Separability in Asymmetric Phase-Covariant Cloning

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(Dated: April 23, 2008)

Here, asymmetric phase-covariant quantum cloning machines are defined and trade-off between qualities of their outputs and its impact on entanglement properties of the outputs are studies. In addition, optimal families among these cloners are introduced and also their entanglement properties are investigated. An explicit proof of optimality is presented for the case of qubits, which is based on the no-signaling condition. Our optimality proof can also be used to derive an upper bound on trade-off relations for a more general class of optimal cloners which clone states on a specific orbit of the Bloch sphere. It is shown that the optimal cloners of the equatorial states, as in the case of symmetric phase-covariant cloning, give rise to two separable clones, and in this sense these states are unique. For these cloners it is shown that total output is of GHZ-type.

PACS numbers: 03.67.-a, 03.67.Mn, 89.70.+c

I. INTRODUCTION

In quantum information theory the so-called no-cloning theorem prohibits ideal copying of a quantum state [1, 2, 3]. Although it may seem a restrictive fact, it is an advantage over classical information theory. Because security in quantum cryptography is largely attributed to impossibility of exact copying of quantum data. It is impossible to clone quantum information exactly, however, it is useful to know how well one can achieve this goal. As well, there are inevitable needs to investigate this since, for example, storage and retrieval of quantum information on quantum computers are essentially related to copying. Hence quantum cloning enters into the scope of real experiments. Also, as is seen any success in good cloning makes quantum cryptography more at risk. For example in BB84 protocol [4] there is a link between optimal cloning of equatorial states and optimal eavesdropping attack. There have been extensive studies on this subject that illuminate some aspects of quantum cloning [5, 6, 7, 8, 9, 10, 11, 12, 13].

In this paper, we investigate a family of asymmetric cloning machines for d-dimensional states in the form of \(|\psi\rangle = \sqrt{d} \sum_k e^{i\phi_k} |k\rangle\). However asymmetric machines are important in their own merits, their occurrence in the context of quantum information theory can also be attributed to the situations in which one of clones needs to be a bit better than the other, or when there may be an internal flaw in hardware of symmetric cloner that makes two copies non-identical. Also in studying these machines various no-cloning inequalities, which are consequences of the quantum uncertainty principle, are relevant and obtain practical meanings. We also investigate these machines in the sense of entanglement produced in their outputs.

The paper is structured as follows. In Sec. II we review universal asymmetric cloning machines, based on trivial asymptomization of the original universal cloning machine of Bužek and Hillery [5]. Indeed, this is nothing more than a simple re-explanation of the asymmetric cloners firstly introduced by Cerf [14, 15, 16]. Its general properties are reviewed and a trade-off relation for the qualities of the two clones is derived. In Sec. III after some general remarks on d-dimensional asymmetric phase-covariant cloners, we study a special purpose asymmetric machine for cloning \(x - y\) equatorial states of the Bloch sphere; phase-covariant qubit cloners. Next, we investigate optimality of such asymmetric machines. Then we analyze the separability of the output copies and show that among all inputs only equatorial states give rise to two separable outputs, that is, output clones of an optimal phase-covariant cloner are unentangled. In addition, it is shown that total outputs of these machines are of GHZ-type. The paper is concluded in Sec. IV.

II. UNIVERSAL ASYMMETRIC CLONING MACHINE

In this section we are going to devise an asymmetric cloning machine which is universal, that is, it treats all inputs in the same way. For the sake of simplicity, here, we restrict ourselves to the case of duplicators, i.e. 1→2 universal cloners. However, extension to triplicators is also straightforward. The question of how well one can design an approximate duplicator of a qubit (or qudit), provided that the quality of the two outputs be independent of the input states, has been investigated by Bužek and Hillery [5, 6] and the others [7, 8, 9, 10, 11, 12, 13].

At first, we briefly review d-dimensional universal cloning machines following Bužek and Hillery [6]. Consider the unitary transformation

\[
|i\rangle_A |0\rangle_B |\Sigma\rangle X \rightarrow \mu |i\rangle_A |i\rangle_B |i\rangle_X \\
+ \nu \sum_{j \neq i}^{d-1} \left( |i\rangle_A |j\rangle_B + |j\rangle_A |i\rangle_B \right) |j\rangle_X, (1)
\]

in which A and B are, respectively, input and blank qudits, and X is an ancilla that always can be considered as the cloning machine itself which is initially in a fixed state, say \(|\Sigma\rangle\). The set \(|i\rangle_A |\Sigma\rangle_{X_i}\rangle_{i=0}^{d-1}\) is a set of orthonormal basis vectors of the Hilbert space of input (machine); \(\mathcal{H}_{A(X)}\). Without loss of generality, \(\mu\) and \(\nu\) can always be considered to be real parameters. Requiring unitarity of the transformation and the
following conditions: (i) quality of cloning (defined based on fidelity of the copies \( F := \langle \psi | \rho^{(\text{out})} | \psi \rangle \)) does not depend on the particular state which is going to be copied (universality or input state-independence), (ii) the outputs are symmetric, i.e. \( \rho_A^{(\text{out})} = \rho_B^{(\text{out})} \), the following relations can be obtained

\[
\begin{align*}
\rho_{A(B)}^{(\text{id})} &= \eta \rho_{A(B)}^{(\text{id})} + \frac{1-\eta}{d} 1_{A(B)}, \\
\mu^2 &= 2\eta \mu^2 = \frac{2}{d+1}, \quad \nu^2 = \frac{1}{2(d+1)}, \\
\eta &= \mu^2 + (d-2)\nu^2 = \frac{d+2}{2(d+1)},
\end{align*}
\]

where \(1_{A(B)}\) stands for the \(d \times d\) identity operator on the space of \(\mathcal{H}_{A(B)}\), and \(\eta = \frac{dF-1}{d-1}\) is called shrinking factor. Some points on the above cloning transformation are worth noting. In \(d = 2\), this machine simply reduces to the original universal cloning machine \([5]\), with \(\eta = \frac{2}{3}\) or \(F = \frac{2}{3}\). This machine is proved to be optimal in that it produces maximal fidelity considering its requirements \([9, 10, 11, 12, 13]\). It can be justified that symmetry of the outputs is a consequence of equality of the coefficients of the terms \(\langle i| A| j \rangle X\) and \(\langle i| A| j \rangle X\) in Eq. (2). Thus one can consider it as a starting point for extension to transformations which produce asymmetric output copies. Here, it must be noted that this kind of survey is not something different from the asymmetric cloning introduced by Cerf \([14, 15, 16]\). However, to clarify the subject we use a simpler exposition. Let us start simply by giving different contributions to the two latter terms. Hence, the following cloning transformation can be introduced (which is an isometry)

\[
\begin{align*}
| i \rangle_A | 0 \rangle_B | \sum_j X \rightarrow \mu^2 | i \rangle_A | j \rangle_B | i \rangle_X \\
+ \nu \sum_{j \neq i} | i \rangle_A | j \rangle_B | j \rangle_X + \xi \sum_{j \neq i} \langle j | A | i \rangle_B | j \rangle_X.
\end{align*}
\]

(3)

If a state in the form of \(| \psi \rangle = \sum_i \alpha_i | i \rangle\) is given to the machine as an input, then, the state of the output copy \(A\) becomes

\[
\begin{align*}
\rho_A^{(\text{out})} &= | \psi \rangle_A \langle \psi | [ (d-2)\nu^2 + 2\mu \nu ] + \xi^2 1_A \\
&+ (\mu^2 + \nu^2 - \xi^2 - 2\mu \nu) \sum_i | i \rangle_A [\xi | i \rangle_A | i \rangle_a],
\end{align*}
\]

and similarly for the copy \(B\) (with the replacements \(\xi \leftrightarrow \nu, A \leftrightarrow B\)). The last term in Eq. (4) is obviously state-dependent. If we impose state-independence condition for the cloning machine, by considering the fact that necessary and sufficient conditions for \(F_A\) being state-independent is that for an input state \(\rho^{(\text{id})}\), the output state \(\rho_A^{(\text{out})}\) has a form as

\[
\begin{align*}
\nu_A^{(\text{out})} &= \eta \nu_A^{(\text{id})} + \frac{1-\eta}{d} 1_A,
\end{align*}
\]

(5)

we get the following relations

\[
\begin{align*}
\eta_A &= (d-2)\nu^2 + 2\mu \nu, \\
\mu^2 + (d-1)\nu^2 + \xi^2 &= 1, \\
\frac{1-\eta}{d} &= \xi^2, \\
\mu^2 + \nu^2 - \xi^2 - 2\mu \nu &= 0.
\end{align*}
\]

Eqs. (6b) and (6d), and the corresponding relation for the copy \(B\), \(\mu^2 + \xi^2 - \nu^2 - 2\mu \xi = 0\), give rise to the following result

\[
\mu = \nu + \xi.
\]

(7)

If we introduce the parameterization: \(\nu = r \cos \phi\) and \(\xi = r \sin \phi\), the expressions for \(F_A\) and \(F_B\) take the simple forms as below

\[
\begin{align*}
F_A &= \frac{d \cos^2 \phi + \sin^2 \phi + \sin 2\phi}{d + \sin 2\phi}, \\
F_B &= \frac{d \sin^2 \phi + \cos^2 \phi + \sin 2\phi}{d + \sin 2\phi}.
\end{align*}
\]

(8a)

(8b)

from which by cancelation of \(\phi\) we reach a relation between the fidelities as below

\[
F_A^2 + F_B^2 + 2 \frac{d+2}{d-1} F_A F_B - \frac{2(d+2)^2}{d^2}(F_A + F_B) + \frac{(d-1)(d+3)}{d^2} = 0.
\]

(9)

This is equation of a set of ellipses (in the space of fidelities) that their eccentricities vary with dimension. Using the relation \(\eta = \frac{dF-1}{d-1}\) a corresponding set of ellipses in the space of shrinking factors can be found. Figure 1 illustrates these ellipses for some specific dimensions. As is seen in infinite dimensional case the corresponding ellipse shrinks to the line \(\eta_A + \eta_B = 1\), and also all of ellipses in \((\eta_A = 0, \eta_B = 1)\) point are tangent to \(\eta_B = 1\) line (and similarly, in \((1, 0)\)). Also the slope of the tangents to the ellipses in the points on the symmetry axis \((\eta_A = \eta_B)\) is 1.

In a given dimension, the corresponding ellipse, indeed, induces a kind of complementarity or trade-off between the two copies, since if one fixes one of the parameters the other one is determined too. So we have provided a graph for full spectrum of the qualities, in which special points corresponding to \(\eta_A = \eta_B\) on each graph are representatives of the symmetric cloners.

Now, we can investigate question of optimality of this simple cloning transformation. Similar questions have been studied earlier in \([14, 15, 16, 17, 18, 19]\). As we stressed above, it can be seen that this machine is nothing but a Heisenberg cloning machine which is firstly introduced by Cerf \([15, 16]\). The quantum uncertainty principle gives rise to the so-called no-cloning inequalities which are upper bounds on trade-off relations for the qualities of the two clones. Heisenberg cloning machines generally give two non-identical output copies, each of which comes out of a different Heisenberg channel. It can be simply checked that the cloning transformation \([3]\) satu-
rates the no-cloning inequality, and, therefore, it corresponds to an optimal cloner.

In the next section we relax the universality condition and focus on asymmetric phase-covariant cloning machines.

III. ASYMMETRIC PHASE-COVARIANT CLONING MACHINES

It is evident that if one a priori has a partial knowledge about input states, then utilizing this information, more efficient special purpose cloning machines can be designed. This fact leads to the investigation of some state-dependent cloners, among which the class of phase-covariant cloners lies. These special cloners are designed to clone equatorial states as well as possible (better than the universal cloning). This class, in the case of symmetric cloning, has been studied previously [20, 21, 22, 23, 29]. In this section we investigate asymmetric version of phase-covariant machines.

In Eq. (4) it is seen that the last term, $\sum_i |\alpha_i|^2 |i\rangle|0\rangle$, depends on input state of the machine. In the special case of input states in the form 

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_k e^{i\phi_k} |k\rangle, \quad (0 \leq \phi_k < 2\pi)$$

(10)

which are covariant with respect to rotations of the phases, this term automatically reduces to the identity matrix and, therefore, becomes state-independent. As a result, for this class of inputs one does not need to consider the condition (6d), since this condition was to cancel the contribution of the state-dependent term. Insisting on having this condition results in a less efficient cloner for this particular class of inputs. Anyway, here Eq. (6d) is not necessary and Eq. (4) reduces to 

$$\rho_A^{(out)} = |\psi\rangle_A \langle \psi|\nu^2 (d-2) + 2\mu \nu \rangle + (\xi^2 + \mu^2 + \nu^2 - \xi^2 - 2 \mu \nu) 1_A,$$

(11)

and similarly for the $B$ copy (by the simple replacement of $\nu \leftrightarrow \xi$). Then, shrinking factors of the two output clones are 

$$\eta_A = 2 \mu \nu + (d-2) \nu^2,$$

$$\eta_B = 2 \mu \xi + (d-2) \xi^2.$$ 

(12a, b)

Using the normalization condition (6b), the above equations can be simplified as below

$$\eta_A = (d-2) \nu^2 + 2 \nu \sqrt{1 - (d-1) \nu^2 + \xi^2},$$

(13a)

$$\eta_B = (d-2) \xi^2 + 2 \xi \sqrt{1 - (d-1) \nu^2 + \xi^2}.$$ 

(13b)

As is seen, contrary to the case of universal cloning, here we are left with two free tuning parameters to set.

Now, in this class of cloners an optimal machine is defined as the following: in this machine if we fix the quality of one of the clones—say $A$—then the quality of the other clone is the highest possible value.

From Eq. (13a) one can find $\xi$ in terms of $\nu$ and $\eta_A$. After inserting this value in Eq. (13b), we have 

$$\eta_B = \frac{d-2}{d-1} \left[ 1 - (d-1) \nu^2 - \left( \frac{\eta_A - (d-2) \nu^2}{2 \nu} \right)^2 \right] + \frac{\eta_A - (d-2) \nu^2}{\nu} \sqrt{1 - \left( \frac{\eta_A - (d-2) \nu^2}{2 \nu} \right)^2}.$$ 

(14)

By optimization of $\eta_B(\nu)$ (and assuming $\eta_A=$const.) the value of $\nu_{optimal}(\eta_A)$ can be found for any dimension $d$. Unfortunately, in general, there is not a closed analytical form for this value. However, the correct solutions can be found by a simple numerical examination. Figure 2 shows the trade-off diagrams $(\eta_A$ vs. $\eta_B)$ for some typical dimensions. It can be inferred that in optimal asymmetric cloners the final relation between the two shrinking factors $\eta_A$ and $\eta_B$ is in the form of the equation of an ellipse with an eccentricity depending on dimension. In the symmetrical case, where $\eta_A = \eta_B = (d-2) \nu^2 + 2 \nu \sqrt{1 + 2(d-1) \nu^2}$, a simple algebra shows that 

$$\nu_{optimal}(d) = \frac{1}{2} \sqrt{\frac{(d^2 + 4d - 4)d + 4d - 4}{d^2 + 4d^2 - 8d + 4}},$$

(15)

from which the fidelity of the optimal phase covariant cloning machine is obtained as 

$$F = \frac{1}{d} + \frac{1}{d^2}(d - 2 + \sqrt{d^2 + 4d - 4}),$$

(16)

which is in accordance with the result of [21].

A. Proof of optimality for qubit cloning

In this subsection, we restrict ourselves to the case of $d = 2$ (qubit cloning), and study optimality of the cloning transformations. Optimality of this machine can be proved on the basis of no-signaling condition, which has been used previously in this context [24, 25, 26, 27, 28]. In simple words, this condition states that one cannot exploit quantum entanglement between two spacelike separated parties for superluminal communication. However this condition is very general, a simplified version of that is sufficient for means of this paper.
We show that using this condition (and positive semidefinite-ness of density operators) a trade-off relation between $\eta_A$ and $\eta_B$ is found which is saturated by our cloning transformation. We devise our proof so that it can also be used to derive an upper bound on trade-off relations of clones of orbital states of the Bloch sphere. These states are simply defined to be the states for which we have $\langle \psi | \sigma_z | \psi \rangle = \cos \theta$, for a given $\theta$. Figure 3 shows an illustration of the Bloch sphere along with an orbit having the polar angle $\theta$. The $x$-$y$ equator is a special kind of these orbital states with $\theta = \frac{\pi}{4}$, or equivalently $\langle \psi | \sigma_x | \psi \rangle = 0$. Phase-covariant clones for these special class of states were initially introduced in [29], and lately were considered for optical implementation [30].

An orbital state, with a given value of $\theta$, can be represented as

$$|\psi(\vec{r})\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle \quad (0 \leq \phi < 2\pi),$$

where $\vec{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector representing the state on a given orbit of the Bloch sphere. Imposing the universality condition for the cloning of a given orbit requires the condition

$$\rho^{\text{(out)}}_{AB}(\vec{r} R) = U(R) \otimes U(R) \rho^{\text{(out)}}_{AB}(\vec{r}) U^\dagger(R) \otimes U^\dagger(R),$$

for any $\vec{r}$ on the orbit. In this relation $R \equiv R(\hat{z}, \chi) \in \text{SO}(3)$ is the usual rotation matrix in 3-dimensional space about $z$-axis through an angle $\chi$, and $U(R) = e^{-i\vec{r}\vec{s}} \in \text{SU}(2)$ is the corresponding unitary operation on the Bloch vector (and $\sigma_z = (1, 0, -1)$). We want to clone (asymmetrically and) universally this qubit (independent of the Bloch vector $\vec{r}$ on a given orbit), in such a way that the reduced density matrices of the clones $A(B), \rho^{(out)}_{A(B)}(\vec{r})$, are of the forms

$$\rho^{(out)}_{A(B)} = \eta_{A(B)} \rho^{(id)}_{A(B)} + \frac{1-\eta_{A(B)}}{2} \mathbf{1}_{A(B)}.$$  

This equation is usually referred to as the isotropy condition. The most general form for the combined output of this machine, $\rho^{\text{(out)}}_{AB}(\vec{r})$, can be written as

$$\rho^{(out)}_{AB}(\vec{r}) = \frac{1}{2} \left(1 \otimes 1 + \eta_A \vec{r} \vec{\sigma} \otimes 1 + 1 \otimes \eta_B \vec{r} \vec{\sigma} + \sum_{j,k=x,y,z} t_{jk} \sigma_j \otimes \sigma_k \right),$$

where $t_{jk}$'s are real parameters and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. By using the identities

$$U(R) \sigma_x U^\dagger(R) = \cos \chi \sigma_x + \sin \chi \sigma_y,$$

$$U(R) \sigma_y U^\dagger(R) = \cos \chi \sigma_y - \sin \chi \sigma_x$$

one gets the following relations for $t'_{jk}$ (the parameters of $\rho^{\text{(out)}}_{AB}(\vec{r} R)$)

$$t'_{xx} = \cos^2 \chi \cos^2 \theta + \sin^2 \chi \cos^2 \theta - \sin \chi \cos \chi (t_{xx} + t_{yz})$$

$$t'_{xy} = \cos^2 \chi \cos \theta \sin \theta + \sin \chi \cos \chi (t_{xy} + t_{yy})$$

$$t'_{yx} = \cos \chi \cos \theta \sin \theta + \sin \chi \cos \chi (t_{yx} + t_{yy})$$

$$t'_{yy} = \cos^2 \chi \cos^2 \theta + \sin^2 \chi \cos^2 \theta - \sin \chi \cos \chi (t_{xy} + t_{yy})$$

$$t'_{yz} = \cos \chi \cos \theta \sin \theta + \sin \chi \cos \chi (t_{yx} + t_{yy})$$

$$t'_{xz} = \cos \chi \cos \theta \sin \theta + \sin \chi \cos \chi (t_{xz} + t_{zz})$$

$$t'_{zy} = \cos \chi \cos \theta \sin \theta + \sin \chi \cos \chi (t_{zy} + t_{zz})$$

$$t'_{zz} = \cos \chi \cos^2 \theta + \sin \chi \cos \chi (t_{zx} + t_{zy})$$

We introduce simpler notations for four special equatorial vectors below

$$\hat{x} \equiv (1, 0, 0) \quad | - \hat{x} \rangle \equiv (-1, 0, 0)$$

$$\hat{y} \equiv (0, 1, 0) \quad | - \hat{y} \rangle \equiv (0, -1, 0).$$

No-signaling condition [24, 25, 26, 27, 28], here, reads as

$$\rho^{(out)}_{AB}(\hat{x}) + \rho^{(out)}_{AB}(\hat{y}) = \rho^{(out)}_{AB}(| - \hat{x} \rangle \langle \hat{x}|) + \rho^{(out)}_{AB}(| - \hat{y} \rangle \langle \hat{y}|).$$

Putting $\chi = 0$ for $| \hat{x} \rangle$ results in

$$t'_{xx} = t_{xx}, t'_{xy} = t_{xy}, t'_{yx} = t_{yx}, t'_{yy} = t_{yy}, t'_{yz} = t_{yz}, t'_{zx} = t_{zx}$$

and also $\chi = \frac{\pi}{2}$ for $| \hat{y} \rangle$ results in

$$t'_{xx} = t_{yy}, t'_{xy} = -t_{yx}, t'_{yx} = -t_{xy}, t'_{yy} = t_{xx}, t'_{yz} = t_{xz}, t'_{zx} = -t_{yz}, t'_{zy} = -t_{zy}, t'_{zz} = t_{zz},$$

and finally $\chi = \frac{3\pi}{4}$ for $| - \hat{y} \rangle$ gives

$$t'_{xx} = t_{yy}, t'_{xy} = -t_{yx}, t'_{yx} = -t_{xy}, t'_{yy} = t_{xx}, t'_{yz} = t_{xz}, t'_{zx} = -t_{yz}, t'_{zy} = -t_{zy}, t'_{zz} = t_{zz}.$$

Considering the above equations and Eq. (24), the following conditions are obtained

$$t_{xx} = t_{yy}, t_{xy} = -t_{yx}, t_{yx} = -t_{xy}, t_{yy} = t_{xx}, t_{yz} = t_{xz}, t_{zx} = -t_{yz}, t_{zy} = -t_{zy}, t_{zz} = t_{zz}.$$
From positive semidefiniteness of this density matrix the following conditions [31] are found

\[
-1 + \eta_A^2 + \eta_B^2 - 2\eta_A \eta_B t_{xx} + 2t_{xx}^2 + 2t_{yy}^2 + t_{zz}^2 - 2t_{xx} t_{xx} t_{zx} + 2t_{xy} t_{yzt} + t_{zz}^2 \\
-2t_{xy} t_{zx} t_{zz} + 2t_{xx} t_{zz} + 2t_{xy} t_{zz} + t_{zz}^2 \leq 0,
\]

\[
\det(\rho_{AB}^{(out)}(\hat{\Phi})) \geq 0.
\]

To maximize the value of \(\eta_A^2 + \eta_B^2\), it is seen from Eq. (31) that one must take \(t_{xx} = t_{yy} = t_{zz} = t_{xy} = 0\), which in turn gives

\[
\eta_A^2 + \eta_B^2 \leq 1 - 2(t_{xx}^2 - \eta_A \eta_B t_{xx}).
\]

From this relation it is concluded that to maximize the value of the left hand side one should choose \(t_{xx} = \frac{\eta_A \eta_B}{2}\), hence it follows that

\[
\eta_A^2 + \eta_B^2 \leq 1 + \frac{\eta_A \eta_B}{2}.
\]

These considerations along with Eq. (32), finally give rise to

\[
(\eta_A^2 + \eta_B^2)^2 - 4(\eta_A^2 + \eta_B^2) + 3 \geq 0,
\]

and hence the following upper bound on the qualities of the clones is obtained

\[
\eta_A^2 + \eta_B^2 \leq 1.
\]

In the case of qubit cloning, Eqs. (13a) and (13b) reduce to

\[
\eta_A(\nu, \xi) = 2\nu \sqrt{1 - (\nu^2 + \xi^2)}
\]

\[
\eta_B(\nu, \xi) = 2\xi \sqrt{1 - (\nu^2 + \xi^2)}.
\]

Obviously we must assume \(\nu, \xi \geq 0\) to avoid negative shrinking factors that are unphysical. Thus, the only acceptable value of \(\nu\) which optimizes \(\eta_B\), provided that \(\eta_A = \text{const.}\), is

\[
\nu_{\text{optimal}} = \frac{\eta_A}{\sqrt{2}}.
\]

Therefore, the optimal trade-off relation, now becomes

\[
\eta_A^2 + \eta_B^2 = 1
\]

which is the equation of a unit circle in the \((\eta_A, \eta_B)\) space. This cloner is on the edge of no-signaling condition [26]. In corresponding symmetric cloner, for which \(\eta_A = \eta_B\), Eq. (39) gives

\[
F = \frac{1}{2} + \frac{1}{\sqrt{3}},
\]

which is the same as the known value [22]. Figure 4 compares the trade-off diagrams for both optimal universal and optimal phase-covariant cloners.

**B. Separability properties of the clones**

In this subsection we want to study entanglement properties of the outputs of these cloners. For the symmetric case as has been shown in [22], using the Peres-Horodecki’s positive partial transposition criterion [32, 33], only for optimal phase-covariant cloners two output clones are separable. We presently want to investigate a similar question in the context of asymmetric cloning machines. It can be obtained that in \(d=2\) partial transposition of the density matrix of the combined clones, \(\rho_{AB}^{(out)}|^{TA}\), is as follows

\[
[\rho_{AB}^{(out)}]^{TA} = \frac{1}{2} \begin{pmatrix}
\mu^2 & \mu\xi e^{-i\phi} & \mu\nu e^{i\phi} & 2\nu\xi \\
\mu\xi e^{i\phi} & \nu^2 + \xi^2 & 0 & \mu\nu e^{-i\phi} \\
\mu\nu e^{i\phi} & 0 & \nu^2 + \xi^2 & \mu\xi e^{-i\phi} \\
2\nu\xi & \mu\nu e^{-i\phi} & \mu\xi e^{i\phi} & \mu^2
\end{pmatrix}
\]

in which we have considered \(\alpha_0 = \frac{1}{\sqrt{2}}\) and \(\alpha_1 = \frac{e^{i\phi}}{\sqrt{2}}\), and the computational basis has been used. Eigenvalues of this matrix, in terms of \(\nu\) and \(\xi\) (after the cancelation of \(\mu\) by the normalization condition), are

\[
\frac{1}{4} \left(1 + 2\nu \xi \pm \sqrt{1 + 12\nu^2 \xi - 16\nu^3 \xi + 4\nu^2 \xi^2 - 16\nu \xi^3} \right),
\]

\[
\frac{1}{4} \left(1 - 2\nu \xi \pm \sqrt{1 - 12\nu^2 \xi + 16\nu^3 \xi + 4\nu^2 \xi^2 + 16\nu \xi^3} \right).
\]

Replacing the optimal values of \(\nu = \frac{\eta_A}{\sqrt{2}}\) and \(\xi = \frac{\eta_B}{\sqrt{2}}\), and using Eq. (39), the eigenvalues for the optimal asymmetric phase-covariant cloner are obtained as

\[
\{0, 0, \frac{1}{2} \left(1 - \sqrt{\eta_A^2 (1 - \eta_B^2)} \right), \frac{1}{2} \left(1 + \sqrt{\eta_A^2 (1 - \eta_B^2)} \right) \}.
\]

FIG. 4: Comparison of trade-off ellipses for universal and phase-covariant cloners in \(d=2\).
which, by considering $0 \leq \eta_A \leq 1$, are all non-negative. Therefore, for the optimal asymmetric phase-covariant cloner, like symmetric case, two output copies are separable\(^1\). By numerical analysis of Eq. (42) in other cases, we have found that except in the optimal asymmetric cloners, always at least one of eigenvalues is negative. Thus it is argued that the special class of the optimal phase-covariant cloners is unique in that they are the only cloning machines that give rise to separable output clones. Therefore, this property of the phase-covariant cloners is respected in asymmetric case, too. In the symmetric case, in which $\eta_A = \eta_B = \frac{1}{\sqrt{2}}$, Eq. (43) gives $\{0, 0, \frac{1}{2}, \frac{1}{2}\}$ which coincides to the known value.

Moreover, let us study the entanglement properties of the total output pure state, $|\psi\rangle_{ABX}$. This state clearly has three-party entanglement. Such a similar investigation for the symmetric cloners has been done in \([35]\). Also a comparison of different cloning machines, in the sense of their entanglement properties, can be found in \([36]\).

There exists two different inequivalent classes of three-party entanglement \([37]\), namely $W$- and GHZ-type (shown, respectively, by $|\psi_W\rangle$ and $|\psi_{GHZ}\rangle$) which their representatives are as follows

$$
|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),
$$

$$
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).
$$

For pure states of three qubits, $|\psi\rangle_{ABC}$, there exists a simple criterion to detect to which class an entangled state belongs, which is called 3-tangle (or shortly, tangle) $\tau_{ABC}$ \([38]\). It can be shown that $\tau(|\psi\rangle_W) = 0$, and $\tau(|\psi\rangle_{GHZ}) > 0$. A simple calculation shows that for the case of $\nu$ on an orbit of the Bloch sphere, Eq. (17), one obtains

$$
\tau_{ABX}(\nu, \theta) = 4 \sin^2 \theta \nu^2 (\frac{1}{2} - \nu^2).
$$

Figure 5 shows a plot of this entanglement measure vs. $\theta$, and $\nu$. Also in Fig. 5 tangle is shown for the special case of $x$- $y$ equatorial states (which gain the maximum tangle among all orbits). As is seen, except the two special cases $\nu = 0$ and $\frac{1}{\sqrt{2}}$, the tangle for other asymmetric phase-covariant cloners is greater than zero which indicates that the total outputs of these types of machines are of GHZ-type. The maximum value for the tangle is attained in the case of $\nu = \frac{1}{2}$ (and subsequently, $\xi = \frac{1}{2}$) which is the symmetric case. This property is reasonable in the sense that in the symmetric case two-party entanglement of the two clones is zero (clones are separable) and as well two-party entanglement between the clone $A$ and the machine $X$ is relatively small $(N(\rho_{AX}^{\text{out}})_{\text{symm}} \simeq 0.0346)$, which implies that big portion of entanglement is of three-party\(^2\) type.

\[\text{IV. CONCLUSION}\]

In this paper, we have considered a simple two-parametric family of optimal asymmetric cloners. In this type of machines if the quality of a clone (defined by its fidelity) is given then the quality of the other clone will be the highest possible value. Thus, the qualities are free to be tuned (up to a complementarity relation between the two clones). These clones saturate the so-called no-cloning inequalities, and therefore are optimal.

Next, we have introduced optimal asymmetric cloning transformation for the special class of phase-covariant (equatorial) states; optimal asymmetric phase-covariant cloners. An explicit proof, based upon the no-signaling condition, has been presented for the case of qubit cloning. The proof can easily be generalized to the phase-covariant cloners for orbital states (though explicit form of these cloning transformations has not been mentioned). As well, for the case of equatorial states, the trade-off between clones has been obtained explicitly, and entanglement properties of the clones have been investigated. It has been argued that among all inputs, only the equatorial qubits, and only in the case of an optimal phase-covariant cloner, give rise to separable clones. So, again as in the optimal symmetric phase-covariant cloners, the equatorial states (though explicit form of these cloning transformations has not been mentioned). As well, for the case of equatorial states, the trade-off between clones has been obtained explicitly, and entanglement properties of the clones have been investigated. It has been argued that among all inputs, only the equatorial qubits, and only in the case of an optimal phase-covariant cloner, give rise to separable clones. So, again as in the optimal symmetric phase-covariant cloners, the equatorial

\[\text{Footnotes}\]

1 Or, equivalently, another good measure of entanglement named negativity, defined as: $N(\rho_{AB}) = 2 \max(0, -\lambda_{\text{min}})$, where $\lambda_{\text{min}}$ is the minimal eigenvalue of $\rho_{AB}^T$.\(^3\), vanishes.

2 Following \([38]\), for pure states we have $\tau_{AB} + \tau_{AC} + \tau_{ABC} = \tau_{A(BC)}$. This is a kind of trade-off between two- and three-party entanglements in tripartite systems.
qubits are unique, and asymmetry in qualities of their clones respects this special feature. Also three-party entanglement of the total state of this cloner has been shown to be of GHZ-type, and for symmetric case the value of this entanglement is maximum.

Acknowledgments

A. T. R would like to thank V. Karimipour and P. Zanardi for discussions on quantum cloning, and also acknowledges hospitality of the I.S.I Foundation (Turin) where some part of the work was completed.

Note added.– After completion of this work, we became aware that Lamoureux and Cerf [39] have investigated optimal asymmetric phase-covariant $d$-dimensional cloning by another method.