The Optimal BER Linear Rake Receiver for Alpha-Stable Noise

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Abstract—The optimal linear Rake receiver for the detection of binary signals contaminated by symmetric alpha-stable noise is derived for $1 < \alpha \leq 2$. The bit error rate improvements of the optimal linear Rake receiver over the maximal ratio combiner and the equal gain combiner are also derived in the form of signal-to-noise ratio advantage. The proposed receiver is a very simple form of diversity combiner for signal detection in alpha-stable noise.

Index Terms—Linear Rake receiver, non-Gaussian noise detection, alpha-stable noise.

I. INTRODUCTION

The maximal ratio Rake (MRC) combiner is optimal in the sense of maximizing the signal-to-noise ratio (SNR) regardless of the type of the noise distribution (except for the requirement for the noise distribution to have a finite $2^{nd}$-order moment). However, a MRC Rake receiver is optimal in the sense of minimizing the bit error probability (BER) only if the noise is Gaussian distributed. Otherwise, maximizing the SNR does not necessarily guarantee minimal BER.

The symmetric alpha-stable distribution ($S_{\alpha}S$) is widely suggested for modeling impulsive non-Gaussian noise [1] - [3]. Impulsive noise naturally arises in communication; examples include multiple access interference (MAI) in ultra-wide bandwidth (UWB) systems [4], radar clutter [5] and shot noise. In [6], it is demonstrated that the MAI in UWB systems can be better modeled by an alpha-stable process.

We consider the problem of signal detection using samples from $N$ independent channels (the channels may represent $N$ Rake fingers or $N$ antennas or $N$ frequency multiplexed channels). The optimal detector in the maximal likelihood (ML) sense for this problem will be a non-linear combiner and it will require knowledge of the probability density function (PDF) of the $S_{\alpha}S$ process for any value of $\alpha$. However, no exact closed-form solution is known for the alpha-stable PDF except for three special cases ($\alpha = 0.5, 1$ and $2$) [3]. A simple and easy to implement form of a sub-optimal receiver is a linear Rake combiner illustrated in Fig. 1, which only requires knowledge of the optimal weights, $w_i$, for operation and does not require any non-linear processing at the fingers.

In this paper, we derive the optimal linear Rake combiner detector having the structure shown in Fig. 1, which minimizes the BER for binary signal detection. The derivation is performed for a certain range of $\alpha$ ($1 < \alpha \leq 2$), the shaping parameter of the $S_{\alpha}S$ distribution, while assuming that the additive noises in the channels (fingers) are independent and identically distributed (i.i.d). The range of $\alpha$ for which the solution is found is adequate to include the values of $\alpha$ occurring in practical cases and it covers cases of closely Gaussian and closely Cauchy distributed noise.

It is well known that a linear combiner is not in general the optimal combiner for the detection of signals immersed in alpha-stable noise using samples from multiple channels. However, the optimal Rake combiner derived here is found to give a significant amount of advantage in terms of SNR and it is easy to implement.

The rest of this paper is organized as follows. Section II describes the system model and the derivation of the optimal combiner. In Section III, the SNR advantages of the optimal combiner over MRC and equal gain combiner (EGC) are derived. In Section IV, we consider the performance of linear combiners in a mixture of alpha-stable and Gaussian noise. Numerical results are provided in Section V and Section VI is the conclusion.

II. SYSTEM MODEL AND THE OPTIMAL COMBINER

The Rake combiner operates on the input signal vector $\mathbf{r} = [r_1, r_2, \ldots, r_N]$ where $r_i = s_i + n_i$ where $s_i$ is the signal component and $n_i$ is the additive $S_{\alpha}S$ distributed noise in the $i^{th}$ channel (finger). The noise distribution $S_{\alpha}S(\zeta, \alpha)$ does not have a general closed-form and is most conveniently described

![Fig. 1. A block diagram of the linear Rake combiner with $N$ fingers.](image-url)
by its characteristic function (CF) \[3\]

\[\Phi_n(\omega) = E \{\exp(j \omega n)\} = \exp(-\zeta^\alpha |\omega|^\alpha), \quad 0 \leq \alpha \leq 2\]  \hspace{1cm} (1)

where \(\zeta\) is the shaping parameter and \(\alpha\) is the characteristic exponent and the location parameter is zero. The noise components \(\{n_i\}_{i=1}^N\) are independent and identically distributed (i.i.d) random variables. The linear Rake combiner output \(\gamma\) is given by

\[\gamma = \sum_{i=1}^N w_i r_i = S + n\]  \hspace{1cm} (2)

where \(w = \{w_i\}_{i=1}^N \subseteq \mathbb{R}^N\), \(S = \sum_{i=1}^N w_i s_i\) and \(n = \sum_{i=1}^N w_i n_i\). Since the \(n_i\) are independent random variables (RVs), the CF of \(n\), defined by \(E[\exp(j \omega n)]\), can be written as

\[
\Phi_n(\omega) = \prod_{i=1}^N \Phi_{n_i}(\omega) = \prod_{i=1}^N \exp(-\zeta^\alpha |w_i|^\alpha |\omega|^\alpha)
\]

\[
= \exp\left(-\zeta^\alpha \left[ \sum_{i=1}^N |w_i|^\alpha \right] |\omega|^\alpha \right).
\]

Dividing equation (2) on both sides by a positive constant will not cause any change in the detector’s BER performance. Therefore, we divide (2) by \(\sqrt{\sum_{i=1}^N |w_i|^\alpha}\) to obtain

\[
\gamma' = \sum_{i=1}^N w_i r_i \sqrt{\sum_{i=1}^N |w_i|^\alpha}.
\]

Let \(\gamma' = S' + n'\) where

\[S' = \sum_{i=1}^N \frac{w_i}{\sqrt{\sum_{i=1}^N |w_i|^\alpha}} s_i\]  \hspace{1cm} (5)

and

\[n' = \sum_{i=1}^N \frac{w_i}{\sqrt{\sum_{i=1}^N |w_i|^\alpha}} n_i\]  \hspace{1cm} (6)

Now the CF of \(n'\) is given by

\[
\Phi_{n'}(\omega) = \Phi_n \left( \omega / \sqrt{\sum_{i=1}^N |w_i|^\alpha} \right) = \exp(-\zeta^\alpha |\omega|^\alpha).
\]

The choice of the constant \(\sqrt{\sum_{i=1}^N |w_i|^\alpha}\) to divide (2) makes \(\Phi_{n'}(\omega)\) independent of \(w\). Since a CF uniquely defines its corresponding PDF, the PDF and the cumulative density function (CDF) of \(n'\) are also independent of \(w\). This means that no matter how \(w\) is chosen, the PDF of the noise component in the new decision variable, \(\gamma'\), does not change. The BER for binary signal detection with equiprobable 1 and -1 symbols for the detector in Fig. 1 is given by

\[P_e = P(n' > |S'|) = 1 - F_{\zeta,\alpha}(|S'|)\]  \hspace{1cm} (8)

where \(F_{\zeta,\alpha}(x) = \frac{1}{2} + \frac{1}{2} \pi \int_{-\infty}^{\infty} \exp(-\zeta^\alpha |\omega|^\alpha + j \omega x) / j \omega \, d\omega\) [7] is the distribution function of \(n'\). The problem of finding the optimal linear Rake receiver can now be reduced as

\[
\arg \min_w (P_e) = \arg \min_w (P(n' > |S'|)) = \arg \min_w (1 - F_{\zeta,\alpha}(|S'|)) = \arg \max_w (F_{\zeta,\alpha}(|S'|)) = \arg \max_w (|S'|).
\]  \hspace{1cm} (9)

Note that in (9) we used the fact that \(F_{\zeta,\alpha}(x)\) is a monotonically increasing function of \(x\). By using the Holder’s inequality [8] the following inequality is obtained for \(\alpha > 1\) and \((w_i, s_i) \in \mathbb{R}^2\)

\[
\sum_{i=1}^N w_i s_i \leq \left( \sum_{i=1}^N |w_i|^\alpha \right)^{1/\alpha} \left( \sum_{i=1}^N |s_i|^\alpha (\alpha - 1)/\alpha \right)^{(\alpha - 1)/\alpha}.
\]

Define the set \(S = \{x, y | (x, y) \in \mathbb{R}^2, xy \geq 0\}\). Since \(S \subset \mathbb{R}^2\) we have

\[
\sum_{i=1}^N w_i s_i \leq \left( \sum_{i=1}^N |w_i|^\alpha \right)^{1/\alpha} \left( \sum_{i=1}^N |s_i|^\alpha (\alpha - 1)/\alpha \right)^{(\alpha - 1)/\alpha}.
\]

for \((w_i, s_i) \in S\), which is equivalent to

\[
\sum_{i=1}^N |w_i| s_i \leq \left( \sum_{i=1}^N |w_i|^\alpha \right)^{1/\alpha} \left( \sum_{i=1}^N |s_i|^\alpha (\alpha - 1)/\alpha \right)^{(\alpha - 1)/\alpha}
\]

for \((w_i, s_i) \in \mathbb{R}^2\). Since \(\sum_{i=1}^N |w_i| s_i \leq \sum_{i=1}^N |w_i| s_i|\) we get

\[
|S'| = \frac{1}{\sqrt{\sum_{i=1}^N |w_i|^\alpha}} \leq \left( \sum_{i=1}^N |s_i|^\alpha (\alpha - 1)/\alpha \right)^{(\alpha - 1)/\alpha}.
\]

The equality in (13) is achieved when

\[
w_i = w_i^* = \text{sign}(s_i) |s_i|^{1/(\alpha - 1)}\]  \hspace{1cm} (14)

which gives the optimal weights for linear combining. The corresponding optimal value of \(|S'|\) obtained from (13) is given by

\[
S'_{\text{max}} = \left[ \sum_{i=1}^N |s_i|^\alpha \right]^{\frac{\alpha - 1}{\alpha}}.
\]

III. ADVANTAGE OVER OTHER COMBINING METHODS

Since \(S\) and \(S\) processes do not have a finite \(2^{nd}\) order moment except for \(\alpha = 2\), the usual definition of SNR is not valid. Hence, the SNR must be defined in alternative ways when studying signal detection in alpha-stable noise. A simple definition is the signal to dispersion ratio given by

\[
\text{SNR}_D = \frac{S^2}{2\zeta^2}.
\]  \hspace{1cm} (16)
The SNR can also be defined using the geometric power [2], [3] of the alpha-stable process as

\[
SNR_{GM} = \frac{S^2}{2C_G S_0^2}
\]

(17)

where the geometric power \(S_0\) of an alpha-stable RV \(x\) is given by \(S_0 = e^{E[\ln|\xi|]}\) and \(C_G = e^{EI}\) where \(EI\) is the Euler constant [8, p. xxxii]. One can note that the BER improvement can be measured in the commonly used dB SNR advantage for both these SNR definitions.

1) **SNR advantage over MRC:** If MRC is used as the output signal component will be

\[
S_{MRC} = \sum_{i=1}^{N} \frac{|s_i|^2}{\sqrt{\sum_{i=1}^{N} |s_i|^\alpha}}.
\]

(18)

Therefore, the SNR advantage over MRC is given by

\[
20 \log_{10} \left( \frac{S_{\text{max}}'}{S_{\text{MRC}}} \right) = 20 \log_{10} \left( \frac{\left[ \sum_{i=1}^{N} |s_i|^\frac{\alpha}{\alpha-1} \right]^{\frac{\alpha-1}{\alpha}} \left[ \sum_{i=1}^{N} |s_i|^\alpha \right]^{\frac{1}{\alpha}}}{\sum_{i=1}^{N} |s_i|^2} \right). \tag{19}
\]

2) **SNR Advantage Over EGC:** The output signal component for EGC can be given by

\[
S_{\text{EGC}} = \frac{\sum_{i=1}^{N} |s_i|}{N^{1/\alpha}}. \tag{20}
\]

Therefore, The SNR advantage over EGC is given by

\[
20 \log_{10} \left( \frac{\left[ \sum_{i=1}^{N} |s_i|^\frac{\alpha}{\alpha-1} \right]^{\frac{\alpha-1}{\alpha}} \left[ \sum_{i=1}^{N} |s_i|^\alpha \right]^{\frac{1}{\alpha}}}{\sum_{i=1}^{N} |s_i|^2} \right).
\]

(21)

IV. **PERFORMANCE IN A MIXTURE OF \(S_\alpha S\) NOISE AND GAUSSIAN NOISE**

In the previous sections, we derived the optimal linear Rake combiner for alpha-stable distributed noise. In environments where the ambient noise, which is Gaussian distributed, is not negligible the total noise should be modeled as a mixture of alpha-stable noise and Gaussian noise. In these environments the optimal linear combiner for alpha-stable noise that we derived in Section II will not be the optimal linear combiner for signal detection. However, the task of deriving an optimal linear combiner for signal detection in a mixture of alpha-stable noise and Gaussian noise is difficult. Therefore, it is important to study the performance of the proposed combining scheme in an alpha-stable plus Gaussian mixed noise environment.

In the following exposition, the BER performance of a general linear combiner will be derived. In the mixed noise environment, \(r_i\) can be written as \(r_i = s_i + n_i + n_{G,i}\) where \(n_{G,i}\) is the additive white Gaussian noise (AWGN) in the \(i\)th channel. It is assumed that \(\{n_{G,i}\}_{i=1}^N\) are i.i.d with \(n_{G,i} \sim \text{norm}(0, \sigma^2)\) where \(\text{norm}(\mu, \sigma^2)\) denotes the Gaussian PDF with mean \(\mu\) and variance \(\sigma^2\). Now the modified decision metric \(\gamma'\) can be written as

\[
\gamma' = \frac{\sum_{i=1}^{N} w_i |r_i|^{\alpha}}{\sum_{i=1}^{N} |w_i|^{\alpha}} = S' + n' + n'_G.
\]

where \(n'_G = \sum_{i=1}^{N} \frac{w_i}{\sqrt{\sum_{i=1}^{N} |w_i|^{\alpha}}} n_{G,i}\). The CF of \(n'_G\) can be written as

\[
\Phi_{n'_G}(\omega) = \prod_{i=1}^{N} \Phi_{n_{G,i}} \left( \frac{w_i}{\sqrt{\sum_{i=1}^{N} |w_i|^{\alpha}}} \right) = e^{\exp \left( \frac{-\sigma^2 \omega^2}{2} \sum_{i=1}^{N} \left[ \frac{w_i^2}{\sum_{i=1}^{N} |w_i|^{2/\alpha}} \right] \right)}.
\]

(23)

Since the noise modeled as an alpha-stable process and the ambient noise generally originiate from physically independent processes, the noise components \(n'\) and \(n'_G\) are independent. One can now write the BER of binary signal detection with equiprobable symbols as [7]

\[
P_e = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{n'_G}(\omega) e^{\exp(j\omega|S'|)} |j\omega| dw
\]

\[= \frac{1}{2} - \frac{|S'|}{\pi} \int_{0}^{\infty} e^{\exp \left( -\zeta \omega^\alpha \right)} \left( \frac{-\sigma^2 \omega^2}{2} \sum_{i=1}^{N} \frac{w_i^2}{\sum_{i=1}^{N} |w_i|^2/\alpha} \right) \sin(\omega |S'|/\pi) \omega. \tag{24}
\]

Note that \(S'\) is a function of \(\omega\) and part of the integrand in (24) is also a function of \(\omega\). Therefore, it is difficult to compare the performances analytically for different combining schemes. Comparison of different combining schemes is performed using numerical integration and the results are provided in Section V.

V. NUMERICAL RESULTS

In Fig. 2 the maximum SNR advantage of the optimal linear Rake combiner over MRC and EGC is depicted for different values of the characteristic exponent \(\alpha\) with \(N = 4\). The maximum SNR gain is calculated by numerically optimizing (19) and (21) with respect to \(s = \{|s_i|\}_{i=1}^N\) in the search space of \(A^N\) where \(A = [0,1]\). One is able to reduce the original search space of \(\mathbb{R}^2\) to \(A^N\) by using the scaling and symmetry properties of the optimal \(s\). Fig. 2 shows that the SNR advantage gradually decreases with \(\alpha\). When compared to MRC, the SNR advantage is maximum (\(\sim 3.5\) dB) when \(\alpha\) is close to 1 and it reduces gradually to 0 as \(\alpha\) approaches 2. This is because, when \(\alpha = 2\) the optimal combiner becomes the MRC combiner (see (14)). The number of channels, \(N\), is set at 4 due to the complexity in numerically optimizing the
SNR advantages for larger $N$. The maximum SNR advantage will be larger for larger values of $N$.

Fig. 3 shows the variation of the maximal SNR advantage with $N$ for an arbitrary value of $\alpha = 1.1$. These curves clearly show that the larger the value of $N$, the larger the SNR advantage. In Fig. 4, the BER performance of the optimal Rake receiver is compared with MRC and EGC for $\alpha = 1.1$ and $N = 8$. The signal vector $s$ used in this particular example is $[0, 0.1927, 0.3179, 0.0439, 0.6696, 0.3949, 0.2869, 0.3493, 0.2263]$. Note that this vector is arbitrarily chosen for illustrative purposes. In this example, the SNR advantage of the optimum combiner is approximately $3\,\text{dB}$ over MRC and $5\,\text{dB}$ over EGC.

Fig. 5 shows a comparison of the BERs of EGC, MRC and optimal combining schemes in a fading multipath channel. A Rayleigh fading model is assumed for the channel amplitude, the multipath signals are assumed to be equally spaced in time and an exponential power-delay profile (PDP) is assumed for the channel. If $P_0$ is the energy of the signal in the $1^{st}$ arrival path the energy of the signal in the $i^{th}$ path in this example is given by $P_0 \exp(-\rho (i - 1))$ with $\rho = 0.5$. The signal-to-dispersion ratio, $SNR_D$, is used in comparing the BERs in Fig. 5 and its definition in a multipath fading channel is $SNR_D = \sum_{i=1}^{N} E[|s_i|^2]/\sigma^2$. Fig. 5 shows that the improvements in BERs of the optimal combiner over MRC and EGC are significant even in a fading multipath channel. In this example, the SNR advantage of the optimal combiner over MRC is approximately $2\,\text{dB}$ and this is a significant amount of improvement in wireless communication in terms of reduction in transmitted power.

Figs. 6 and 7 compare the BERs of the three combining schemes (MRC, EGC and the optimal combiner for SαS noise) in a mixed alpha-stable plus Gaussian noise environment in a static channel. The static channel used in Fig. 4 is used here and $\alpha = 1.1$. In these figures, $SNR_D$ denotes the signal-to-dispersion ratio of the SαS noise and $SNR_G$ denotes the signal-to-noise ratio of the Gaussian noise. In Fig. 6, the variation of BERs against $SNR_G$ is shown for a fixed value of $SNR_D = 10\,\text{dB}$. Fig. 6 shows that the optimal combiner outperforms MRC for large values of $SNR_G$ and the MRC is preferable when $SNR_G$ is small compared to $SNR_D$. In Fig. 7, the variation of BERs against $SNR_D$ is shown for a fixed value of $SNR_G = 10\,\text{dB}$. Fig. 7 shows that the optimal combiner is the best combining method for small values of $SNR_D$. These results show that one can choose between the optimal combiner and MRC based on the ratio $2\gamma^2/\sigma^2$ to maximize benefits.

**VI. CONCLUSION**

The optimal linear Rake receiver which minimizes the BER for signal detection in SαS noise has been derived for $1 < \alpha \leq 2$. The SNR advantages of the optimal combiner over MRC and EGC were calculated and compared. The SNR advantage of the optimal structure over MRC and EGC are increasing functions of the number of channels and decreasing functions of the characteristic exponent (within the range $1 < \alpha \leq 2$). Simulation results showed that the performance of the optimal combiner can be significantly better than MRC and EGC in fading multipath channels as well. It is also found that the optimal combiner is a useful combining method even in mixed alpha-stable plus Gaussian noise environments.

**REFERENCES**


Fig. 4. Example BER comparison of optimal, MRC and EGC combiners for a static multipath channel.

Fig. 5. Example BER comparison of optimal, MRC and EGC combiners for a Rayleigh fading multipath channel.

Fig. 6. Variation of BERs of optimal, MRC and EGC combiners with $SNR_G$. A static multipath channel is assumed with mixed alpha-stable plus Gaussian noise.

Fig. 7. Variation of BERs of optimal, MRC and EGC combiners with $SNR_D$. A static multipath channel is assumed with mixed alpha-stable plus Gaussian noise.


