma \)) attractor \( A_\Sigma \).

Note that this theorem is not related to the topology of the symbol space \( \Sigma \). In contrast, to obtain the structure of uniform attractor, we need to consider the topology of the symbol space and need the following assumption.

**Assumption II.** Let \( \Sigma \) be a weakly compact subset of some Banach space, \( \{ T(t) \} \) be weakly continuous on it and \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) be \( (\mathcal{E} \times \Sigma, \mathcal{E}) \) weakly continuous, i.e., for any \( t \geq \tau, \tau \in \mathbb{R} \), the mapping \((u, \sigma) \rightarrow U_\sigma(t, \tau)u \) is weakly continuous from \( \mathcal{E} \times \Sigma \) to \( \mathcal{E} \).

In [LWZ] (see also [Lu]), it is also proved the following theorem (in fact, more general) related to the structure of uniform (w.r.t. \( \tau \in \mathbb{R} \)) attractor.

**Theorem 3.2.** Let \( \Sigma \) be the weak closure of a subset \( \Sigma_0 \) of some Banach space. If the family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) satisfies assumptions of Theorem 3.1, then it and the family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma_0 \) possess, respectively,
compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$ and uniform (w.r.t. $\sigma \in \Sigma_0$) attractor $\mathcal{A}_{\Sigma_0}$. Furthermore, if Assumption II is also valid, then

$$\mathcal{A}_{\Sigma_0} = \mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} K_\sigma(0),$$

and $K_\sigma(s)$ is nonempty for all $\sigma \in \Sigma$.

Here we complement an obvious fact.

**Remark 3.1.** Assumption II can be weaken as $\Sigma$ being compact Hausdorff space and $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ being closed, i.e., for any $t \geq \tau$, $\tau \in \mathbb{R}$, if $x_n \rightarrow x_0$, $\sigma_n \rightarrow \sigma_0$, $U_\sigma_n(t, \tau)x_n \rightarrow y_0$ and $T(t)\sigma_n \rightarrow \delta_0$ in $\Sigma$ then $U_{\sigma_0}(t, \tau)x_0 = y_0$ and $T(t)\sigma_0 = \delta_0$. First note that the related results in [LWZ] (especially Theorem 3.1 and Theorem 3.2) are valid if $\Sigma$ being compact Hausdorff space. Second, as we know in the theory of autonomous system, the closeness of a semigroup of solution operators guarantees the invariance of global attractor (see [BV]), and this can be generalized to a family of processes of solution operators without any difficulty. In this paper, weak continuity ($\Sigma$ being compact Hausdorff space) is sufficient.

### 3.1 Compactness

Note that, for the $\sigma_0$ given in Theorem 2.1, every $\sigma = (f, g) \in H_0(\sigma_0)$ satisfies (1)-(3) with the same constant and

$$\|g\|_{L_2^\Sigma}^2 \leq \|g_0\|_{L_2^\Sigma}^2. \quad (6)$$

Hence, the system (RDS) with symbol $\sigma$ defines a process and we have a family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in H_0(\sigma_0)$.

**Lemma 3.1.** For any given $\sigma_0 = (f_0(v, s), g_0(s))$ such that $g_0 \in L_2^\Sigma(\mathbb{R}; V')$ and $f_0$ satisfies conditions (1)-(3), the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in H_0(\sigma_0)$ associated to the system (RDS) has a compact uniformly (w.r.t. $\sigma \in H_0(\sigma_0)$) absorbing set.

**Proof.** It is known (cf. [CV5, p285]) that a weak solution $u(t)$ of (RDS) satisfies

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2 + (a\nabla u, \nabla u) + (f(u(t), t), u(t)) = \langle g(t), u(t) \rangle,$$

for almost $t \geq \tau$. We find that, by (1),

$$\frac{d}{dt} |u(t)|^2 + \beta \|u(t)\|^2 + 2 \sum_{i=1}^N \gamma_i \|u^i\|_{L_p^\Sigma}^p \leq C + \beta^{-1} \|g(t)\|_{V'}, \quad (7)$$

which implies that, by Gronwall lemma,

$$|u(t)|^2 + 2 \int_\tau^t \sum_{i=1}^N \gamma_i \|u^i\|_{L_p^\Sigma}^p \, ds \leq |u(\tau)|^2 e^{-\lambda_1(t-\tau)} + C \lambda_1^{-1} \left(1 - e^{-\lambda_1(t-\tau)}\right) + \beta^{-1}(1 + \lambda_1^{-1}) \|g\|_{L_2^\Sigma}^2, \quad \lambda = \lambda_1 \beta. \quad (8)$$
Then it is easy to know from (8) and (6) that there exists a constant \( \rho_0 > 0 \) such that

\[
B_0 = \{ u \in H \mid |u| \leq \rho_0 \} \tag{9}
\]

is a bounded uniformly (w.r.t. \( \sigma \in H_0(\sigma_0) \)) absorbing set of the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in H_0(\sigma_0) \), that is, for any \( \tau \in \mathbb{R} \) and bounded \( B \) in \( H \), there exists \( T_0 = T_0(\tau, B) > \tau \) satisfying

\[
B_t := \bigcup_{\sigma \in H_0(\sigma_0)} U_\sigma(t, \tau)B \subset B_0, \quad \forall \ t \geq T_0. \tag{10}
\]

Thanks to (8)-(10), we have, by integrating (7),

\[
\beta \int_{t_1}^{t_2} \|u(s)\|^2 ds \leq C, \quad \forall \ t_2 > t_1 \geq T_0. \tag{11}
\]

This means

\[
B_{[t_1, t_2]} = \{ u(t) \mid u(t) = U_\sigma(t, \tau)u_\tau, \ u_\tau \in B, \sigma \in H_0(\sigma_0) \} \mid_{t \in [t_1, t_2]}
\]

is bounded in \( L^2(t_1, t_2; V) \).

It follows from (RDS) that

\[
\partial_t u = a\Delta u - f + g \in L^{q_1}(t_1, t_2; H^{-r_1}(\Omega)) \times \cdots \times L^{q_N}(t_1, t_2; H^{-r_N}(\Omega)) := L^q(t_1, t_2; H^{-r}(\Omega))
\]

with

\[
1/p_k + 1/q_k = 1, \ r_k := \max\{1, n(1/2 - 1/p_k)\}, \ k = 1, \cdots, N.
\]

Hence

\[
\partial_t B_{[t_1, t_2]} := \{ \partial_t u \mid u \in B_{[t_1, t_2]} \}
\]

is bounded in \( L^q(t_1, t_2; H^{-r}(\Omega)) \). By the embedding theorem (cf. Theorem II.1.4 in [CV5], Theorem 8.1 in [Rb], Theorem III.2.3 in [Te2]), we know that \( B_{[t_1, t_2]} \) is precompact in \( L^2(t_1, t_2; H) \).

Let \( u_1, u_2 \) be the solutions of (RDS) with symbols \( \sigma_1 = (f_1, g_1) \), \( \sigma_2 = (f_2, g_2) \in H_0(\sigma_0) \), respectively. The difference \( w = u_1 - u_2 \) satisfies the system:

\[
\partial_t w - a\Delta w + f_1(u_1, t) - f_2(u_2, t) = g_1 - g_2, \ w(\tau) = u_1(\tau) - u_2(\tau).
\]

It follows that

\[
\frac{d}{dt} |w(t)|^2 \leq \frac{1}{2\beta} \|g_1 - g_2\|^2_{L^p} + \sum_{i=1}^{N} \|f_i^1(u_2, s) - f_i^2(u_2, s)\|_{L^p} \|w^i(s)\|_{L^p}.
\]
Integrating it from \( t_1 \) to \( t_2 \), we find that
\[
|w(t_2)|^2 \leq |w(t_1)|^2 + \frac{1}{2\beta} \int_{t_1}^{t_2} \|g_1 - g_2\|^2_{V'}, \, ds \\
+ \int_{t_1}^{t_2} \sum_{i=1}^N \|f_i^1(u_2, s) - f_i^2(u_2, s)\|_{L^q_0} \|w'(s)\|_{L^p} \, ds.
\] (12)

Since \( g_0 \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \), for any \( 0 < \varepsilon < 1 \), there exists a \( \eta \leq \varepsilon < 1 \), such that
\[
\sup_{t \in \mathbb{R}} \int_t^{t+\eta} \|g_0(s)\|^2_{V'} \, ds \leq \varepsilon.
\] (13)

Note that similar to (6), we have
\[
\int_t^{t+\eta} \|g(s)\|^2_{V'} \, ds \leq \int_t^{t+\eta} \|g_0(s)\|^2_{V'} \, ds,
\] (14)

for every \( \sigma \in \mathcal{H}_0(\sigma_0) \). Let \( T_1 = T_0 + 1 \). Since \( T - \eta > T_0 \), for any \( T > T_1 \), we know that \( B_{[T-\eta, T]} \) is precompact in \( L^2(T-\eta, T; H) \). Therefore, there exist finite number of \( u_1, \cdots, u_m \in B_{[T-\eta, T]} \) such that, for any \( u \in B_{[T-\eta, T]} \) there is some \( j \in \{1, \cdots, m\} \) satisfying
\[
\int_{T-\eta}^T |u - u_j|^2 \, ds \leq \eta \varepsilon.
\]

This implies that there exists \( \tilde{\tau} \in [T - \eta, T] \) such that
\[
|u(\tilde{\tau}) - u_j(\tilde{\tau})|^2 \leq \varepsilon.
\] (15)

(For a similar argument to obtain (15) see [ZN].) By (2), (13), (14), (8), (15) and Hölder inequality, (12) implies that,
\[
|u(T) - u_j(T)|^2 \leq C \left( \varepsilon + \sum_{i=1}^N \left( \int_{T-\eta}^T \|f^i(u_j, s) - f^i_j(u_j, s)\|_{L^q_0}^q \, ds \right)^{1/q_i} \right) \\
\leq C \left( \varepsilon + \sum_{i=1}^N \int_{T-\eta}^T \left( |u_j^i|_{L^p_0}^p \right) \, ds + \eta |\Omega| \right) \\
\leq C \varepsilon.
\]

Thus \( B_T \) is precompact in \( H \). Let \( T_2 > T_0(T_0(\tau, B), B_0) + 1 \). Then, replacing \( B \) with \( B_0 \) above, we know that \( B_{0T_2} \) is precompact in \( H \). Obviously, \( B_{0T_2} \) is a bounded uniformly (w.r.t. \( \sigma \in \mathcal{H}_0(\sigma_0) \)) absorbing set of \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \mathcal{H}_0(\sigma_0) \).
3.2 Weak continuity

In this subsection, we assume further that \( f_0 \in C_{tr,c}(\mathbb{R}; \mathcal{M}) \). Since the weak convergence of \( C(\mathbb{R}; \mathcal{M}) \) is equivalent to boundedness and pointwise convergence (cf. [Di, Me]), every \( \sigma \in \mathcal{H}_w(\sigma_0) \) satisfies (1)-(3) with the same constant by Theorem 2.2 and (6) by Proposition V.4.2 in [CV5]. Hence the system (RDS) with symbol \( \sigma \) defines a process and we have a family of processes \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \mathcal{H}_w(\sigma_0) \). Note that \( \mathcal{H}_w(\sigma_0) \) is a compact Hausdorff space. We now verify the \( (H \times \mathcal{H}_w(\sigma_0), H) \) weak continuity.

Lemma 3.2. For any given \( \sigma_0 = (f_0(v, s), g_0(s)) \) such that \( g_0 \in L^2_t(\mathbb{R}; V') \) and \( f_0 \in C^{p,u}_{tr,c}(\mathbb{R}; \mathcal{M}) \) satisfying conditions (1)-(3), the family of processes \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \mathcal{H}_w(\sigma_0) \) associated to the system (RDS) is \( (H \times \mathcal{H}_w(\sigma_0), H) \) weakly continuous, that is, if \( u_{\tau_n} \to u_\tau \) weakly in \( H \) and \( \{\sigma_n = (f_n, g_n)\} \subset \mathcal{H}_w(\sigma_0) \), \( \sigma = (f, g) \in \mathcal{H}_w(\sigma_0) \), \( \sigma_n \to \sigma \) in \( C^{p,u}(\mathbb{R}; \mathcal{M}) \times L^2_{loc}(\mathbb{R}; V') \), then for any fixed \( t \geq \tau, \tau \in \mathbb{R} \),

\[
U_{\sigma_n}(t, \tau) u_{\tau_n} \to U_\sigma(t, \tau) u_\tau, \text{ weakly in } H. \quad (16)
\]

Proof. The proof is analogous with that of Lemma 2.1 in [Ro]. To deal with the convergence of nonlinearity, we will follow the way of the Galerkin method for proving the existence of weak solution of (RDS) (cf. [CV5]).

Let \( u_n(t) = U_{f_n}(t, \tau) u_{\tau_n} \) and \( u(t) = U_\sigma(t, \tau) u_\tau \). Then \( u_n(t) \) is the solution of the following equation

\[
\partial_t u_n(t) = \alpha \Delta u_n(t) - f_n(u_n(x, t), t) + g_n(t) \quad (17)
\]

Let \( T > \tau \). As the derivation in previous subsection, we conclude that

\[
\begin{align*}
\{u_n\}_n & \text{ is bounded in } L^\infty(\tau, T; H) \cap L^2(\tau, T; V); \\
\{\partial_t u_n\}_n & \text{ is bounded in } L^9(\tau, T; H^{-r}(\Omega)); \\
\{u_n\}_n & \text{ is precompact in } L^2(\tau, T, H).
\end{align*}
\]

Meanwhile, it follows from (8) that

\[
\{u_n\}_n \text{ is bounded in } L^p(\tau, T; L^p(\Omega)).
\]

Combining with (2), we have

\[
\{f_n(u_n(x, s), s)\}_n \text{ is bounded in } L^q(\tau, T; L^q(\Omega)). \quad (19)
\]

\footnote{If \( K \) is a weakly compact set in a Banach space \( X \) and \( X' \) contains a countable total set, then the \( \mathcal{K}^{\text{weak}} \) is metrizable. Recall that a set \( L \subset X' \) is called total if \( l(x) = 0 \) for every \( l \in L \) implies \( x = 0 \) (see [Di, p18]). Note that \( \mathcal{H}_w(\sigma_0) \) is a compact set in \( C^{p,p}(\mathbb{R}; \mathcal{M}) \), i.e., a weakly compact set in \( C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N) \) (cf. Theorem VII.1 in [Di]). \( C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N) \) contains a total set of Dirac \( \delta \)-measures on rational points of \( \mathbb{R}^N \). Hence \( \mathcal{H}_w(\sigma_0) \) is metrizable with metric deducing the topology of \( (\mathcal{H}_w(\sigma_0), C^{p,p}(\mathbb{R}; \mathcal{M})) \). Since the embedding \( C^{p,u}(\mathbb{R}; \mathcal{M}) \hookrightarrow C^{p,p}(\mathbb{R}; \mathcal{M}) \) is continuous, \( (\mathcal{H}_w(\sigma_0), C^{p,u}(\mathbb{R}; \mathcal{M})) \) is a compact Hausdorff space.}
It follows that, by a diagonal process, we can extract a subsequence, still denote by \( \{u_n\}_n \), such that

\[
\begin{align*}
    u_n &\to \tilde{u} \quad \text{weak-star in } L^\infty(\tau, T; H), \\
    &\quad \text{weakly in } L^2(\tau, T; V), \\
    &\quad \text{weakly in } L^p(\tau, T; L^p(\Omega)), \\
    &\quad \text{strongly in } L^2(\tau, T; H), \\
\end{align*}
\]

and

\[
\begin{align*}
    \Delta u_n &\to \Delta \tilde{u} \quad \text{weakly in } L^2(\tau, T; V'), \\
    \partial_t u_n &\to \partial_t \tilde{u} \quad \text{weakly in } L^q(\tau, T; H^{-r}(\Omega)), \\
    f_n(u_n(x, s), s) &\to w(s) \quad \text{weakly in } L^q(\tau, T; L^q(\Omega)),
\end{align*}
\]

for some \( \tilde{u} \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^p(\tau, T; L^p(\Omega)), \)

and some

\[w(s) \in L^q(\tau, T; L^q(\Omega)).\]

Passing the limit in (17), we have the following equality

\[
\partial_t \tilde{u} = a \Delta \tilde{u} - w + g
\]

in the distribution sense of the space \( \mathcal{D}'(\tau, T; H^{-r}(\Omega)) \). On the other hand, by the convergence of (20), we know that \( u_n \to \tilde{u} \) in \( C_w(\mathbb{R}; H) \), which means that \( \tilde{u}(\tau) = u_\tau \).

Now we verify that \( f(\tilde{u}(x, s), s) = w(s) \), which will imply that \( \tilde{u}(s) \) is a solution of (RDS). Hence, by uniqueness, \( \tilde{u} = u \). Thanks to the strong convergence in (20), we know that, passing to a subsequence if necessary, \( u_n(x, s) \to \tilde{u}(x, s) \) for almost every \((x, s) \in \Omega \times [\tau, T]\). Now

\[
\begin{align*}
    \|f_n(u_n(x, s), s) - f(\tilde{u}(x, s), s)\|_{\mathbb{R}^N} \\
    \leq \|f_n(u_n(x, s), s) - f_n(\tilde{u}(x, s), s)\|_{\mathbb{R}^N} \\
    &\quad + \|f_n(\tilde{u}(x, s), s) - f(\tilde{u}(x, s), s)\|_{\mathbb{R}^N}
\end{align*}
\]

By Proposition 2.1, all \( f_n \) satisfy (5) with the same function \( \theta \). Hence we obtain that

\[
\|f_n(u_n(x, s), s) - f(\tilde{u}(x, s), s)\|_{\mathbb{R}^N} \to 0 \quad \text{as } n \to \infty,
\]

for almost every \((x, s) \in \Omega \times [\tau, T]\). On the other hand, by (19) and Lemma II.1.2 in [CV5], we find that \( f_n(u_n(x, s), s) \to f(\tilde{u}(x, s), s) \) weakly in \( L^q(\tau, T; L^q(\Omega)) \). Therefore, \( f(\tilde{u}(x, s), s) = w(s) \).

Thanks to the strong convergence in (20) again, we know that for almost every \( t \geq \tau \), \( u_n(t) \) strongly converges to \( u(t) \) in \( H \). Thus

\[
(u_n(t), v) \to (u(t), v), \quad a.e. \ t \geq \tau, \quad v \in (C_c^\infty(\Omega))^N.
\]
However, it follows from (18) that \( \{u_n(t), v\}_n \) is uniformly bounded. Therefore, if it is locally equicontinuous, then

\[
(u_n(t), v) \rightarrow (u(t), v), \quad \forall t \geq \tau, \; v \in (C^\infty_c(\Omega))^N.
\]

Then we obtain (16) from (21) by (18) and the fact that \( (C^\infty_c(\Omega))^N \) is dense in \( H \).

Note that, for all \( v \in (C^\infty_c(\Omega))^N \) and \( \delta \geq 0, \)

\[
(u_n(t + \delta) - u_n(t), v) = \int_t^{t+\delta} \langle \partial_t u_n(s), v \rangle \, ds \\
\leq C\delta \|v\|_{H^r} \|\partial_t u_n\|_{L^q(t+\delta, H^r)} \\
\leq C\delta \|v\|_{H^r}.
\]

Hence \( \{u_n(t), v\}_n \) is locally equicontinuous.

\[\square\]

3.3 Proof of Theorem 2.1

Proof. For the \( \sigma_0 \) given in Theorem 2.1, take \( \Sigma = H_0(\sigma_0) = \{\sigma_0(\cdot + h) | h \in \mathbb{R}\} \). Note that the family of processes \( \{U_\sigma(t, \tau)\}, \; \sigma \in H_0(\sigma_0) \) satisfies the translation identity. Therefore, the uniform (w.r.t. \( \tau \in \mathbb{R} \)) attracting property of \( \{U_\sigma(t, \tau)\} \) is equivalent to the uniform (w.r.t. \( \sigma \in H_0(\sigma_0) \)) attracting property of \( \{U_\sigma(t, \tau)\}, \; \sigma \in H_0(\sigma_0) \). In particular, \( A_\sigma \) coincides with \( A_{H_0(\sigma_0)}(\sigma) \) of \( \{U_\sigma(t, \tau)\}, \; \sigma \in H_0(\sigma_0) \). Hence the first part result of Theorem 2.1 follows from Lemma 3.1 and Theorem 3.1.

Now let \( f_0 \in C^u_{r.c.}(\mathbb{R}; M) \). The family of processes \( \{U_\sigma(t, \tau)\}, \; \sigma \in H_0(\sigma_0) \) also satisfies the translation identity. Note that Lemma 3.1 is also valid for \( \{U_\sigma(t, \tau)\}, \; \sigma \in H_0(\sigma_0) \). Take \( \Sigma_0 = H_0(\sigma_0) = \{\sigma_0(\cdot + h) | h \in \mathbb{R}\} \) and \( \Sigma = H_0(\sigma_0) \). Then the second part result of Theorem 2.1 follows from Lemma 3.2 and Theorem 3.2.

\[\square\]

4 Proofs of properties of the related symbols

Proof of Theorem 2.2. The equivalence of (i) and (ii) is due to Arzelà-Ascoli compactness criterion in \( \mathcal{M} \). Hence the necessity is obvious by using condition (i).

We now prove the sufficiency by using condition (ii). Fix \( R > 0 \). Since \( \varphi(s) \in C^u_{r.c.}(\mathbb{R}; M) \), for any sequence \( \{\varphi(v, s + h_n) | h_n \in \mathbb{R}, n \in \mathbb{N}\} \), there exists a subsequence, still denoted by \( \varphi(v, s + h_n) \), and \( \varphi_0(v, s) \in C(\mathbb{R}; M) \) such that \( \varphi(v, s + h_n) \rightarrow \varphi_0(v, s) \) in \( \mathbb{R}^N \) for every \( (v, s) \in Q(R) \). Now fix \( s \in \mathbb{R} \) and denote \( B_R = \{v \in \mathbb{R}^N \mid \|v\|_{\mathbb{R}^N} \leq R\} \). For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\|\varphi_0(v_1, s) - \varphi_0(v_2, s)\|_{\mathbb{R}^N} \leq \epsilon/3, \quad \forall v_1, v_2 \in B_R, \|v_1 - v_2\|_{\mathbb{R}^N} \leq \delta,
\]

and the function \( \theta(l, R) \) in (5) is not larger than \( \epsilon/3 \) for \( l \leq \delta \). For such \( \delta \) there exists finite number of points \( v_1, \ldots, v_m \) in \( B_R \) being a \( \delta \)-net of \( B_R \), that is, for
any $v \in B_R$, there exists some $v_i$ satisfying $\|v - v_i\|_{\mathbb{R}^N} \leq \delta$. We have, by (5), (22) and pointwise convergence,

\[
\begin{align*}
\|\varphi(v, s + h_n) - \varphi_0(v, s)\|_{\mathbb{R}^N} \\
\leq & \|\varphi(v, s + h_n) - \varphi(v_i, s + h_n)\|_{\mathbb{R}^N} \\
+ & \|\varphi(v_i, s + h_n) - \varphi_0(v_i, s)\|_{\mathbb{R}^N} \\
+ & \|\varphi_0(v_i, s) - \varphi_0(v, s)\|_{\mathbb{R}^N} \\
\leq & \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,
\end{align*}
\]

for sufficient large $n$. This means

\[
\max_{\|v\|_{\mathbb{R}^N} \leq R} \|\varphi(v, s + h_n) - \varphi_0(v, s)\|_{\mathbb{R}^N} \rightarrow 0, \quad \forall s \in \mathbb{R}, R > 0,
\]

which says that $\varphi(v, s + h_n) \rightarrow \varphi_0(v, s)$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$. Therefore $\varphi \in C^{p,u}_{tr.c}(\mathbb{R}; \mathcal{M})$.

\[\square\]

Proof of Theorem 2.3. The inclusions are obvious. Now we prove the closeness. Let the notation $\mathcal{C}$ represent $C(\mathbb{R}; \mathcal{M})$ or $C^{p,u}(\mathbb{R}; \mathcal{M})$ or $C^{p,p}(\mathbb{R}; \mathcal{M})$ and the notation $\mathcal{C}_{tr.c}$ represent the class of translation compact functions in $\mathcal{C}$. Suppose that $\{\varphi_n\} \subset \mathcal{C}$ and $\varphi_n \rightarrow \varphi$ in $C_0(\mathbb{R}; \mathcal{M})$. Hence, for any $\epsilon > 0$, there exists fixed $n_0$ such that

\[
\|\varphi_{n_0}(v, s) - \varphi(v, s)\|_{\mathbb{R}^N} \leq \epsilon, \quad (23)
\]

for all $(v, s)$ in the cylinder $Q(R)$. On the other hand, for every sequence $\{h_n\} \subset \mathbb{R}$, $\{\varphi_{n_0}(v, s + h_n)\}$ has a subsequence, still denoted by $\varphi_{n_0}(v, s + h_n)$, that converges in $\mathcal{C}$ with the limit in $C(\mathbb{R}; \mathcal{M})$. We shall prove the sequence $\{\varphi(v, s + h_n)\}$ converges in $\mathcal{C}$ and has limit in $C(\mathbb{R}; \mathcal{M})$. Note that

\[
\begin{align*}
\|\varphi(v_1, s_1 + h_n) - \varphi(v_0, s_0 + h_n)\|_{\mathbb{R}^N} \\
\leq & \|\varphi(v_1, s_1 + h_n) - \varphi_{n_0}(v_1, s_1 + h_n)\|_{\mathbb{R}^N} \\
+ & \|\varphi_{n_0}(v_1, s_1 + h_n) - \varphi_{n_0}(v_0, s_0 + h_n)\|_{\mathbb{R}^N} \\
+ & \|\varphi_{n_0}(v_0, s_0 + h_n) - \varphi(v_0, s_0 + h_n)\|_{\mathbb{R}^N}.
\end{align*}
\]

Thanks to (23), the continuity and boundedness of $\varphi$ follow from those of $\varphi_{n_0}$ by taking $h_n = 0$; moreover, we know that, by taking $v_1 = v_0$ and replacing $s_1 + h_n$ and $s_0 + h_n$ with $s + h_n$ and $s + h_n$ respectively, $\{\varphi(v, s + h_n)\}$ is a Cauchy sequence in $\mathcal{C}$ since $\{\varphi_{n_0}(v, s + h_n)\}$ is a Cauchy sequence in $\mathcal{C}$.

Suppose that $\phi(v, s)$ is the limit of $\{\varphi(v, s + h_n)\}$ in $C^{p,p}(\mathbb{R}; \mathcal{M})$ and $\phi_{n_0}(v, s) \in C(\mathbb{R}; \mathcal{M})$ is the limit of $\{\varphi_{n_0}(v, s + h_n)\}$ in $\mathcal{C}$. We have

\[
\begin{align*}
\|\phi(v_1, s_1) - \phi(v_0, s_0)\|_{\mathbb{R}^N} \\
\leq & \|\phi(v_1, s_1) - \varphi(v_1, s_1 + h_n)\|_{\mathbb{R}^N} \\
+ & \|\varphi(v_1, s_1 + h_n) - \varphi(v_0, s_0 + h_n)\|_{\mathbb{R}^N} \\
+ & \|\varphi(v_0, s_0 + h_n) - \phi(v_0, s_0)\|_{\mathbb{R}^N}.
\end{align*}
\]

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On the other hand

\[
\|\varphi_{n_0}(v_1, s_1 + h_n) - \varphi_{n_0}(v_0, s_0 + h_n)\|_{\mathbb{R}^N} \\
\leq \|\varphi_{n_0}(v_1, s_1 + h_n) - \phi_{n_0}(v_1, s_1)\|_{\mathbb{R}^N} \\
+ \|\phi_{n_0}(v_1, s_1) - \phi_{n_0}(v_0, s_0)\|_{\mathbb{R}^N} \\
+ \|\phi_{n_0}(v_0, s_0) - \varphi_{n_0}(v_0, s_0 + h_n)\|_{\mathbb{R}^N}. \tag{26}
\]

Thus the continuity of \(\phi\) follows easily from that of \(\phi_{n_0}\) and (23)-(26). That is, \(\phi \in \mathcal{C}(\mathbb{R}; \mathcal{M})\).

Similar to (24), we find

\[
\|\varphi(v, s + h_n) - \varphi(v, s + h_m)\|_{\mathbb{R}^N} \\
\leq \|\varphi(v, s + h_n) - \phi_{n_0}(v, s + h_n)\|_{\mathbb{R}^N} \\
+ \|\phi_{n_0}(v, s + h_n) - \phi_{n_0}(v, s + h_m)\|_{\mathbb{R}^N} \\
+ \|\phi_{n_0}(v, s + h_m) - \varphi(v, s + h_m)\|_{\mathbb{R}^N}. \tag{27}
\]

By (23) and the fact that \(\phi_{n_0}(v, s)\) is the limit of \(\{\varphi_{n_0}(v, s + h_n)\}\) in \(\mathcal{C}\), (27) implies that \(\phi(v, t)\) is the limit of \(\{\varphi(v, t + t_n)\}\) in \(\mathcal{C}\). Therefore, \(\varphi \in \mathcal{C}_{r.c.c.}\).

The proof is complete. \(\square\)

**Proof of Proposition 2.1.** The proof is essentially similar to that of Proposition V2.3 in [CV5].

Let \(\varphi_1 \in \mathcal{H}_w(\varphi)\). Then \(\varphi(s + h_n) \to \varphi_1(s)\) in \(C^{p.u.}(\mathbb{R}; \mathcal{M})\) for some \(\{h_n\}_n \in \mathbb{R}\). Obviously, \(\varphi(s + h_n + h_n) \to \varphi_1(s + h)\) in \(C^{p.u.}(\mathbb{R}; \mathcal{M})\), for any \(h \in \mathbb{R}\). Hence we have proved (iii). We also have that \(\{\varphi_1(s + h) \mid h \in \mathbb{R}\} \subset \mathcal{H}_w(\varphi)\). Due to the compactness of \(\mathcal{H}_w(\varphi)\) in \(C^{p.u.}(\mathbb{R}; \mathcal{M})\) we obtain that \(\mathcal{H}_w(\varphi_1) \subset \mathcal{H}_w(\varphi)\) and \(\mathcal{H}_w(\varphi_1)\) is compact in \(C^{p.u.}(\mathbb{R}; \mathcal{M})\). Moreover, by Theorem 2.2, inequality (5) holds for \(\varphi_1\) as for \(\varphi\) with the same function \(\theta\). Therefore, (i) and (ii) are true. \(\square\)

**Example** Now we construct a real function that is continuous on \(\mathbb{R}\) and weakly translation compact but not strongly translation compact in \(C(\mathbb{R} ; \mathbb{R})\). Let

\[
\phi_t(s) = \begin{cases} 
  s, & \text{if } 0 \leq s \leq \frac{t}{2 - t}, \\
  t - s, & \text{if } \frac{t}{2 - t} \leq s \leq t, \\
  \frac{1}{1 - t}, & \text{if } 1 \leq s \leq 2.
\end{cases}
\]

Now let

\[
\varphi(s) = \begin{cases} 
  \phi_{1/n}(s - n^2), & \text{if } n^2 \leq s \leq n^2 + 2, n \in \mathbb{N}, \\
  0, & \text{otherwise}.
\end{cases}
\]

Note that \(\varphi(s)\) is bounded and not uniformly continuous. We claim that \(\varphi\) is what we want. For any sequences \(\{\varphi(s + h_n) \mid h_n \in \mathbb{R}, n \in \mathbb{N}\}\) of \(\mathcal{H}_0(\varphi)\),
consider the restriction of \( \{ \varphi(s + h_n) \mid h_n \in \mathbb{R}, n \in \mathbb{N} \} \) to a bounded interval \([a, b]\). Without loss generality, we assume that \( \{h_n\} \) has no limit point. There will have two cases:

(i) \( \{ \varphi(s + h_n) \} \) converges pointwise to zero;

(ii) there exists a subsequence \( \{ \varphi(s + h_{n_k}) \} \) such that \( a - 2 \leq n^2 - h_{n_k} \leq n^2 + 2 - h_{n_k} \leq b + 2 \) for some \( n = n(h_{n_k}) \).

If case (i) is valid for any bounded interval \([a, b]\), then \( \{ \varphi(s + h_n) \} \) converges pointwise to zero on \( \mathbb{R} \). Otherwise, case (ii) is valid for some \([a, b]\). However, if case (ii) is valid, then there exists a convergent subsequence of \( \{1 - 1/n(h_{n_k}) + n^2(h_{n_k}) - h_{n_k}\}_k \), still denoted by \( \{1 - 1/n(h_{n_k}) + n^2(h_{n_k}) - h_{n_k}\}_k \), with limit \( t^* \) in \([a - 1, b + 1]\). Hence, we know that \( \{ \varphi(s + h_{n_k}) \} \) converges pointwise to \( \phi_1(s - t^* + 1) \) on \([t^* - 1, t^* + 1]\). Note that \( \{ \varphi(s + h_{n_k}) \} \) converges pointwise to zero outside of \([t^* - 1, t^* + 1]\). Therefore, in both cases, \( \{ \varphi(s + h_n) \} \) has a pointwise convergent subsequence with the limit being a continuous function. That is \( \{ \varphi(s + h_n) \} \) is weakly sequentially compact in \( C(\mathbb{R}; \mathbb{R}) \). By Eberlein-Šmulian theorem (see [Me, p248]), it is weakly compact in \( C(\mathbb{R}; \mathbb{R}) \).

References


