Interactive Evolutionary Multiobjective Optimization using Dominance-based Rough Set Approach

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Abstract—We present basic ideas related to application of Dominance-based Rough Set Approach (DRSA) in interactive Evolutionary Multiobjective Optimization (EMO). In the proposed methodology, the preference information elicited by the decision maker in successive iterations consists in sorting some solutions in the current population into “relatively good” and “others”, or in comparing some pairs of solutions with respect to preference. The “if ..., then ...” decision rules are then induced from this preference information using Dominance-based Rough Set Approach (DRSA). These rules are used within EMO in order to focus on populations of solutions satisfying the preferences of the decision maker, speeding up convergence to the most preferred region of the Pareto-front. The resulting interactive schemes, corresponding to the two types of preference information, are called DRSA-EMO and DRSA-EMO-PCT, respectively. The proposed methodology permits also to take into account robust concerns in multiobjective optimization.

I. INTRODUCTION

Practical decision making usually involves consideration of multiple conflicting objectives. For example, product mix involves multiple objectives of the type: profit, time machine, sales, market share, net present value, resource consumption, and so on. As, in general, there does not exist a single solution which optimizes simultaneously all objectives, one has to search for Pareto-optimal solutions. A solution is Pareto-optimal (also called efficient or non-dominated) if there is no other feasible solution which would be at least as good on all objectives while being strictly better on at least one objective. Finding the whole set of Pareto-optimal solutions (also called Pareto-set or Pareto-front) is usually computationally hard. This is why a lot of research has been devoted to heuristic search of an approximation of the Pareto-front. Among the heuristics proposed to this end, Evolutionary Multiobjective Optimization (EMO) procedures appeared to be particularly efficient (see, e.g., [5], [6]).

The underlying reasoning behind the EMO search of an approximation of the Pareto-front is that, in the absence of any preference information, all Pareto-optimal solutions have to be considered equivalent. On the other hand, if the decision maker (DM) (alternatively called user) is involved in the multiobjective optimization process, then the preference information provided by the DM can be used to focus the search on the most preferred part of the Pareto-front. This idea stands behind Interactive Multiobjective Optimization (IMO) methods proposed long time before EMO has emerged (see, e.g., [25], [26], [32]).

Recently, it became clear that merging the IMO and EMO methodologies should be beneficial for the multiobjective optimization process [2]. Several approaches have been proposed in this context:

- [11], [8] and [33] are based on various ways to guide the search towards a user-specified reference point,
- [4] proposes the guided MOEA in which the user is allowed to specify preferences in the form of maximally acceptable trade-offs, like “one unit improvement in objective $i$ is worth at most $a_{ij}$ units in objective $j$”,
- [9] proposes an interactive decision support system called I-MODE that allows the DM to interactively focus on interesting region(s) of the Pareto-front using several tools, such as weighted sum approach, utility function based approach, Chebycheff function approach or trade-off information,
- [23] suggests a procedure which asks the user to rank a few solutions, and from this derives constraints for linear weighting of the objectives consistent with the given ordering; these constraints are used within EMO to check whether there exists a feasible linear weighting, such that solution $x$ is preferable to solution $y$,
- [28] proposes an interactive evolutionary algorithm that allows the user to provide preference information about pairs of solutions during the run, and then computes the “most compatible” weighted sum of objectives by means of linear programming; this weighted sum is used as a single substitute objective for some generations of the evolutionary algorithm,
- [24] applies preference information from pairwise comparisons of solutions for sampling sets of scalarizing functions by drawing a random weight vector for each single iteration, and uses this for selection and local search,
- [3] proposes to apply robust ordinal regression [22], [10] to take into account the whole set of additive utility functions compatible with preference comparisons of some pairs of solutions by the DM in order to guide exploration of the Pareto-front using a necessary preference relation, which holds when a solution is at least as good as another solution for all compatible utility functions.
This paper goes in this direction, by considering combination of EMO with Dominance-based Rough Set Approach (DRSA). DRSA is a methodology of reasoning about partially inconsistent and preference-ordered data (see [13], [15], [16], [29], [30], [31]), which has already been applied successfully to IMO [17].

DRSA aims at obtaining a representation of the DM’s preferences in terms of easily understandable “if ..., then ...” decision rules, on the basis of some exemplary decisions (past decisions or simulated decisions) made by the DM. The exemplary decisions can be:

1) assignments of selected alternatives to some ordered classes, such as “bad”, “medium”, “good”, or
2) specifications of some holistic preferences on selected pairs of alternatives.

In case 1), the induced decision rules are of the form:

“If on criterion $i_1$ alternative $x$ has an evaluation at least $\alpha_1$, and on criterion $i_2$ $x$ has an evaluation at least $\alpha_2$, ..., and on criterion $i_h$ $x$ has an evaluation at least $\alpha_h$, then alternative $x$ is at least medium”.

In case 2), decision rules are of the form:

“If on criterion $i_1$ alternative $x$ is at least strongly better than alternative $y$, and on criterion $i_2$ $x$ is at least weakly better than $y$, ..., and on criterion $i_h$ $x$ is at least indifferent to $y$, then alternative $x$ is comprehensively weakly preferred to $y$”.

DRSA can also take into account uncertainty in decision problems [19], [21]; then, the induced decision rules are of the form:

“If on criterion $i_1$ alternative $x$ has an evaluation at least $\alpha_1$ with probability at least $p_{i_1}$, and on criterion $i_2$ $x$ has an evaluation at least $\alpha_2$ with probability at least $p_{i_2}$, ..., and on criterion $i_h$ $x$ has an evaluation at least $\alpha_h$ with probability at least $p_{i_h}$, then alternative $x$ is at least medium”

or

“If on criterion $i_1$ alternative $x$ is at least strongly better than alternative $y$ with probability at least $p_{i_1}$, and on criterion $i_2$ $x$ is at least weakly better than $y$ with probability at least $p_{i_2}$, ..., and on criterion $i_h$ $x$ is at least indifferent to $y$ with probability at least $p_{i_h}$, then alternative $x$ is comprehensively weakly preferred to $y$”.

Application of DRSA to decision under uncertainty within interactive EMO is important because it takes into account robustness concerns in the multiobjective optimization. In fact, we proposed already two computer implementations of robust optimization methods combining DRSA and interactive EMO. These two methods are DARWIN (Dominance-based rough set Approach to handling Robust Winning solutions in IInteractive multiobjective optimization) [18] and DARWIN-PCT (DARWIN using Pairwise Comparison Tables) [20].

In this paper, we outline a general methodology of interactive EMO involving DRSA. We are proposing two general schemes, called DRSA-EMO and DRSA-EMO-PCT, with respect to which DARWIN and DARWIN-PCT are two instances of implementation. The methodology we are proposing involves application of decision rules in EMO, which are induced from easily elicited preference information by DRSA. In result, EMO is focusing the search of the Pareto-front on the most preferred region. More specifically, DRSA is used for structuring preference information obtained through interaction with the user, and then a set of decision rules representing user’s preferences is induced from this information; these rules are used to rank solutions in the current population of EMO, which has an impact on the selection and crossover.

We believe that integration of DRSA and EMO is particularly promising for two reasons:

1) The preference information required by DRSA is very basic and easy to elicit by the DM. All that the DM is asked for is to assign solutions to preference ordered classes, such as “good”, “medium” and “bad”, or compare pairs of non-dominated solutions from a current population in order to reveal whether one is preferred over the other. The preference information is provided every $k$ iterations ($k$ depends on the problem and the willingness of the user to interact with the system. In our studies, $k$ ranges from 10 to 30).

2) The decision rules are transparent and easy to interpret for the DM. The preference model supplied by decision rules is a “glass box”, while many other competitive multiple criteria decision methodologies involve preference models that are “black boxes” for the user. The “glass box” model improves the quality of the interaction and makes that the DM accepts well the resulting recommendation.

The paper is organized as follows. The next Section outlines the DRSA methodology and, more precisely, DRSA applied to ordinal classification and DRSA applied to pairwise comparisons made according to a preference relation. Then, Section III presents basic steps of DRSA to ordinal classification combined with interactive EMO, and introduces a method called DRSA-EMO. The following Section presents DRSA to pairwise comparisons combined with interactive EMO, and introduces another method called DRSA-EMO-PCT. The last Section contains conclusions.

II. DOMINANCE-BASED ROUGH SET APPROACH (DRSA)

A. Dominance-based Rough Set Approach to ordinal classification

DRSA is a methodology of multiple criteria decision analysis aiming at obtaining a representation of the DM’s preferences in terms of easily understandable “if ..., then ...” decision rules, on the basis of some exemplary decisions (past decisions or simulated decisions) given by the DM. In this Subsection, we present the DRSA to sorting problems, because in the dialogue stage of our interactive method this multiple criteria decision problem is considered. In this case, exemplary decisions are sorting examples, i.e. objects (solutions, alternatives, actions) described by a set of criteria and assigned to preference ordered classes. Criteria and the class assignment considered within DRSA correspond to the...
condition attributes and the decision attribute, respectively, in the classical Rough Set Approach [27]. For example, in multiple criteria sorting of cars, an example of decision is an assignment of a particular car evaluated on such criteria as maximum speed, acceleration, price and fuel consumption to one of three classes of overall quality: “bad”, “medium”, “good”.

Let us consider a set of criteria \( F = \{ f_1, \ldots, f_n \} \), the set of their indices \( t = \{ 1, \ldots, n \} \), and a finite universe of objects (solutions, alternatives, actions) \( U \) such that, without loss of generality, \( f_i : U \rightarrow \mathbb{R} \) for each \( i = 1, \ldots, n \), and, for all objects \( x, y \in U \), \( f_i(x) \geq f_i(y) \) means that “\( x \) is at least as good as \( y \) with respect to criterion \( i \)”, which is denoted as \( x \succeq y \). Therefore, we suppose that \( \succeq_1 \) is a complete preorder, i.e. a strongly complete and transitive binary relation, defined on \( U \) on the basis of evaluations \( f_i(\cdot) \). Note that in the context of multiobjective optimization, \( f_i(\cdot) \) corresponds to objective functions. Furthermore, we assume that there is a decision attribute \( d \) which makes a partition of \( U \) into a finite number of decision classes called sorting, \( CL = \{ CL_1, \ldots, CL_m \} \), such that each \( x \in U \) belongs to one and only one class \( CL_t \), \( t = 1, \ldots, m \). We suppose that the classes are preference ordered, i.e. for all \( r, s = 1, \ldots, m \), such that \( r > s \), the objects from \( CL_r \) are preferred to the objects from \( CL_s \). More formally, if \( \succeq_r \) is a comprehensive weak preference relation on \( U \), i.e. if for all \( x, y \in U \), \( x \succeq_r y \) reads “\( x \) is at least as good as \( y \)”, then we suppose

\[
[x \in CL_r, y \in CL_s, r > s] \Rightarrow x \succeq_r y,
\]

where \( x \succeq_r y \) means \( x \succeq y \) and not \( y \succeq x \). The above assumptions are typical for consideration of a multiple criteria sorting problem (also called ordinal classification with monotonicity constraints).

In DRSA, the explanation of the assignment of objects to preference ordered decision classes is made on the base of their evaluation with respect to a subset of criteria \( P \subseteq I \). This explanation is called approximation of decision classes with respect to \( P \). Indeed, in order to take into account the order of decision classes, in DRSA the classes are not considered one by one but, instead, unions of classes are approximated: upward union from class \( CL_t \) to class \( CL_m \), denoted by \( CL^+_{t,m} \), and downward union from class \( CL_t \) to class \( CL_1 \), denoted by \( CL^-_{t,1} \), i.e.:

\[
CL^+_{t,m} = \bigcup_{s \geq t} CL_s, \quad CL^-_{t,1} = \bigcup_{s \leq t} CL_s, \quad t = 1, \ldots, m.
\]

The statement \( x \in CL^+_{t,m} \) reads “\( x \) belongs to at least class \( CL_t \)”, while \( x \in CL^-_{t,1} \) reads “\( x \) belongs to at most class \( CL_t \)”. Let us remark that \( CL^+_{t,m} = CL^+_{m,m} = U \), \( CL^-_{m,1} = CL_1 \) and \( CL^+_{t,m} = CL_{t,m} \). Furthermore, for \( t=2, \ldots, m \), we have:

\[
CL^+_{t-1} = U - CL^+_{t-1} \quad \text{and} \quad CL^-_{t} = U - CL^-_{t-1}.
\]

In the above example concerning multiple criteria sorting of cars, the upward unions are: \( CL^+_{medium} \), that is the set of all the cars classified at least “medium” (i.e. the set of cars classified “medium” or “good”), and \( CL^+_{good} \), that is the set of all the cars classified at least “good” (i.e. the set of cars classified “good”), while the downward unions are: \( CL^-_{medium} \), that is the set of all the cars classified at most “medium” (i.e. the set of cars classified “medium” or “bad”), and \( CL^-_{bad} \), that is the set of all the cars classified at most “bad” (i.e. the set of cars classified “bad”). Notice that, formally, also \( CL^+_{bad} \) is an upward union, as well as \( CL^-_{good} \) is a downward union, however, since “bad” and “good” are extreme classes, these two unions boil down to the whole universe \( U \).

The key idea of the rough set approach is explanation (approximation) of knowledge generated by the decision attributes, by granules of knowledge generated by condition attributes.

In DRSA, where condition attributes are criteria and decision classes are preference ordered, the knowledge to be explained is the assignment of objects to upward and downward unions of classes, and the granules of knowledge are sets of objects contained in dominance cones defined in the space of evaluation criteria.

We say that \( x \) dominates \( y \) with respect to \( P \subseteq I \) (shortly, \( x \ P-dominates \ y \)), denoted by \( x P y \), if for every criterion \( i \in P \), \( f_i(x) \geq f_i(y) \). The relation of \( P \)-dominance is reflexive and transitive, that is, it is a partial preorder.

Given a set of criteria \( P \subseteq I \) and \( x, y \in U \), the granules of knowledge used for approximation in DRSA are:

- a set of objects dominating \( x \), called \( P \)-dominating set, \( D^+_P(x) = \{ y \in U : x P y \} \),
- a set of objects dominated by \( x \), called \( P \)-dominated set, \( D^-_P(x) = \{ y \in U : x P y \} \).

Let us recall that the dominance principle requires that an object \( x \) dominating object \( y \) with respect to considered criteria (i.e. \( x \) having evaluations at least as good as \( y \) on all considered criteria) should also dominate \( y \) on the decision (i.e. \( x \) should be assigned to at least as good decision class as \( y \)).

The \( P \)-lower approximation of \( CL^+_{t,m} \), denoted by \( P(CL^+_{t,m}) \), and the \( P \)-upper approximation of \( CL^+_{t,m} \), denoted by \( \overline{P}(CL^+_{t,m}) \), are defined as follows \( (t=1, \ldots, m) \):

\[
\begin{align*}
P(CL^+_{t,m}) &= \{ x \in U : D^+_P(x) \subseteq CL^+_{t,m} \}, \\
\overline{P}(CL^+_{t,m}) &= \{ x \in U : D^-_P(x) \cap CL^+_{t,m} \neq \emptyset \}.
\end{align*}
\]

Analogously, one can define the \( P \)-lower approximation and the \( P \)-upper approximation of \( CL^-_{t,1} \) as follows \( (t=1, \ldots, m) \):

\[
\begin{align*}
P(CL^-_{t,1}) &= \{ x \in U : D^-_P(x) \subseteq CL^-_{t,1} \}, \\
\overline{P}(CL^-_{t,1}) &= \{ x \in U : D^+_P(x) \cap CL^-_{t,1} \neq \emptyset \}.
\end{align*}
\]

The \( P \)-boundaries of \( CL^+_{t,m} \) and \( CL^-_{t,1} \), denoted by \( BnP(CL^+_{t,m}) \) and \( BnP(CL^-_{t,1}) \), respectively, are defined as follows \( (t=1, \ldots, m) \):

\[
\begin{align*}
BnP(CL^+_{t,m}) &= \overline{P}(CL^+_{t,m}) - P(CL^+_{t,m}), \\
BnP(CL^-_{t,1}) &= \overline{P}(CL^-_{t,1}) - P(CL^-_{t,1}).
\end{align*}
\]
The dominance-based rough approximations of upward and downward unions of decision classes can serve to induce a generalized description of sorting decisions in terms of “if ..., then ...” decision rules. For a given upward or downward union of classes, $C_{I}^{+}$ or $C_{I}^{-}$, the decision rules induced under a hypothesis that objects belonging to $P(C_{I}^{+})$ or $P(C_{I}^{-})$ are positive examples (that is objects that have to be matched by the induced decision rules), and all the others are negative (that is objects that have to be not matched by the induced decision rules), suggest a certain assignment to “class $C_{t}$ or better”, or to “class $C_{s}$ or worse”, respectively. On the other hand, the decision rules induced under a hypothesis that objects belonging to $P(C_{I}^{+})$ or $P(C_{I}^{-})$ are positive examples, and all the others are negative, suggest a possible assignment to “class $C_{t}$ or better”, or to “class $C_{s}$ or worse”, respectively. Finally, the decision rules induced under a hypothesis that objects belonging to the intersection $P(C_{I}^{+}) \cap P(C_{I}^{-})$ are positive examples, and all the others are negative, suggest an assignment to some classes between $C_{t}$ and $C_{s}$, if they satisfy certain conditions.

Given the preference information in terms of sorting examples, it is meaningful to consider the following five types of decision rules:

1. certain DS2-decision rules, providing lower profiles (i.e. sets of minimal values for considered criteria) of objects belonging to $P(C_{I}^{+})$, $P = \{i_{1}, \ldots , i_{p}\} \subseteq I$:
   - if $f_{i_{1}}(x) \geq r_{i_{1}}$ and ... and $f_{i_{p}}(x) \geq r_{i_{p}}$,
     then $x \in C_{I_{i_{1}}}^{+}$,
   - $t = 2, \ldots , m, r_{i_{1}}, \ldots , r_{i_{p}} \in \mathbb{R}$;

2. possible DS2-decision rules, providing upper profiles of objects belonging to $P(C_{I}^{-})$, $P = \{i_{1}, \ldots , i_{p}\} \subseteq I$:
   - if $f_{i_{1}}(x) \geq r_{i_{1}}$ and ... and $f_{i_{p}}(x) \geq r_{i_{p}}$,
     then $x$ possibly belongs to $C_{I_{i_{1}}}^{-}$,
   - $t = 2, \ldots , m, r_{i_{1}}, \ldots , r_{i_{p}} \in \mathbb{R}$;

3. certain DS2-decision rules, providing upper profiles (i.e. sets of maximal values for considered criteria) of objects belonging to $P(C_{I}^{-})$, $P = \{i_{1}, \ldots , i_{p}\} \subseteq I$:
   - if $f_{i_{1}}(x) \leq r_{i_{1}}$ and ... and $f_{i_{p}}(x) \leq r_{i_{p}}$,
     then $x \in C_{I_{i_{1}}}^{-}$,
   - $t = 1, \ldots , m - 1, r_{i_{1}}, \ldots , r_{i_{p}} \in \mathbb{R}$;

4. possible DS2-decision rules, providing upper profiles of objects belonging to $P(C_{I}^{-})$, $P = \{i_{1}, \ldots , i_{p}\} \subseteq I$:
   - if $f_{i_{1}}(x) \leq r_{i_{1}}$ and ... and $f_{i_{p}}(x) \leq r_{i_{p}}$,
     then $x$ possibly belongs to $C_{I_{i_{1}}}^{-}$,
   - $t = 1, \ldots , m - 1, r_{i_{1}}, \ldots , r_{i_{p}} \in \mathbb{R}$;

5. approximate DS2-decision rules, providing simultaneously lower and upper profiles of objects belonging to $C_{I_{s}} \cup C_{I_{s+1}} \cup \ldots \cup C_{I_{t}}$, without possibility of discerning to which class:
   - if $f_{i_{1}}(x) \geq r_{i_{1}}$ and ... and $f_{i_{k}}(x) \geq r_{i_{k}}$ and $f_{i_{k+1}}(x) \leq r_{i_{k+1}}$ and ... and $f_{i_{p}}(x) \leq r_{i_{p}}$,
     then $x \in C_{I_{s}} \cup C_{I_{s+1}} \cup \ldots \cup C_{I_{t}}$,
   - $\{i_{1}, \ldots , i_{p}\} \subseteq I, s, t \in \{1, \ldots , m\}, s < t, r_{i_{1}}, \ldots , r_{i_{p}} \in \mathbb{R}$.

B. DRSA applied to Pairwise Comparison Tables (PCT)

DRSA can also be used to pairwise comparisons made according to a preference relation $\succeq$ on $A$ and a negative weak preference relation $x \succeq y$ on $A$ such that, for a pair of objects $(x, y) \in A \times A$, $x \succeq y$ means that $x$ is at least as good as $y$ and $x \succeq y$ means that it is not true that $x$ is at least as good as $y$. The only assumptions with respect to (wrt) these relations are that $\succeq$ is reflexive and $\succeq$ is irreflexive, and they are incompatible in the sense that for all $x, y \in A$ it is not possible that $x \succeq y$ and $x \succeq y$.

For each pair of reference objects $(x, y) \in A \times A$, the DM can select one of the three following possibilities:

1) Object $x$ is as good as $y$, i.e., $x \succeq y$.
2) Object $x$ is not as good as $y$, i.e., $x \succeq y$.
3) The two objects are incomparable at the present stage, in the sense that neither $x \succeq y$ nor $x \succeq y$ can be asserted.

Let $\succeq \cup \succeq = \mathcal{B}$, with $\text{card}(\mathcal{B}) = u$. We also suppose that objects from $A$ are described by a finite set of criteria $C = \{g_{1}, \ldots , g_{u}\}$. Without loss of generality, for each $g_{i} \in C$ we suppose that $g_{i} : A \rightarrow R$, such that, for each $x, y \in A$, $g_{i}(x) \geq g_{i}(y)$ means that $x$ is at least as good as $y$ wrt criterion $g_{i}$, which is denoted by $x \succeq_{i} y$. For each criterion $g_{i} \in C$, we also suppose that there exists a quaternary relation $\succeq_{i}^{\ast}$ defined on $A$, such that, for each $x, y, w, z \in A$, $(x, y) \succeq_{i}^{\ast} (w, z)$ means that, wrt $g_{i}$, $x$ is preferred to $y$ at least as strongly as $w$ is preferred to $z$. We assume that for each $g_{i} \in C$, the quaternary relation $\succeq_{i}^{\ast}$ is monotonict wrt to evaluations on criterion $g_{i}$, such that, for all $x, y, w, z \in A$,

$$g_{i}(x) \geq g_{i}(w) \text{ and } g_{i}(y) \leq g_{i}(z) \Rightarrow (x, y) \succeq_{i}^{\ast} (w, z).$$

We shall denote by $\succeq_{i}^{\ast}$ and $\succeq_{i}^{*}$ the asymmetric and the symmetric part of $\succeq_{i}^{\ast}$, respectively, i.e., $(x, y) \succeq_{i}^{\ast} (w, z)$ if $(x, y) \succeq_{i}^{\ast} (w, z)$ and not $(w, z) \succeq_{i}^{\ast} (x, y)$, and $(x, y) \succeq_{i}^{\ast} (w, z)$ if $(x, y) \succeq_{i}^{\ast} (w, z)$ and $(w, z) \succeq_{i}^{\ast} (x, y)$. The quaternary relation $\succeq_{i}^{\ast}$, $g_{i} \in C$, is supposed to be a complete preorder on $A \times A$. For each $(x, y) \in A \times A$, $C_{i}(x, y) = \{(w, z) \in A \times A : (w, z) \succeq_{i}^{\ast} (x, y)\}$ is the equivalence class of $(x, y)$ wrt $\succeq_{i}^{\ast}$. Intuitively, for each $(x, y), (w, z) \in A \times A$, $(w, z) \in C_{i}(x, y)$ means that $w$ is preferred to $z$ with the same strength as $x$ is preferred to $y$. We suppose also that, for each $x, y \in A$ and $g_{i} \in C$, $(x, x) \sim_{i}^{\ast} (y, y)$ and, consequently, $(y, y) \in C_{i}(x, y)$. Assuming that they are finite, we denote the equivalence classes of $\sim_{i}^{\ast}$ by $\sim_{i, 0}^{\ast}, \sim_{i, 1}^{\ast}, \ldots , \sim_{i, 0}^{\ast}, \sim_{i, 1}^{\ast}, \ldots , \sim_{i, 0}^{\ast}, \ldots , \sim_{i, 0}^{\ast}$, such that
• for all \(x, y, w, z \in A\), \(x \geq^h_i y\), \(w \geq^k_i z\), and \(h \geq k\) implies
\[(x, y) \geq^*_i (w, z),\]
• for all \(x \in A\), \(x \geq^0_i x\).

We call strength of preference of \(x\) over \(y\) the equivalence class of \(\geq^*_i\) to which pair \((x, y)\) belongs. For each \(g_i \in C\), we denote by \(H_i\) the indices of the equivalence classes of \(\geq^*_i\), i.e.
\[H_i = \{\alpha_i, \alpha_i - 1, \ldots, -1, 0, 1, \ldots, \beta_i - 1, \beta_i\}.\]

Therefore, there exists a function \(f : A \times A \rightarrow H_i\), such that, for all \(x, y \in A\), \(x \geq^{f(x,y)}_{i \in [1, \ldots, n]} y\), i.e., for all \(x, y \in A\) and \(g_i \in C\), function \(f\) gives the strength of preference of \(x\) over \(y\) wrt \(g_i\). Taking into account the dependence of \(\geq^*_i\) on evaluations by criterion \(g_i \in C\), there also exists a function \(f' : R \times R \times C \rightarrow H_i\), such that \((f(x, y), g_i) = f'(g_i(x), g_i(y), g_i)\) and, consequently, \(x P CT\) is defined as follows:
\[x P CT \Leftrightarrow (f(x, y), g_i), (g_i(x), g_i(y), g_i)\]  
Due to monotonicity of \(\geq^*_i\) wrt to evaluations on \(g_i\), we have that \(f'(g_i(x), g_i(y), g_i)\) is non-decreasing wrt \(g_i(x)\) and non-increasing wrt \(g_i(y)\). Moreover, for each \(x \in A\), \(f'(g_i(x), g_i(y), g_i) = 0\).

An \(u \times (n + 1)\) Pairwise Comparison Table (PCT) is then built up on the basis of this information. The first \(n\) columns correspond to the criteria from set \(C\), while the \(u\) rows correspond to the pairs from \(B\), such that, if the DM judges that two objects are incomparable, then the corresponding pair does not appear in PCT. The last, i.e. the \((n + 1)\)-th, column represents the comprehensive binary preference relation \(\geq\) or \(\geq^c\). For each pair \((x, y)\) in \(B\), and for each criterion \(g_i \in C\), the respective strength of preference \(f'(g_i(x), g_i(y), g_i)\) is put in the corresponding column.

In terms of rough set theory, the pairwise comparison table is defined as a data table \(PCT = \{B, C \cup \{d\}, H_C \cup \{\geq, \geq^c\}, f\}\), where \(B \subseteq A \times A\) is a non-empty set of exemplary pairwise comparisons of reference objects, \(H_C = \{g_i \in C\}, D\) is a decision corresponding to the comprehensive pairwise comparison resulting in \(\geq\) or \(\geq^c\), and \(f : B \times (C \cup \{d\}) \rightarrow H_C \cup \{\geq, \geq^c\}\) is a total function, such that \(f(x, y, g_i) \in H_i\) for every \((x, y) \in B\) and for each \(g_i \in C\), and \(f(x, y, d) \in \{\geq, \geq^c\}\) for every \((x, y) \in B\). Thus, binary relations \(\geq\) and \(\geq^c\) induce a partition of \(B\). In fact, PCT can be seen as a decision table, since the set of considered criteria \(C\) and the decision \(D\) are distinguished.

On the basis of preference relations \(\geq^h, h \in H_i, g_i \in C\), upward cumulated preference relations \(\geq^h\), and downward cumulated preference relations \(\leq^h\), can be defined as follows: for all \(x, y \in A\),
\[x \geq^h_i y \Rightarrow x \geq^k_i y \text{ with } k \geq h,\]
\[x \geq^h_i y \Rightarrow x \geq^k_i y \text{ with } k \leq h.\]

Given \(P \subseteq C\) (\(P \neq \emptyset\)), \((x, y), (w, z) \in A \times A\), the pair of objects \((x, y)\) is said to dominate \((w, z)\) wrt criteria from \(P\) (denoted by \((x, y)D_P(w, z)\)), if \(x\) is preferred to \(y\) at least as strongly as \(w\) is preferred to \(z\) wrt each \(g_i \in P\), i.e.,
\[xD_P y \Leftrightarrow (x, y) \geq^*_i (w, z) \text{ for all } g_i \in P,\]
or, equivalently,
\[xD_P y \Leftrightarrow (f(x, y), g_i) \geq (f(w, z), g_i) \text{ for all } g_i \in P.\]

Since \(\geq^*_i\) is a complete preorder for each \(g_i \in C\), the intersection of complete preorders is a partial preorder, and \(D_P = \bigcap_{g_i \in P} \geq^*_i\), \(P \subseteq C\), then the dominance relation \(D_P\) is a partial preorder on \(A \times A\).

Let \(R \subseteq P \subseteq C\) and \((x, y), (w, z) \in A \times A\); then the following implication holds:
\[(x, y)D_P (w, z) \Rightarrow (x, y)D_R (w, z).\]

Given \(P \subseteq C\) and \((x, y) \in B\), the P-dominating set, denoted by \(D^+_P (x, y)\), and the P-dominated set, denoted by \(D^-_P (x, y)\), are defined as follows:
\[D^+_P (x, y) = \{(w, z) \in B : (w, z)D_P (x, y)\},\]
\[D^-_P (x, y) = \{(w, z) \in B : (x, y)D_P (w, z)\}.\]

The P-dominating sets and the P-dominated sets are granules of knowledge that can be used to express P-lower and P-upper approximations of the comprehensive weak preference relations \(\geq\) and \(\geq^c\), respectively:
\[P^c = \{(x, y) \in B : (x, y)D^+_P (x, y)\},\]
\[P^c = \{(x, y) \in B : (x, y)D^-_P (x, y)\}.

The P-boundaries (P-doubtful regions) of \(\geq\) and \(\geq^c\) are defined as
\[B_{nP} = \{x \in B : x \geq y_{1, p}\},\]
\[B_{nP} = \{x \in B : x \geq^c y_{1, p}\}.

In fact, we have that \(B_{nP} = B_{nP} \cap C\).

Using the rough approximations of \(\geq\) and \(\geq^c\), it is possible to induce a generalized description of the preference information contained in PCT in terms of suitable decision rules, having the following syntax:

1. certain \(D_{\geq}\)-decision rules, supported by pairs of objects from the P-lower approximation of \(\geq\) only:
\[x \geq y_{1, i, p} \Leftrightarrow x \geq y_{1, i, p},\]
\[P = \{y_{1, i, p} \in C\} \text{ and } (h_{1, i, p})(x) \in H_{1, i} \times \ldots \times H_{i, p}\].

2. certain \(D_{\geq^c}\)-decision rules, supported by pairs of objects from the P-lower approximation of \(\geq^c\) only:
\[x \geq^c y_{1, i, p} \Leftrightarrow x \geq^c y_{1, i, p},\]
\[P = \{y_{1, i, p} \in C\} \text{ and } (h_{1, i, p})(x) \in H_{1, i} \times \ldots \times H_{i, p}\].

3. approximate \(D_{\geq^c}\)-decision rules, supported by pairs of objects from the P-boundary of \(\geq\) and \(\geq^c\) only:
\[x \geq y_{1, i, p} \Leftrightarrow x \geq y_{1, i, p},\]
\[x \geq^c y_{1, i, p} \Leftrightarrow x \geq^c y_{1, i, p},\]
\[P = \{y_{1, i, p} \in C\} \text{ and } (h_{1, i, p})(x) \in H_{1, i} \times \ldots \times H_{i, p}.\]
III. DRSA-EMO: DRSA APPLIED TO INTERACTIVE EMO

In this section, we propose a new interactive EMO scheme involving DRSA, and called DRSA-EMO. The method consists of a sequence of steps alternating calculation and elicitation of DM’s preferences. In the calculation stage, a population of feasible solutions is generated. This population of solutions is evaluated by multiple objective functions. The DM is asked to sort the solutions according to his/her preferences into “relatively good” and “others”. This information is then processed by DRSA producing a set of “if . . . then . . . ” decision rules representing DM’s preferences. Then, an EMO stage starts with generation of a new population of feasible solutions. The solutions from the new population are evaluated again by the multiple objective function. The “if . . . then . . . ” decision rules induced in the previous stage are then matched to the new population. In result of this rule matching, the solutions from the new population are ranked from the best to the worst. This is a starting point for selection and crossover of parent solutions, followed by a possible mutation of the offspring solutions. A half of the population of parents and all the offsprings form then a new population of solutions for which a new iteration of EMO starts. The process is iterated until the termination condition of EMO is satisfied. Then, the DM evaluates again the solutions from the last population and either the method stops because the most satisfactory solution was found, or a new EMO stage is launched with DRSA decision rules induced from DM’s sorting of solutions from the last population into “relatively good” and “others”.

DRSA-EMO is composed of two embedded loops: the exterior interactive loop, and the interior evolutionary loop. These loops are described in the next subsections.

A. The exterior interactive loop of DRSA-EMO

Consider the following MultiObjective Optimization (MOO) problem:

\[
\begin{align*}
\max & \quad [f_1(x), \ldots , f_n(x)] \\
\text{subject to:} & \\
& \quad g_1(x) \leq b_1, \\
& \quad \ldots \\
& \quad g_\omega(x) \leq b_\omega,
\end{align*}
\]

where \( x = [x_1, \ldots , x_k] \) is a vector of decision variables, called solution, \( f_j(x), j = 1, \ldots , n \), are real-valued objective functions, \( g_i(x), i = 1, \ldots , \omega \), are real-valued functions of the constraints, and \( b_i, i = 1, \ldots , \omega \), are right-hand sides of the constraints.

The exterior interactive loop of DRSA-EMO is composed of the following steps.

\textbf{Step 1.} Generate a set of feasible solutions \( X \) to the MOO problem, using a Monte Carlo method.

\textbf{Step 2.} Evaluate each solution \( x \in X \) in terms of considered objective functions \( [f_1(x), \ldots , f_n(x)] \).

\textbf{Step 3.} Present to the DM the solutions from \( X \) in terms of values of objective functions.

\textbf{Step 4.} If the DM finds a satisfactory solution in \( X \), then STOP, otherwise go to \textbf{Step 5}.

\textbf{Step 5.} Ask the DM to indicate a subset of “relatively good” solutions in set \( X \)

\textbf{Step 6.} Apply DRSA to the current set \( X \) of solutions sorted into “relatively good” and “others”, in order to induce a set of decision rules with the following syntax: “if \( f_{j_1}(x) \geq \alpha_{j_1} \) and . . . and \( f_{j}\_e(x) \geq \alpha_{j}\_e \), then solution \( x \) is relatively good”. The decision rules represent DM’s preferences on the set of solutions \( X \).

\textbf{Step 7.} Activate an EMO procedure guided by DRSA decision rules [Steps a to k of the interior loop].

B. The interior evolutionary loop of DRSA-EMO

The interior loop of DRSA-EMO is an evolutionary search procedure guided by DRSA decision rules obtained in \textbf{Step 6} of the exterior loop.

\textbf{Step a.} Generate a new set of feasible solutions \( X \) to the MOO problem, using a Monte Carlo method.

\textbf{Step b.} Evaluate each solution \( x \in X \) in terms of considered objective functions \( [f_1(x), \ldots , f_n(x)] \).

\textbf{Step c.} If termination condition is fulfilled, then show the solutions to the DM, otherwise go to \textbf{Step e}.

\textbf{Step d.} If the DM finds in the current set \( X \) a satisfactory solution, then STOP, otherwise go to \textbf{Step e}.

\textbf{Step e.} Compute a primary score of each solution \( x \in X \), based on the number of DRSA rules matching \( x \).

\textbf{Step f.} Compute a secondary score of each solution \( x \in X \), based on the crowding distance of \( x \) from other solutions in \( X \).

\textbf{Step g.} Rank solutions \( x \in X \) lexicographically, using the primary and the secondary score.

\textbf{Step h.} Make Monte Carlo selection of parents, taking into account the ranking of solutions obtained in \textbf{Step g}.

\textbf{Step i.} Recombine parents to get offsprings.

\textbf{Step j.} Mutate offsprings.

\textbf{Step k.} Update the set of solutions \( X \) by putting in \( X \) a half of best ranked parents and all offsprings. Go back to \textbf{Step b}.

In \textbf{Step e}, the primary score is calculated as follows. Let \( \text{Rule} \) be a set of DRSA rules obtained in \textbf{Step 6} of the exterior loop. Then, \( \text{Rule}(x) \subseteq \text{Rule} \) is a subset of rules \( \text{Rule}_h \) matched by solution \( x \in X \):

\[
\text{Rule}(x) = \{ \text{Rule}_h \in \text{Rule} : \text{Rule}_h \text{ is matched by solution } x \}.
\]
For each rule \( h \in \text{Rule} \), the set of solutions matching it is defined as:
\[
X(\text{rule}_h) = \{ x \in X : x \text{ is matching rule}_h \}.
\]
Then, each rule \( h \in \text{Rule} \) gets a weight related to the number of times it is matched by a solution:
\[
w(\text{rule}_h) = (1 - \delta)^{\text{card}(X(\text{rule}_h))},
\]
where \( \delta \) is a decay of rule weight, e.g., \( \delta = 0.1 \). The above formula gives higher weights to rules matching less solutions – this permits to maintain diversity with respect to rules.

The primary score of solution \( x \in X \) is then defined as:
\[
\text{Score}(x) = \sum_{\text{rule}_h \in \text{Rule}(x)} w(\text{rule}_h).
\]

In Step 1, the secondary score of each solution \( x \in X \) is calculated in the same way as the crowding distance in the NSGA-II method [7], i.e., it is defined as the sum of distances between a solution’s neighbors on either side in each dimension of the objective space. Individuals with a large crowding distance are preferred, as they are in a less crowded region of the objective space, and favoring them aims at preserving diversity in the population.

In Step 2, the probability of selecting solution \( x \in X \) as a parent is:
\[
Pr(x) = \left( \frac{\text{card}(X) - \text{rank}(x) + 1}{\text{card}(X)} \right)^\gamma \left( \frac{\text{card}(X) - \text{rank}(x)}{\text{card}(X)} \right)^\gamma,
\]
where \( \text{rank}(x) \) is a rank of solution \( x \) in the ranking, and \( \gamma \geq 1 \) is a coefficient of elitism, e.g., \( \gamma = 2 \). When \( \gamma \) is increasing, then the probability of choosing solutions with a higher rank is increasing.

If solutions \( x = [x_i], y = [y_i] \in X \) are selected parents, then the offspring (child) solution \( z \) is obtained in Step j from recombination of \( x \) and \( y \), e.g., as:
\[
z_i = \lambda_i x_i + (1 - \lambda_i) y_i,
\]
where multipliers \( \lambda_i \in [0, 1] \) are chosen randomly.

The probability of mutation of an offspring \( z \) in Step j of iteration \( t \) is:
\[
Pr(t) = \epsilon(1 - \omega)^{t-1},
\]
where \( \epsilon \) is the initial mutation probability, and \( \omega \) is the decay rate of the mutation probability, e.g., \( \epsilon = 0.5 \), and \( \omega = 0.1 \). One can see, that this probability is decreasing in successive iterations. The mutation picks randomly one variable \( x_i, i = 1, \ldots, n \), and replaces it with another value randomly generated within the range of variation of \( x_i \).

The termination condition considered in Step c can be, e.g., a fixed number of iterations.

IV. DRSA-EMO-PCT: DRSA APPLIED TO INTERACTIVE EMO USING PAIRWISE COMPARISON TABLES

In this section, we propose a new EMO method based on DRSA approximation of a preference relation called DRSA-EMO-PCT: Dominance-based Rough Set Approach to Evolutionary Multiobjective Optimization using Pairwise Comparison Tables. More precisely, comparing to DRSA-EMO, we propose to change the preference information required from the DM and, consequently, to change its use in the procedure. Instead of asking the DM which solutions are for him/her “relatively good”, we ask the DM to compare some solutions pairwise. From this preference information some decision rules are inferred using DRSA. They are of the form:

“If \( f_{j_1}(x) - f_{j_1}(y) \geq \alpha_{j_1} \) and ... and \( f_{j_p}(x) - f_{j_p}(y) \geq \alpha_{j_p} \),
then solution \( x \) is at least as good as solution \( y \)”.

These decision rules are then used to define a comprehensive preference relation in the current population of solutions. For example, denoting by \( rule(x \succ y) \) and \( rule(y \succ x) \) the number of rules for which \( x \succ y \) and \( y \succ x \), respectively, solution \( x \) is preferred to solution \( y \) if \( NFS(x) > NFS(y) \), with \( NFS(z) = \sum_{w \in \text{rule}(z, w) - \text{rule}(w, z)} \) being the Net Flow Score of solution \( z \). This preference relation is used within NSGA-II [7] instead of the dominance relation to define consecutive fronts in the population.

With respect to the crowding distance used in NSGA-II, it is replaced by a diversity measure which avoids normalization of the values of objective functions. This diversity measure gives the distance between solution \( x \) and solution \( y \), calculated as:
\[
dist_{rule}(x, y) = rule(x \succ y) + rule(y \succ x),
\]
where \( rule(x \succ y) \) is the number of rules for which \( x \succ y \), and \( rule(y \succ x) \) is the number of rules for which \( y \succ x \).

\begin{algorithm}
\caption{DRSA-EMO-PCT}
\begin{algorithmic}
\State Generate initial population of feasible solutions randomly
\State Elicit user’s preferences \{Present to the user some pairs of solutions from the population and ask for a preference comparison\}
\State Determine primary ranking taking into account preferences between solutions obtained using decision rules \{Will replace dominance ranking in NSGA-II\}
\State Determine secondary ranking \{Order solutions within a preference front, using the diversity measure \( \text{dist}_{rule}(x, y) \)\}
\Repeat
\State Mating selection and offspring generation
\If {Time to ask DM}
\State Elicit user’s preferences
\EndIf
\State Determine primary ranking
\State Determine secondary ranking
\State Environmental selection
\Until{Stopping criterion met}
\State Return all preferred solutions according to primary ranking
\end{algorithmic}
\end{algorithm}

V. CONCLUSIONS

DRSA-EMO and DSA-EMO-PCT are interactive EMO schemes involving preferences of the DM represented by
“if . . . , then . . . ” decision rules induced form preference information by Dominance-based Rough Set Approach (DRSA). As proved in [16], [29], the set of “if . . . , then . . . ” decision rules is the most general and the most comprehensible preference (aggregation) model.

In DRSA-EMO and DRSA-EMO-PCT, the DM gives preference information by answering easy questions, and obtains transparent feedback in a learning oriented perspective (see [1]).

DRSA-EMO and DRSA-EMO-PCT can also take into account robustness concerns because DRSA can handle a plurality of scenarios in case of decision under uncertainty [19], [21]. While DRSA-EMO and DRSA-EMO-PCT are considered as two general methodological frameworks for interactive EMO methods, they have already two first computer implementations called DARWIN [18] and DARWIN-PCT [20], respectively. DRSA-EMO and DRSA-EMO-PCT can further be customized to a large variety of Operational Research problems, from location and routing to scheduling and supply chain management.

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