Decision Support

Implementations of PACMAN

Silvia Angilella *, Alfio Giarlotta

Department of Economics and Quantitative Methods, Faculty of Economics, University of Catania, Corso Italia 55, I-95129 Catania, Italy

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Abstract

Passive and Active Compensability Multicriteria ANalysis (PACMAN) is a multiple criteria methodology based on a decision maker oriented notion of compensation, called compensability. An important feature of PACMAN is a possible asymmetry of the connected decision procedure, since compensability is determined for each ordered pair of criteria, distinguishing the compensating criterion from the compensated one. Here we give a notion of implementation of PACMAN, which allows a concrete modelization of a multiple criteria decision problem. We study regular implementations of PACMAN and their monotonicity properties. We also examine several regular implementations, which satisfy some additional properties. Particular emphasis is given to a regular implementation of PACMAN that produces the lexicographic ordering.

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1. Introduction

In Multiple Criteria Decision Aid (MCDA), several procedures have been proposed to compare alternatives or criteria pairwise, that is, taking into account two of them a time. The principal idea underlying a pairwise approach is that a human mind is able to process only few conceptual units at time, as it is confirmed by some psychological studies [2]. Therefore, multiple criteria procedures that require the Decision Maker (DM) only to deal with pairwise comparisons appear to be more reliable as for what concerns adherence to her/his scheme of preferences. Methodologies based on a pairwise approach can be divided into two groups, depending on the goal that they wish to achieve.

(i) The objective of the first class of methodologies is to obtain a numerical evaluation of all feasible alternatives, which are successively ranked in a total preorder. To this aim the DM compares pairs of criteria and/or alternatives in terms of their difference of attractiveness. Then the DM gives a qualitative measure of this difference, on the basis of some semantical categories predefined by the adopted methodology. This allows the analyst to obtain a set of weights for the given criteria. Finally, a weighted sum utility model is used to evaluate each alternative. Examples of methodologies that belong to this group are the analytic hierarchy process [22,23] and MACBETH [1].

(ii) The methodologies belonging to the second class have the goal of establishing a fundamental system of preferences [20] within the set of alternatives. These methodologies rely on the so-called Pairwise Criterion Comparison Approach (PCCA) [16]. The PCCA is a procedure whose first stage consists of a comparison of all feasible alternatives with

* Corresponding author. Tel.: +39 0957537737; fax: +39 0957537510.
E-mail addresses: angisil@unict.it (S. Angilella), giarlott@unict.it (A. Giarlotta).

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This paper is devoted to a careful analysis of some theoretical and procedural aspects of a methodology that belongs to group (ii): Passive and Active Compensability Multicriteria Analysis (PACMAN). PACMAN is a multiple criteria procedure based on a DM-oriented notion of compensation, called compensability. Compensability is a piece of intercriteria preference information, \textit{ad hoc} furnished by the DM; it expresses “the possibility that an advantage on one criterion can offset a disadvantage on another one”. It is apparent that the notion of compensability among criteria is closely related to that of weight; for example, if a small difference of evaluations on a criterion can compensate even a large difference of opposite sign on another criterion, it seems reasonable to consider the former criterion more important than the latter. Nevertheless, there are several conceptual differences between the notion of \textit{active and passive compensatory power} of a criterion on one hand and that of \textit{importance/weight} on the other hand: see [6,7] for a careful discussion of this topic.

The PACMAN decision procedure is composed of three successive phases:

(I) \textit{compensability analysis}, which is the procedure aimed at evaluating intercriteria relations via the construction of compensatory functions for each ordered pair of criteria; in this analysis, we distinguish the compensating (or active) criterion from the compensated (or passive) one;

(II) pairwise comparison of alternatives via the construction of \textit{active binary indices} and \textit{passive binary indices} at several levels of aggregation; these indices, which express the degree of active and passive preference of an alternative over another one, are built by directly taking into account the results of compensability analysis;

(III) determination, for each couple of alternatives, of two relations of \textit{compensated preference}, obtained by comparing active and passive global indices; this relation of compensated preference is then used to establish a fundamental system of preferences on the set of alternatives.

In this paper, we examine aspects of PACMAN that are connected to the applicability of the procedure to concrete multi-criteria decision problems. Our goal is to balance an \textit{abstract flexibility} of PACMAN with a \textit{concrete implementability} of the methodology. The abstract flexibility of PACMAN is related to the fact that this multi-criteria methodology provides the analyst with a very general decisional framework, which needs to be adapted to the DM’s scheme of preferences. In fact, PACMAN allows the DM to choose among several categories of approaches to compensation. More specifically, the DM has to decide which category of implementation better fits his/her scheme of preferences, and then determine the form and value of the functions and parameters connected to the specific implementation he has chosen. Among these parameters there are the so-called auxiliary functions (ordinal, cardinal or mixed), aggregation functions, preference thresholds, etc. One could argue that this flexibility of the methodology gives too much freedom to the DM and might even generate confusion in his/her mind. On the contrary, we believe that one of the tasks of the analyst is to take advantage of this flexibility and propose to the DM the category of approach that better fits his/her scheme of preferences.

In this paper we exhibit several types of implementations of PACMAN, in order to address and somehow compensate the apparently excessive flexibility of our methodology. In fact, we exhibit a wide selection of decisional settings, which might suit several categories of DMs. To this aim, in this paper we introduce the notion of a \textit{regular implementation} of PACMAN, with the purpose of narrowing the choice of the decisional settings to some predefined implementations. An obvious consequence of this choice is an easier comprehension of the whole decisional process by the DM. Furthermore, we show that some classical models (such as the lexicographic, semi-lexicographic, etc.) can be obtained by suitably choosing the parameters of the implementation.

In this paper, we link the abstract flexibility of PACMAN to its implementability by (i) introducing some axioms that each implementation should satisfy, and (ii) giving a limited yet representative range of choices for the functions and parameters that are used in the modelization.

The paper is organized as follows. Sections 2–4 are devoted to a brief analysis of the three phases of PACMAN. In Section 5 we introduce the notion of implementation of PACMAN and the axioms of regularity. In Section 6 we study regular implementations and their monotonicity properties related to the notion of dominance. Section 7 deals with some regular implementations of PACMAN that satisfy additional properties. In Section 8 we show that the lexicographic model is representable by a suitable implementation of PACMAN. Section 9 groups conclusions and future direction of research. In the Appendix we give several numerical examples that illustrate lexicographic implementations of PACMAN.
2. Phase I: Compensability analysis

Compensability analysis is the procedure that allows the Decision Aider (DA) to evaluate analytically the DM’s aptitude to compensate among criteria. This procedure is extensively described in [7], so here we only give an overview. We start with some basic definitions.

Let \(A = \{a, b, c, \ldots\}\) be a set of \(|A| = m \geq 2\) alternatives, which has to be evaluated on the basis of a consistent [20] family of criteria \(G = \{g_i : i \in J\}, J = \{1, \ldots, n\}, n \geq 2\). In our approach we make the assumption that each criterion \(g_i : A \rightarrow \mathbb{R}\) in \(G\) is an interval scale of measurement [19]. For sake of simplicity, we sometime abuse notation and identify a criterion \(g_i\), with its index \(i\); we also use a similar identification for the sets \(G\) and \(J\). Furthermore, whenever we consider a pair of criteria \((i, j) \in J^2\), we always assume that \(i \neq j\), even without explicit mention.

We usually identify an alternative \(a \in A\) with the vector of its evaluations by the set \(G\) of criteria, i.e., \(a = (a_1, a_2, \ldots, a_n)\), where \(a_i = g_i(a)\) for each \(i \in J\). We also make the non-restrictive assumption that on each criterion there are at least two levels that evaluate a particular attribute (or class of attributes). For each criterion \(i \in J\), let \(x_i, \beta_i \in \mathbb{R}\), with \(x_i < \beta_i\), be two reference evaluations provided by the DM: \(x_i\) and \(\beta_i\) are, respectively, the minimum and maximum value on criterion \(i\). These reference evaluations are usually fixed \(a\ priori\) by the DM.

Next, we define the normalized difference function \(\Delta_i : A \times A \rightarrow \mathbb{R}\) on criterion \(i \in J\) by \(\Delta_i(a, b) = (a_i - b_i)/(\beta_i - x_i)\) for each \((a, b) \in A \times A\). Note that for each \(i \in J\) and \((a, b) \in A \times A\), we have \(-1 \leq \Delta_i(a, b) \leq 1\). Since by hypothesis criteria are interval scales, normalized difference functions provide a meaningful tool to compare criteria. Therefore, for example, if alternative \(a \in A\) is evaluated better than alternative \(b \in A\) by criterion \(i \in J\) (i.e., \(a_i > b_i\)), then the positive normalized difference \(\Delta_i(a, b)\) gives a rough measure of the local (i.e., with respect to criterion \(i \in J\)) strength of \(a\) over \(b\).

At a global level (i.e., with respect to all of \(J\)), a normalized difference interacts with all the other differences of evaluations, showing a double effect:

- active, since it gives some contribution to the (possible) overall preference of \(a\) over \(b\);
- passive, since it determines a resistance to the (possible) overall preference of \(b\) over \(a\).

Therefore, a partial preference of \(a\) over \(b\) on criterion \(i \in J\) may enlarge both the set of arguments to accept a global preference of \(a\) over \(b\) and the set of arguments to reject a global preference of \(b\) over \(a\).

According to a PCCA philosophy, these compensatory effects are measured for each (ordered) pair of different criteria \((i, j) \in J \times J\). This is accomplished by the construction of a so-called compensatory function \(\text{CF}_{ij}\) for each \((i, j) \in J \times J\), \(i \neq j\). In the sequel we sketch very briefly the procedure to build these functions. (For an extensive treatment of this topic, see [7].)

For each \((i, j) \in J^2\) the DM determines the degree of confidence that a positive normalized difference on the active criterion \(i\) compensates a negative normalized difference on the passive criterion \(j\). This evaluation is translated into a numerical form via the construction of a compensatory function \(\text{CF}_{ij}\), which associates to each pair of normalized differences \((\Delta_i, \Delta_j) \in [0, 1] \times [-1, 0]\) the level of credibility that the positive difference \(\Delta_i\) totally compensates the negative difference \(\Delta_j\). Successively, this function is extended in frontier by continuity, thus obtaining a compensatory function \(\text{CF}_{ij} : [0, 1] \times [-1, 0] \rightarrow [0, 1]\).

Some examples of compensatory functions are shown in Fig. 1. The first diagram represents the so-called reference compensatory function, defined by

![Fig. 1. Some compensatory functions.](image-url)
\[ \text{CF}_{\rho j}(A_i, A_j) := \frac{A_i + A_j + 1}{2} \]

for each \((A_i, A_j) \in [0, 1] \times [-1, 0]\). Its graph is the plane passing through the points \((0, 0, \frac{1}{2})\), \((1, 0, 1)\) and \((0, -1, 0)\). The reference compensatory function is particularly important in compensability analysis, since some suitable variations might be used by default in case of lack of information. (See Section 8 in [7] for some considerations on this point.)

It is possible that a compensatory function is constant in a subset of the domain with a positive Lebesgue measure. For example, the second diagram in Fig. 1 represents a compensatory function such that the degree of credibility that a positive normalized difference on the active criterion \(g_i\) compensates a negative normalized difference on the passive criterion \(g_j\) is equal to 1 for large active differences and small passive differences \((\text{large over small})\), it is equal to 0 for small active differences and large passive differences \((\text{small over large})\), and it has intermediate values in between. These type of functions may be useful to model refinements of the so-called lexicographic semiorder [12,24].

It might also happen that \(\text{CF}_{\rho j}\) is identically equal to 1 on the whole domain. This is typical of a situation in which criterion \(i\) is overwhelmingly more important than criterion \(j\) (or, as it is usually said, criterion \(i\) precedes \(j\) in a lexicographic order; see Section 8 for an extensive discussion of this topic). In this circumstance, it is often the case that the dual compensatory function \(\text{CF}_{\rho i}\) is identically equal to 0 on the whole domain.

Compensability analysis allows the DA to assign to each criterion \(g_i \in G\) a pair of indices, which evaluate the compensatory strength of \(g_i\) over the other criteria in \(G\) according to the DM’s preference structure. This compensatory strength can be somehow assimilated to a weight, but it explicitly separates active and passive aspects. For each criterion \(g_i \in G\), we define its active compensatory power \(p_i^+\) and its passive compensatory power \(p_i^-\) as follows (see [6,7]):

\[
\begin{align*}
    p_i^+ &= \frac{\sum_{j=1}^{n} \left( \int_0^1 \int_0^1 \text{CF}_{\rho j}(A_i, A_j) dA_i dA_j \right)}{n - 1}, \\
    p_i^- &= 1 - \frac{\sum_{j=1}^{n} \left( \int_0^1 \int_0^1 \text{CF}_{\rho j}(A_i, A_j) dA_i dA_j \right)}{n - 1}.
\end{align*}
\]

Observe that \(p_i^+, p_i^- \in [0, 1]\). For example, if \(\text{CF}_{\rho j}\) is the reference compensatory function for each \(i, j \in J\), then we have \((p_i^+, p_i^-) = \left(\frac{1}{2}, \frac{1}{2}\right)\).

Compensability analysis also enables the DA to obtain an estimation of the DM’s aptitude to compensate among the criteria in \(G\). We define a DM compensability index as follows:

\[
\gamma_{\text{DM}}(J) = \frac{\sum_{i \in J} \left( \int_0^1 \int_0^1 \text{CF}_{\rho j}(A_i, A_j) dA_i dA_j \right)}{n(n - 1)}.
\]

Note that \(\gamma_{\text{DM}}(J) \in [0, 1]\). In particular, we say that a DM is totally compensatory if \(\gamma_{\text{DM}}(J) = 1\), totally non-compensatory if \(\gamma_{\text{DM}}(J) = 0\), and half-compensatory if \(\gamma_{\text{DM}}(J) = \frac{1}{2}\). For example, in case that all compensatory functions are equal to the reference compensatory function, the DM is half-compensatory.

3. Phase II: Binary indices

For each couple of alternatives \(a, b \in A\), we build some binary preference indices at different levels of aggregation. According to a PCCA philosophy, at the first step these preference indices are built considering two criteria a time; successively, we aggregate these indices in a suitable way. At each stage, we always distinguish active and passive indices. More specifically, for each couple of alternatives we build:

- active and passive elementary indices (related to each ordered pair of distinct criteria);
- active and passive partial indices (related to each criterion);
- active and passive global indices (related to the whole set of criteria).

At each level the DM plays a fundamental role, since he is asked to choose some functions (called auxiliary functions at the first level, aggregation functions at the other two), which are used for the estimation of active and passive effects. Theoretically, allowing the DM to choose among several options is in accordance with the general philosophy of our DM-oriented approach. Practically, in order to find a reasonable balance between the abstract flexibility and the concrete implementability of PACMAN, most of the estimating functions will often be selected in a rather standard way. In the sequel, we examine how active and passive binary indices are built at each stage of the modelization, and the properties that they must satisfy.
3.1. Elementary indices $\Pi_{ij}^+, \Pi_{ij}^-, \Pi_{ij}^x, \Pi_{ij}^y$

For each couple of different criteria $i, j \in J$ and different alternatives $a, b \in A$, eight active and passive elementary indices are built. Four of these indices are related to the ordered pair $(a, b)$, namely:

$$
\Pi_{ij}^+(a, b), \Pi_{ij}^-(a, b), \Pi_{ij}^x(a, b), \Pi_{ij}^y(a, b).
$$

(The first and the third indices are active, the second and the fourth passive.) Other four indices are related to the ordered pair $(b, a)$, namely:

$$
\Pi_{ij}^+(b, a), \Pi_{ij}^-(b, a), \Pi_{ij}^x(b, a), \Pi_{ij}^y(b, a).
$$

The active index $\Pi_{ij}^+(a, b)$ and the passive index $\Pi_{ij}^-(a, b)$ are defined as follows (similar definitions, mutatis mutandis, for the others):

$$
\Pi_{ij}^+(a, b) := \begin{cases} 
\Phi_{ij}^+(A_i(a, b), A_j(a, b)) & \text{if } A_i(a, b) > 0, \\
0 & \text{if } A_i(a, b) \leq 0
\end{cases}
$$

and

$$
\Pi_{ij}^-(a, b) := \begin{cases} 
\Phi_{ij}^-(A_i(a, b), A_j(a, b)) & \text{if } A_i(a, b) > 0, \\
0 & \text{if } A_i(a, b) \leq 0
\end{cases}
$$

where $\Phi_{ij}^+ : (0, 1] \times [-1, 1] \to [0, 1]$ and $\Phi_{ij}^- : [-1, 1] \times (0, 1] \to [0, 1]$ are two auxiliary functions satisfying the following conditions:

(A1) (monotonicity) $\Phi_{ij}^+$ and $\Phi_{ij}^-$ are weakly increasing in each argument;

(A2) (active independence) $\Phi_{ij}^+$ restricted to $(0, 1] \times [0, 1]$ is strictly positive and depends on the first argument only;

(A3) (passive independence) $\Phi_{ij}^-$ restricted to $[0, 1] \times (0, 1]$ is strictly positive and depends on the second argument only.

(Recall that condition A1 for the function $\Phi_{ij}^+$ means that for each $x, x_1, x_2 \in (0, 1]$ and for each $y, y_1, y_2 \in [-1, 1]$, if $x_1 < x_2$ then $\Phi_{ij}^+(x_1, y) \leq \Phi_{ij}^+(x_2, y)$, and if $y_1 < y_2$ then $\Phi_{ij}^+(x, y_1) \leq \Phi_{ij}^+(x, y_2)$; dually for the function $\Phi_{ij}^-$.)

The index $\Pi_{ij}^+(a, b)$ evaluates the active contribution of criterion $i$ over criterion $j$ with respect to the ordered pair of alternatives $(a, b)$. Specifically, this index is an estimation of the active compensatory effect of the difference $A_i(a, b)$ onto the difference $A_j(a, b)$. Clearly, this effect is null whenever $A_i(a, b)$ is negative or zero. Otherwise, this effect is measured by the function $\Phi_{ij}^+$, which explains why the weak monotonicity axiom A1 must hold for it. Note that usually the auxiliary function will be a composition of functions, being $\Phi_{ij}^+ = \Phi_{ij}^+(\text{CF}_{ij})$. (See Section 5 for some standard definitions of auxiliary functions.)

The reason to impose axiom A2 (and its dual A3) is that whenever the passive difference $A_j(a, b)$ is either null or positive, then a positive active difference $A_i(a, b)$ has nothing to compensate; therefore, its compensatory effect is intrinsically evaluated by itself. This explains why the measure of the compensatory effect of a positive active difference $A_i(a, b)$ onto a non-negative passive difference $A_j(a, b)$ has a strictly positive value and depends only on the active difference $A_i(a, b)$.

The index $\Pi_{ij}^-(a, b)$ looks at criterion $i$ from a passive point of view, estimating the passive resistance of the difference $A_i(a, b)$ with respect to the difference $A_j(a, b)$. Again, this resistance is null in the case $A_i(a, b) \leq 0$. If $A_i(a, b)$ is positive, this passive effect is evaluated by the function $\Phi_{ij}^-$. In particular, in case of differences of the same sign, the measure of this resistance is strictly positive and does not depend on $A_j(a, b)$ but only on $A_i(a, b)$. As for active auxiliary functions, also passive auxiliary functions are usually composition of functions, being $\Phi_{ij}^- = \Phi_{ij}^-(\text{CF}_{ij})$.

In general, no functional relation needs to exist between $\Phi_{ij}^+$ and $\Phi_{ij}^-$. Nevertheless, since both functions are related to the results of compensability analysis, they will often be somehow connected to each other. As a consequence, in the above circumstance also the indices $\Pi_{ij}^+$ and $\Pi_{ij}^-$ will be interrelated.

For example, in Section 7 we introduce particular implementations of PACMAN, called co-symmetric, for which active and passive auxiliary functions related to the same couple of distinct criteria are defined in the same way and the compensatory function on which they depend are strictly interrelated. (See properties CS1 and CS2 in Definition 5.) As a consequence of the definition, the equality $\Pi_{ij}^+(a, b) = \Pi_{ij}^-(a, b)$ holds for any pair of alternatives $(a, b) \in A \times A$, whenever a co-symmetric implementation is employed. (See Lemma 3(i).)

**Example 1.** Let $a, b \in A$ be two alternatives such that $a_i \geq b_i$ for each $i \in J$. Then, for each $i, j \in J$ we have $\Pi_{ij}^+(b, a) = \Pi_{ij}^-(b, a) = 0$, regardless of the choice of the auxiliary function. In particular, if $a_i = b_i$ for each $i \in J$, then all eight elementary indices associated to the couple $(a, b)$ are equal to zero.
3.2. Active and passive partial indices $\Pi_i^+$ and $\Pi_i^-$

For each criterion $i \in J$ and for each couple of distinct alternatives $a, b \in A$, we build four active and passive partial indices, namely:

$$
\Pi_i^+(a, b), \Pi_i^-(a, b), \Pi_i^+(b, a), \Pi_i^-(b, a).
$$

(The first and the third indices are active, the second and the fourth passive.) The first two partial indices are defined, respectively, by

$$
\Pi_i^+(a, b) := \Phi_i^+\left(\Pi_{i,k}^+(a, b) : k \in J \setminus \{i\}\right) \quad \text{and} \quad \Pi_i^-(a, b) := \Phi_i^-\left(\Pi_{i,k}^-(a, b) : h \in J \setminus \{i\}\right),
$$

where $\Phi_i^+: [0, 1]^n \to [0, 1]$ and $\Phi_i^-+: [0, 1]^n \to [0, 1]$ are two aggregation functions satisfying the following conditions:

(B1) (monotonicity) $\Phi_i^+$ and $\Phi_i^-$ are weakly increasing in each argument;
(B2) (boundedness) $\Phi_i^+$ and $\Phi_i^-$ have values between the minimum and the maximum of their arguments;
(B3) (idempotence) if all the arguments of $\Phi_i^+$ (respectively, $\Phi_i^-$) have the same value, then also $\Phi_i^+$ (respectively, $\Phi_i^-$) has this value.

The indices $\Pi_i^+(a, b)$ and $\Pi_i^-(a, b)$ estimate, respectively, the active contribution of criterion $i \in J$ to the relation of compensated preference of $a$ over $b$, and the passive resistance of $i \in J$ to the relation of compensated preference of $b$ over $a$.

Note that it is theoretically possible that different criteria $g_i$ and $g_j$ have different active or passive aggregation functions, i.e., $\Phi_i^+ \neq \Phi_j^+$ or $\Phi_i^- \neq \Phi_j^-$. Practically, this will hardly happen. For example, for regular implementations of PACMAN, defined in Section 5, the equalities $\Phi_i^+ = \Phi_i^-$ and $\Phi_i^- = \Phi_i^-$ hold for each $i, j \in J$.

On the other hand, since the two functions $\Phi_i^+$ and $\Phi_i^-$ aggregate, respectively, active and passive effects related to a criterion $g_i$, it is conceivable that we have $\Phi_i^+ \neq \Phi_i^-$ for some $i \in J$. For example, the logic of aggregation might require to average all active effects on one side, and take the maximum passive effect on the other side (similarly to a veto effect, see [20]). Nevertheless, in particular cases (see Section 7), the equality $\Phi_i^+ = \Phi_i^-$ holds for each $i \in J$.

**Example 2.** If $a, b \in A$ are such that $a_i \geq b_i$ for each $i \in J$, then Example 1 and idempotence (axiom B3) yield $\Pi_i^+(b, a) = \Pi_i^-(b, a) = 0$ for each $i \in J$. In particular, if $a$ and $b$ are evaluated in exactly the same way by all criteria, then we have $\Pi_i^+(a, b) = \Pi_i^-(a, b) = \Pi_i^+(b, a) = \Pi_i^-(b, a) = 0$ for each $i \in J$.

3.3. Active and passive global indices $\Pi^+$ and $\Pi^-$

For each couple of distinct alternatives $a, b \in A$, four active and passive global indices are built, namely:

$$
\Pi^+(a, b), \Pi^-(a, b), \Pi^+(b, a), \Pi^-(b, a).
$$

(The first and third indices are active, the second and fourth passive.) The first two indices are defined as follows:

$$
\Pi^+(a, b) := \Phi^+\left(\Pi_i^+(a, b) : i \in J\right) \quad \text{and} \quad \Pi^-(a, b) := \Phi^-\left(\Pi_i^-(a, b) : i \in J\right),
$$

where $\Phi^+: [0, 1]^n \to [0, 1]$ and $\Phi^-+: [0, 1]^n \to [0, 1]$ are two aggregation functions, which satisfy the same properties as $\Phi_i^+$ and $\Phi_i^-$, i.e., monotonicity (axiom B1), boundedness (axiom B2) and idempotence (axiom B3). The two indices $\Pi^+(a, b)$ and $\Pi^-(a, b)$ measure, respectively, active contribution of the whole set $J$ to the relation of compensated preference of $a$ over $b$, and passive resistance of $J$ to the relation of compensated preference of $b$ over $a$. In general, for the same reasons as above, the two aggregation functions $\Phi^+$ and $\Phi^-$ may be different. On the other hand, in special cases the equality $\Phi^+ = \Phi^-$ holds. For example, $\Phi^+$ and $\Phi^-$ might both be a weighted sum of their arguments, where the respective weights $\lambda_i^+ = \lambda_i^+(p_i^+)$ and $\lambda_i^- = \lambda_i^-(p_i^-)$ are defined by the same type of law for each $i \in J$ (cf. Definition 7 in Section 7, and Definition 9 in Section 8).

**Example 3.** If $a, b \in A$ are such that $a_i \geq b_i$ for each $i \in J$, then we have $\Pi^+(b, a) = \Pi^-(b, a) = 0$, using Example 2 and idempotence. In particular, if $a$ and $b$ are evaluated in the same way by all criteria, then all four global indices related to them are equal to zero.

4. Phase III: Modelization of preferences

We determine the relation between any two distinct alternatives $a$ and $b$ in two stages: first we obtain two relations of compensated preference for each couple of alternatives $a, b \in A$, and successively we use them to establish a relational system of preferences on the set $A$ [18,25].
4.1. Compensated preference

For each $a, b \in A$, compensated preference of $a$ over $b$ (respectively, $b$ over $a$) is established in two steps:

(i) first it is evaluated by comparing the active global index $\Pi^+(a,b)$ and the passive global index $\Pi^-(b,a)$ (respectively, the active global index $\Pi^+(b,a)$ and the passive global index $\Pi(a,b)$);

(ii) then it is accepted ($\mathcal{T}^+$), doubted ($\mathcal{T}^-$) or rejected ($\mathcal{T}^\varnothing$) on the basis of a sensitivity threshold $\varepsilon \in [0, 1]$ fixed a priori by the DM.

For step (i), the evaluations of the two relations of compensated preferences are obtained by computing the net global indices $\Pi(a,b)$ and $\Pi(b,a)$ as follows:

$$\Pi(a,b) := \Pi^+(a,b) - \Pi^-(b,a) \quad \text{and} \quad \Pi(b,a) := \Pi^+(b,a) - \Pi^-(a,b).$$

Note that $\Pi(a,b), \Pi(b,a) \in [-1, 1]$. The index $\Pi(a,b)$ gives a numerical estimation of the net result of all the arguments favoring a compensated preference of $a$ over $b$ and all the arguments contrasting it. A dual interpretation has the index $\Pi(b,a)$. The whole procedure related to a couple of alternatives $a, b \in A$ is summarized by the flowchart given in Figs. 2 and 3, where it is assumed that $|J| = 3$.

For step (ii), let $\varepsilon \in [0, 1]$ be a suitable sensitivity threshold fixed by the DM. Then two binary relation of compensated preference related the couple of alternatives $\{a, b\}$ are established on the basis of the global net indices and the sensitivity threshold as follows:

$$a \mathcal{T}^+ b \iff \Pi(a,b) \in (\varepsilon, 1],$$

$$a \mathcal{T}^- b \iff \Pi(a,b) \in [-\varepsilon, \varepsilon],$$

$$a \mathcal{T}^\varnothing b \iff \Pi(a,b) \in [-1, -\varepsilon),$$

$$b \mathcal{T}^+ a \iff \Pi(b,a) \in (\varepsilon, 1],$$

$$b \mathcal{T}^- a \iff \Pi(b,a) \in [-\varepsilon, \varepsilon],$$

$$b \mathcal{T}^\varnothing a \iff \Pi(b,a) \in [-1, -\varepsilon).$$

Example 4. Let $a, b \in A$ be such that $a_i \succeq b_i$ for each $i \in J$. Since $\Pi(b,a) \leq 0$ by Example 3, then either $b \mathcal{T}^+ a$ or $b \mathcal{T}^\varnothing a$. In particular, if $a_i = b_i$ for each $i \in J$, then $\Pi(a,b) = \Pi(b,a) = 0$, hence $a \mathcal{T}^+ b$ and $b \mathcal{T}^- a$.

4.2. Relational system of preferences

In PACMAN we use a relational system of preferences of the type $\{P, Q, I, R\}$, where $P$ denotes strict preference, $Q$ weak preference, $I$ indifference and $R$ incomparability. We use the binary relation of compensated preference to establish the relation between each couple of distinct alternatives $a, b \in A$ as follows:

$$\pi(a,b) = \Pi^+(a,b) - \Pi^-(b,a)$$

Fig. 2. Evaluation of compensated preference of $a$ over $b$ in the case $|J| = 3$. 
Observe that $P$ and $Q$ are asymmetric, whereas $I$ and $R$ are symmetric. Therefore, $\{P, Q, I, R\}$ constitutes a fundamental system of preferences \[20\].

Graphically, the relation between two generic alternatives $a, b \in A$ is determined from the values of two reciprocal net indices $\Pi(a, b)$ and $\Pi(b, a)$ as shown in Fig. 4.

**Example 5.** Let $a, b \in A$ be such that $a_i = b_i$ for each $i \in J$. Then Example 4 yields $alb$, regardless of the choice of the sensitivity threshold $\varepsilon$.

**5. Implementations of PACMAN and regularity conditions**

In order to make PACMAN implementable (i.e., applicable to concrete multi-criteria decision problems), we need to select the functions and parameters used at the different stages of the modelization. The next definition introduces a formal notion of implementation.
Definition 1. An implementation of PACMAN is a selection of the properties (or the definition) of the following items:

- all compensatory functions $CF_{\phi_j}$;
- all auxiliary functions $\phi^+_j$ and $\phi^-_j$;
- all aggregation functions $\Phi^+_j$ and $\Phi^-_j$;
- the two aggregation functions $\Phi^+$ and $\Phi^-$;
- the sensitivity threshold $\varepsilon$.

In the next section we shall prove that, under suitably general conditions, all implementations of PACMAN do satisfy some natural monotonicity properties. These general conditions are condensed into the notion of regularity of an implementation, which is the main topic of this section. Before introducing regular implementations of PACMAN, we define some typical auxiliary functions, called ordinal, cardinal and mixed. For each $i, j \in J$, a rather natural choice for the active auxiliary function $\Phi^+_j : [0, 1] \times [-1, 1] \rightarrow [0, 1]$ and the passive auxiliary function $\Phi^-_j : [-1, 1] \times (0, 1] \rightarrow [0, 1]$ is the following:

$$
\Phi^+_j(A_i, A_j) = \begin{cases} 
1 & \text{if } A_j \geq 0, \\
CF_{\phi_j}(A_i, A_j) & \text{if } A_j < 0,
\end{cases}
$$

$$
\Phi^-_j(A_i, A_j) = \begin{cases} 
1 & \text{if } A_j \geq 0, \\
1 - CF_{\phi_i}(-A_j, -A_i) & \text{if } A_j < 0.
\end{cases}
$$

In this case, the two elementary indices $\Pi^+_{\phi_j}(a, b)$ and $\Pi^-_{\phi_i}(a, b)$ will assume the following form:

$$
\Pi^+_{\phi_j}(a, b) = \begin{cases} 
1 & \text{if } A_i(a, b) > 0 \land A_j(a, b) \geq 0, \\
CF_{\phi_j}(A_i(a, b), A_j(a, b)) & \text{if } A_i(a, b) > 0 \land A_j(a, b) < 0, \\
0 & \text{if } A_i(a, b) \leq 0,
\end{cases}
$$

$$
\Pi^-_{\phi_i}(a, b) = \begin{cases} 
1 & \text{if } A_i(a, b) > 0 \land A_j(a, b) \geq 0, \\
1 - CF_{\phi_i}(-A_j, -A_i) & \text{if } A_i(a, b) > 0 \land A_j(a, b) < 0, \\
0 & \text{if } A_i(a, b) \leq 0.
\end{cases}
$$

Note that this selection of the auxiliary functions $\Phi^+_j$ and $\Phi^-_j$ does take into account the two differences $A_i(a, b)$ and $A_j(a, b)$, but only indirectly. In fact, in case of concordant evaluations on the two criteria (first law), both active and passive effects are constantly equal to 1; in case of discordant evaluations (second law), the two effects depend only on the degrees of confidence $CF_{\phi_j}$ and $CF_{\phi_i}$. Hence we regard this selection of $\Phi^+_j$ and $\Phi^-_i$ as an ordinal choice.

A different philosophy underlies the selection of the following functions:

$$
\Phi^+_j(A_i, A_j) = \begin{cases} 
A_i & \text{if } A_j \geq 0, \\
A_i CF_{\phi_j}(A_i, A_j) & \text{if } A_j < 0,
\end{cases}
$$

$$
\Phi^-_i(A_i, A_j) = \begin{cases} 
A_i & \text{if } A_j \geq 0, \\
A_i(1 - CF_{\phi_i}(-A_j, -A_i)) & \text{if } A_j < 0.
\end{cases}
$$

If we employ these auxiliary functions, then the correspondent elementary indices are

$$
\Pi^+_{\phi_j}(a, b) = \begin{cases} 
A_i(a, b) & \text{if } A_i(a, b) > 0 \land A_j(a, b) \geq 0, \\
A_i(a, b) CF_{\phi_j}(A_i(a, b), A_j(a, b)) & \text{if } A_i(a, b) > 0 \land A_j(a, b) < 0, \\
0 & \text{if } A_i(a, b) \leq 0,
\end{cases}
$$

$$
\Pi^-_{\phi_i}(a, b) = \begin{cases} 
A_i(a, b) & \text{if } A_i(a, b) > 0 \land A_j(a, b) \geq 0, \\
A_i(a, b)(1 - CF_{\phi_i}(-A_j, -A_i)) & \text{if } A_i(a, b) > 0 \land A_j(a, b) < 0, \\
0 & \text{if } A_i(a, b) \leq 0.
\end{cases}
$$

Note that with this selection of auxiliary functions, the difference $A_i(a, b)$ is taken directly into account, since its active and passive effects are a certain percentage of it (the whole difference $A_i(a, b)$, in case of concordant evaluations). Hence, we regard this selection as a cardinal choice for $\Phi^+_j$ and $\Phi^-_i$.

Finally, a third selection for the auxiliary functions, which is intermediate between the first two, is the following:

$$
\Phi^+_j(A_i, A_j) = \begin{cases} 
1 & \text{if } A_j \geq 0, \\
A_i CF_{\phi_j}(A_i, A_j) & \text{if } A_j < 0,
\end{cases}
$$

$$
\Phi^-_i(A_i, A_j) = \begin{cases} 
1 & \text{if } A_j \geq 0, \\
A_i(1 - CF_{\phi_i}(-A_j, -A_i)) & \text{if } A_j < 0.
\end{cases}
$$
hence

\[
\Pi_{i,j}^{+}(a, b) = \begin{cases} 
1 & \text{if } \Delta_i(a, b) > 0 \land \Delta_j(a, b) \geq 0, \\
\Delta_i(a, b) \text{CF}_{i,j}(\Delta_i(a, b), \Delta_j(a, b)) & \text{if } \Delta_i(a, b) > 0 \land \Delta_j(a, b) < 0, \\
0 & \text{if } \Delta_i(a, b) \leq 0,
\end{cases}
\]

\[
\Pi_{i,j}^{-}(a, b) = \begin{cases} 
1 & \text{if } \Delta_i(a, b) > 0 \land \Delta_j(a, b) \geq 0, \\
\Delta_i(a, b) (1 - \text{CF}_{i,j}(-\Delta_i(a, b), -\Delta_j(a, b))) & \text{if } \Delta_i(a, b) > 0 \land \Delta_j(a, b) < 0, \\
0 & \text{if } \Delta_i(a, b) \leq 0.
\end{cases}
\]

The philosophy underlying this choice is similar to that of the MAPPAC methodology, which can be seen as a compromise between outranking methods and MAUT [17]. Indeed, in this case the values of elementary indices are determined on the basis of different approaches, according to whether the evaluations by means of the two criteria are concordant or discordant. Specifically, in case of concordant evaluations (first law), the indices underline partial dominance (active and passive, respectively) with a value of 1, in analogy to non-compensatory outranking methods, and to the ELECTRE methods [3,21] in particular. On the other hand, in case of discordant evaluations (second law), the value of these indices is a percentage of the relevant difference, in analogy to the MAUT approach. Note that the difference \( \Delta_i(a, b) \) is taken into account either directly (in case of discordance) or indirectly (in case of concordance). Therefore, we regard this selection of the functions \( \Phi_{i,j}^{+} \) and \( \Phi_{i,j}^{-} \) as a mixed choice.

The next result ensures that ordinal, cardinal and mixed auxiliary functions are suitable choices for implementations of PACMAN. Its proof is a straightforward computation and is left to the reader.

**Lemma 1.** Ordinal, cardinal and mixed auxiliary functions satisfy axioms A1, A2 and A3.

We are now ready to introduce the notion of a regular implementation.

**Definition 2.** An implementation of PACMAN is regular if it satisfies the following properties:

(R1) each auxiliary function is either ordinal or cardinal or mixed;
(R2) for each \( i, j \in J \), we have \( \Phi_i^+ = \Phi_j^+ \) and \( \Phi_i^- = \Phi_j^- \);
(R3) all aggregation functions \( \Phi_i^+, \Phi_i^-, \Phi_i^+, \Phi_i^- \) assume a positive value whenever at least one of their arguments is positive.

Condition R1 restraints the choice for auxiliary functions to one of the three listed types. Note that R1 does not imply that the active and passive auxiliary functions \( \Phi_{i,j}^+ \) and \( \Phi_{i,j}^- \) related to each couple \( \{g_i, g_j\} \) of criteria are all of the same type. Indeed, it might happen that the most suitable choice of auxiliary functions is ordinal for some pairs of criteria, cardinal or mixed for some others.

Condition R2 requires that all active partial aggregation functions are the same. This is a rather natural restriction, since a logic of active aggregation that allows a different treatment for different criteria might affect the final results in a way that is both determinant and obscure. Similar considerations can be done for passive partial aggregation functions. Note that, for reasons already emphasized in Section 3.2, condition R2 does not imply that the equality \( \Phi_i^+ = \Phi_i^- \) holds for some \( i \in J \).

Finally, observe that unless a minimum-like operator is employed as a partial or global aggregation function, each implementation of PACMAN satisfies condition R3. Indeed, this condition is quite reasonable, since it is rather unlikely that the logic of aggregation requires the usage of a minimum-like operator as an aggregation function. (On the other hand, it is conceivable that a maximum-like operator is chosen by the DM, e.g., for some passive aggregation functions; but such a choice would not affect regularity, since it satisfies R3.)

6. Dominance and regularity

Regular implementations of PACMAN satisfy some monotonicity properties, which are related to the notion of dominance. In this section we state and prove these properties. We first recall the notion of dominance.

**Definition 3.** Let \( G = \{g_i : i \in J\} \) be a given set of evaluation criteria. For each couple of alternatives \( a, b \in A \), we use the following notation:

\[ J_{a,b}^{+} := \{i \in J : a_i > b_i\}, \quad J_{a,b}^{-} := \{i \in J : a_i = b_i\}, \quad J_{a,b}^{=} := \{i \in J : a_i < b_i\}. \]

Furthermore, we set

\[ J_{a,b}^\geq := J_{a,b}^{+} \cup J_{a,b}^{=} \quad \text{and} \quad J_{a,b}^{\leq} := J_{a,b}^{=} \cup J_{a,b}^{-}. \]
Note that \( J_{a,b}^+ = J_{a,b}^-, J_{a,b}^- = J_{b,a}^- \), and \( J_{a,b}^+ = J_{b,a}^- \). We say that \( a \) dominates \( b \) (with respect to \( G \)), and we denote by \( aD_gb \), if \( J = J_{a,b}^+ \) and \( |J_{a,b}^+| \geq 1 \). Equivalently, \( a \) dominates \( b \) if \( a_i \geq b_i \) for each \( i \in J \), with at least one of the inequalities being strict.

Whenever the set of criteria \( G \) is understood, we simplify notation and write \( aDb \) in place of \( aD_gb \).

The notion of dominance is quite important in a multiple criteria setting, since it is independent of the DM’s preference structure. Those alternatives of \( A \) that are dominated by others are usually removed from \( A \). Nevertheless, a consistent multi-criteria methodology should satisfy some obvious monotonicity properties with respect to dominance: this is the content of Theorem 1. We need a preliminary result, which we prove first.

**Lemma 2.** Assume that a regular implementation of PACMAN is used, and let \( a, b \in A \) be such that \( aDb \). We have:

(i) \( \Pi^+(a,b) > 0, \Pi^-(a,b) > 0 \) and \( \Pi^+(b,a) = \Pi^-(b,a) = 0; \)

(ii) for each \( c \in A \), we have:

- \( \Pi^+(a,c) \geq \Pi^+(b,c) \) and \( \Pi^-(a,c) \geq \Pi^-(b,c); \)
- \( \Pi^+(c,a) \leq \Pi^+(c,b) \) and \( \Pi^-(c,a) \leq \Pi^-(c,b). \)

**Proof.** We prove (i). For each \( i \in J_{a,b}^+ \) and \( j \in J \setminus \{i\} \), we have either \( \Pi_{i,j}^+(a,b) = 1 \) (in case that an ordinal or a mixed auxiliary function \( \Phi_{i,j}^o \) is used) or \( \Pi_{i,j}^+(a,b) = \Delta_i(a,b) \) (in case that a cardinal auxiliary function \( \Phi_{i,j}^c \) is used); in any case, \( \Pi_{i,j}^+(a,b) \geq 0 \). Therefore, condition R3 of regularity yields \( \Pi_i^+(a,b) > 0 \) for each \( i \in J_{a,b}^+ \). Since \( |J_{a,b}^+| \geq 1 \), it follows that \( \Pi^+(a,b) > 0 \), again using regularity of the implementation. Reasoning as above, we also obtain \( \Pi^-(a,b) > 0 \). Finally, Example 3 yields \( \Pi^+(b,a) = \Pi^-(b,a) = 0 \). (Note that regularity is not needed to prove these equalities.)

To prove (ii), let \( c \in A \). We only show that the inequality \( \Pi^+(a,c) \geq \Pi^+(b,c) \) holds; the other inequalities can be proved in a similar way. We start by showing that for each \( i, j \in J \) we have

\[
\Pi_{i,j}^+(a,c) \geq \Pi_{i,j}^+(b,c). \tag{*}
\]

The inequality is obvious if \( i \in J_{b,c}^+ \), since in this case we have \( \Delta_i(b,c) \leq 0 \) and so \( \Pi_{i,j}^+(b,c) = 0 \). Thus, assume that \( i \in J_{b,c}^-, i.e., \Delta_i(a,c) \geq \Delta_i(b,c) > 0 \). Two cases: (1) \( j \in J_{b,c}^- \); (2) \( j \in J_{b,c}^+ \). In case (1), we have \( \Delta_j(a,c) \geq \Delta_j(b,c) > 0 \). It follows that either \( \Pi_{i,j}^+(a,c) = \Pi_{i,j}^+(b,c) = 1 \) (if we use an ordinal or mixed auxiliary function) or \( \Pi_{i,j}^+(a,c) = \Pi_{i,j}^+(b,c) = \Delta_i(a,c) > \Delta_i(b,c) = \Pi_{i,j}^+(b,c) \) (if we use a cardinal auxiliary function). Therefore, inequality (*) holds. Next, we examine case (2), i.e., \( \Delta_i(b,c) < 0 \). Since \( \Delta_j(a,c) \geq \Delta_j(b,c) \), we can have the following subcases: (2a) \( j \in J_{b,c}^+ \); (2b) \( j \in J_{b,c}^- \). In subcase (2a), we have \( \Delta_j(b,c) \leq \Delta_j(a,c) < 0 \). If we use an ordinal auxiliary function, then the definition of elementary indices yields

\[
\Pi_{i,j}^+(a,c) = \text{CF}_{i,j}^o(\Delta_i(a,c), \Delta_j(a,c)) \geq \text{CF}_{i,j}^o(\Delta_i(b,c), \Delta_j(b,c)) = \Pi_{i,j}^+(b,c),
\]

where the inequality holds because of the monotonicity (in both arguments) of the compensatory function \( \text{CF}_{i,j}^o \). On the other hand, if we use a cardinal or a mixed auxiliary function, then we have

\[
\Pi_{i,j}^+(a,c) = \Delta_i(a,c)\text{CF}_{i,j}^c(\Delta_i(a,c), \Delta_j(a,c)) \geq \Delta_i(b,c)\text{CF}_{i,j}^c(\Delta_i(b,c), \Delta_j(b,c)) = \Pi_{i,j}^+(b,c),
\]

where the inequality holds for the same reason as above. In subcase (2b), the chain of inequalities \( \Delta_j(a,c) \geq 0 > \Delta_j(b,c) \) implies

\[
\Pi_{i,j}^+(a,c) = 1 \geq \Pi_{i,j}^+(b,c),
\]

if we use an ordinal or a mixed auxiliary function, and

\[
\Pi_{i,j}^+(a,c) = \Delta_i(a,c) \geq \Delta_i(b,c)\text{CF}_{i,j}^o(\Delta_i(b,c), \Delta_j(b,c)) = \Pi_{i,j}^+(b,c),
\]

if we use a cardinal auxiliary function. This proves that inequality (*) holds in all cases. Now the claim follows from a straightforward application of the monotonicity property (axiom B1) of all active partial aggregation functions \( \Phi_i^+ \) first, and of the active global aggregation function \( \Phi^+ \) afterwards. \( \Box \)

**Theorem 1.** Assume that a regular implementation of PACMAN is used. For each \( a, b, c \in A \), we have:

(i) if \( aDb \) and \( bDc \), then \( aDc \);

(ii) if \( aDb \) and \( bDc \), then either \( aDc \) or \( aQc \);

(iii) if \( aDb \) and \( bDc \), then either \( aPc \) or \( aQc \) or \( aRc \);

(iv) if \( aDb \) and \( bDc \), then either \( aPc \) or \( aQc \) or \( aRc \).
**Proof.** We prove (i) and (iv). (Parts (ii) and (iii) can be proved in a similar way.) For (i), let \( a, b, c \in A \) be such that \( aDb \) and \( bPc \). We have, using Lemma 2(ii):

\[
\begin{align*}
bPc & \iff b\Upsilon^c \land c\Upsilon^a \\
& \iff \Pi^+(b,c) - \Pi^-(c,b) \in (-1,1) \land \Pi^+(c,b) - \Pi^-(b,c) \in [-1,1] \\
& \implies \Pi^+(a,c) - \Pi^-(c,a) \in (-1,1) \land \Pi^+(c,a) - \Pi^-(a,c) \in [-1,1]
\end{align*}
\]

This completes the proof. \( \square \)

**Corollary 1.** Assume that a regular implementation of PACMAN is used. For each \( a, b \in A \), if \( aDb \) and \( bRc \), then either \( aPb \) or \( aQb \) or \( aRb \).

**Proof.** The claim is an immediate consequence of Theorem 1(iii) and Example 5. \( \square \)

Since regularity properties are rather natural requirements, in this paper we assume that all implementations of PACMAN are regular.

7. Some regular implementations of PACMAN

In this section we examine several regular implementations of PACMAN, which satisfy some additional properties. The most important of these regular implementations is called co-symmetric. Before defining it and studying its properties, we need to introduce the notion of complementarity for criteria of \( G \) (see Section 4 in [7]).

**Definition 4.** A complementary couple of criteria is a set of two criteria \( \{g_i, g_j\} \subseteq G \) such that the related compensatory functions \( \text{CF}_{g_i} \) and \( \text{CF}_{g_j} \) satisfy the following equation for each \( A_i, A_j \in [0, 1] \):

\[
\text{CF}_{g_i}(A_i, -A_j) + \text{CF}_{g_j}(A_j, -A_i) = 1.
\]

An absolutely complementary criterion is a criterion \( g_k \in G \) such that for each \( i \in J \setminus \{k\} \), the couple \( \{g_k, g_i\} \) is complementary. An absolutely complementary DM is a decision maker such that each criterion \( g_i \in G \) is absolutely complementary. (Equivalently, a DM is absolutely compensatory if all reciprocal compensatory functions satisfy Eq. (\( \Diamond \)).)

**Remark 1.** An absolutely complementary DM is half-compensatory (see Corollary 7.2 in [7]).

**Example 6.** We give some instances of compensatory functions such that the DM is absolutely complementary.

(i) If all compensatory functions \( \text{CF}_{g_i} \) are equal to the reference compensatory function (see first diagram in Fig. 1), then it is easy to check that each couple of criteria is complementary. Thus, the DM is absolutely complementary, hence half-compensatory by Remark 1. (Note that in this case we have \( (p_i^0, p_i^1) = \left(\frac{1}{2}, \frac{1}{2}\right) \) for all \( i \in J \).

(ii) If for each \( i, j \in J \) we have either \( \text{CF}_{g_i} \equiv 0 \) and \( \text{CF}_{g_j} \equiv 1 \), or \( \text{CF}_{g_i} \equiv 1 \) and \( \text{CF}_{g_j} \equiv 0 \), then all couples of criteria are complementary, and therefore the DM is absolutely complementary. The so-called lexicographic compensatory functions (introduced in the next section) belong to this category.

Next we introduce a regular implementation of PACMAN that satisfies some symmetry properties.

**Definition 5.** A regular implementation of PACMAN is co-symmetric (complementary-symmetric) if the following conditions hold for each \( i, j \in J \):
We end this section by defining some other regular implementations of PACMAN. They will be useful in the sequel.

Assume that a co-symmetric implementation of PACMAN is used. For each Corollary 3. The result follows from Theorems 1 and 2.

Proof. The result follows Corollary 1 and Theorem 2.

We end this section by defining some other regular implementations of PACMAN. They will be useful in the sequel.
Definition 6. A regular implementation of PACMAN is uniform if the auxiliary functions are all of the same type. Specifically, it is ordinal (respectively, cardinal, mixed) if the selection of all auxiliary functions is ordinal (respectively, cardinal, mixed).

Even if it is theoretically possible that a regular implementation employs more than one type of auxiliary functions, the most natural case is that of a uniform implementation.

Definition 7. A regular implementation of PACMAN is weighted if the following conditions hold for each $i \in J$:

1. $\Phi^i$ and $\Phi^i$ are the average of their arguments;
2. $\Phi^i$ and $\Phi^i$ are the weighted sum of their arguments, where the weights $\lambda^i = \lambda^i(p^i) > 0$ and $\lambda^i = \lambda^i(p^i) > 0$ are strictly increasing functions of, respectively, $p^i$ and $p^i$.

Roughly speaking, a weighted implementation is how PACMAN interprets a weighted sum utility model.

Definition 8. A regular implementation of PACMAN is crisp if the sensitivity threshold in the final stage is $\epsilon := 0$.

Observe that in a crisp implementation of PACMAN the relation $Q$ of weak preference is empty (cf. Fig. 4).

8. A lexicographic implementation of PACMAN

In the process of modeling multi-dimensional preferences, lexicographic structures arise quite naturally. These types of structures are linked to the existence of evaluation criteria that are “overwhelmingly more important” than others. Since the above circumstance is not unusual in some real life situations, lexicographic models have been widely studied in economic theory and related fields, with particular attention to utility theory: see, e.g., [4,5,8,9,11] and references therein for an extensive account of this subject.

In this section we show that a suitable implementation of PACMAN allows one to obtain the lexicographic ordering. More specifically, we prove that the lexicographic model is representable by a regular implementation of PACMAN, which is co-symmetric, ordinal, weighted, and crisp.

The fact that the lexicographic model is representable by PACMAN is of some interest for its theoretical implications. In fact, it is true that one can obtain a lexicographic modelization of a specific multi-criteria decision problem by suitably selecting the criteria weights to be used in a weighted sum utility model. On the other hand, it is obvious that there exists no universal selection of criteria weights that allows one to obtain a lexicographic modelization of an arbitrary multi-criteria decision problem using a weighted sum utility model. In PACMAN the selection of the weights used in the weighted sum is universal, in the following sense: once that the criteria are ordered lexicographically, we can obtain a lexicographic ordering on an arbitrary set of alternatives by using global aggregation functions that are weighted sums.

Recall that, given a set $A \subseteq \mathbb{R}^n$, the lexicographic order on $A$ is the total order $\preceq_{\text{lex}}$ defined as follows: for each $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $A$, let $b \prec_{\text{lex}} a$ if there exists an index $\ell \leq n$ such that $b_\ell < a_\ell$ and $a_i = b_i$ for all $i$ satisfying $i < \ell$; the notation $b \preceq_{\text{lex}} a$ stands for $b \preceq_{\text{lex}} a$ or $a = b$. In the following, we use the asymmetric part $\prec_{\text{lex}}$ of the linear order $\preceq_{\text{lex}}$. Furthermore, whenever we refer to the lexicographic order $\prec_{\text{lex}}$ with respect to a set of criteria $G = \{g_1, g_2, \ldots, g_n\}$, we always assume that the criteria are ordered in decreasing order of importance, which is denoted by $g_1 \gg g_2 \gg \cdots \gg g_n$.

As observed in Section 5.3 in [7], compensability analysis can be naturally used to model a lexicographic ordering of criteria. Note that if $g_i \gg g_j$, then not only any negative difference on $g_i$ cannot be compensated at all by a positive difference on $g_j$, but also any positive difference on $g_i$ totally compensates a negative difference on $g_j$. Therefore, in this case the related compensatory functions $\text{CF}_{gj}$ and $\text{CF}_{gj}$ are constant and identically equal to, respectively, 1 and 0: we call this situation a lexicographic modelization of compensability. Note that in a lexicographic modelization of compensability, active and passive compensatory powers have a simple value. Namely, if $g_1 \gg g_2 \gg \cdots \gg g_n$, then for each $i \in J$ we have $(p^i, p^i) = (0, 1)$ and $(p^i, p^i) = (0, 0)$.

Observe also that a lexicographic modelization of compensability implies that the DM is absolutely complementary, and so half-compensatory (see Remark 1 and Example 6). This does not clash with the invertebrate habit of thinking of the lexicographic model as non-compensatory. Indeed, the equality $\gamma_{\text{DM}} = \frac{1}{2}$ simply means that the DM (and not the model) is willing to partially compensate among criteria. In other words, this is a consequence of the fact that we look at the concept of compensation from the point of view of the DM and not of the model. In a way, this situation even sheds light on the natural asymmetry of the notion of compensability. In fact, if $g_i$ is more important than $g_j$ in the lexicographic order, then the compensatory effect of $g_i$ over $g_j$ is null, but the compensatory effect of $g_j$ over $g_i$ is total; therefore, the DM is partially compensatory.

We now introduce a regular implementation of PACMAN that will allow us to represent the lexicographic order.
Definition 9. A regular implementation of PACMAN is lexicographic if it satisfies the following conditions:

(L1) it uses a lexicographic modelization of compensability;
(L2) it is ordinal;
(L3) it is weighted, and for each \( i \in J \) the weights \( \lambda^-_i \) and \( \lambda^+_i \) used in the definition of \( \Phi^- \) and \( \Phi^+ \) are given by
\[
\lambda^-_i = \frac{t}{t+1} \lambda^+_i := \frac{[t(t+1)]^{d_i}}{t+1}, \quad \text{where} \quad t \geq 2;
\]
(L4) it is crisp.

Henceforth we use the subscription “\( \text{lex} \)” to emphasize that we are using a lexicographic implementation of PACMAN. For instance, the notations \( a_P a b \) and \( a_{\text{lex}} b \) stand for, respectively, \( a_P b \) and \( a b \), in the case that a lexicographic implementation of PACMAN is employed. A similar meaning have the notations \( \Phi^-_{\text{lex}}, \Phi^+_{\text{lex}}, \Phi_{\text{lex}}, \xi_{\text{lex}}, \gamma^Y_{\text{lex}}, \gamma^N_{\text{lex}}, \gamma^V_{\text{lex}}, \gamma^N_{\text{lex}} \).

Remark 2. Despite the apparently complicated formula used for the weights \( \lambda^-_i \) and \( \lambda^+_i \) (property L3 in Definition 9), their value is naturally derived from those of the passive and active compensatory powers \( p^-_i \) and \( p^+_i \). For example, the value of \( \lambda^-_i \) is obtained as follows:
\[
\lambda^-_i = \lambda_1^-(p^-_i) := \frac{t^{(n-1)p^-_i}}{\sum_{r=1}^{n} t^{(n-1)p^-_i}} = \frac{t^{(n-1)p^-_i}}{\sum_{r=1}^{n} t^{(n-1)p^-_i}} = \frac{(t+1)^{p^-_i} - 1}{t+1}.
\]
A similar computation holds for \( \lambda^+_i = \lambda_1^+(p^+_i) \). Note also that \( \lambda^-_i \) and \( \lambda^+_i \) do satisfy the requirements of a weighted implementation, since they are strictly increasing functions of the passive and active compensatory powers \( p^-_i \) and \( p^+_i \), respectively.

Example 7. Let \( G \) be composed of seven evaluation criteria, ordered lexicographically by \( g_1 \gg g_2 \gg \cdots \gg g_7 \). Assume that the parameter \( t \geq 2 \) used in property L3 of Definition 9 to obtain the weights \( \lambda^-_i \) and \( \lambda^+_i \) is \( t := 2 \). Then \( \lambda^-_i \) and \( \lambda^+_i \) have the following value for each \( i \in J \):
\[
\lambda^-_i = \lambda^+_i = \frac{2^{t^{-i}}}{2^{t+1}}.
\]
Lexicographic implementations belong to a special category of regular implementations.

Lemma 4. A lexicographic implementation of PACMAN is co-symmetric.

Proof. Condition CS1 follows from condition L1 and Example 6(iii). Condition CS2 is a consequence of conditions L1 and L2. By condition L3, a lexicographic implementation is weighted; thus condition CS3 holds because of condition W1. Finally, condition CS4 follows from condition W2 of a weighted implementation and from the definition of \( \lambda^+_i \) and \( \lambda^-_i \) as given in condition L3. \( \square \)

If we use a lexicographic implementation of PACMAN, then all binary indices have a simple form. The next result computes these indices in a special case, namely, \( b \sim_{\text{lex}} a \) but alternative \( a \) is evaluated better than alternative \( b \) only on criterion \( g_i \), whereas \( a_i < b_i \) on all criteria \( g \), less important than \( g_i \).

Lemma 5. Assume that a lexicographic implementation of PACMAN is used. Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be elements of \( A \) such that there exists \( \ell \leq n \) such that \( a_i = b_i \) for each \( 1 \leq i < \ell, a_\ell > b_\ell \), and \( a_i < b_i \) for each \( \ell < i \leq n \). Then the values of the active and passive binary indices associated to \( a, b \in A \) are the following:

(i) Elementary indices:
\[
\Pi^+_{\beta_i}(a, b) = \Pi^-_{\beta_i}(a, b) = \begin{cases} 0 & \text{if } i \neq \ell \text{ and } j \neq i, \\ 1 & \text{if } i = \ell \text{ and } j \neq i, \\ 0 & \text{if } i \leq \ell \text{ and } j \neq i, \end{cases}
\]
\[
\Pi^+_{\beta_i}(b, a) = \Pi^-_{\beta_i}(b, a) = \begin{cases} 0 & \text{if } i > \ell \text{ and } j = \ell, \\ 1 & \text{if } i > \ell \text{ and } j \neq i, \ell. \end{cases}
\]
(ii) Partial indices:

\[ \Pi^+_i(a, b) = \Pi_i(a, b) = \begin{cases} 0 & \text{if } i \neq \ell, \\
1 & \text{if } i = \ell, \end{cases} \]

\[ \Pi^+_i(b, a) = \Pi_i(b, a) = \begin{cases} 0 & \text{if } i \leq \ell, \\
\frac{n^i - 2}{n - 1} & \text{if } i > \ell. \end{cases} \]

(iii) Global indices:

\[ \Pi^+(a, b) = \Pi^-(a, b) = \frac{(t - 1)n^{\ell - \ell}}{t^\ell - 1}, \]

\[ \Pi^+(b, a) = \Pi^-(b, a) = \frac{t^{\ell - \ell} - 1}{n - 1}. \]

Proof. The proof of (i) is a straightforward computation and is left to the reader. Next, we prove (ii). Since we are using a lexicographic implementation of PACMAN, we have

\[ \Pi^+_i(a, b) = \Phi^+_i(\Pi^+_i(a, b))_{\varphi_i} = \begin{cases} \Phi^+_i(0, 0, \ldots, 0) = 0 & \text{if } i \neq \ell, \\
\Phi^+_i(1, 1, \ldots, 1) = 1 & \text{if } i = \ell, \end{cases} \]

\[ \Pi^+_i(b, a) = \Phi^+_i(\Pi^+_i(b, a))_{\varphi_i} = \begin{cases} \Phi^+_i(0, 0, \ldots, 0) = 0 & \text{if } i \leq \ell, \\
\Phi^+_i(1, 1, 0, 1, \ldots, 1) = \frac{n^i - 2}{n - 1} & \text{if } i > \ell, \end{cases} \]

where the elementary indices \( \Pi^+_i(a, b) \) and \( \Pi^+_i(b, a) \) are computed using part (i). A joint application of Lemmas 4 and 3(ii) yields \( \Pi^+_i(a, b) = \Pi^+_i(a, b) \) and \( \Pi^+_i(b, a) = \Pi^+_i(b, a) \). The claim follows.

Finally, we show that (iii) holds. Indeed, the equalities in (ii) yield

\[ \Pi^+(a, b) = \Phi^+_l(\Pi^+_i(a, b))_{\ell \leq n} = \lambda_i \Pi^+_i(a, b) = \frac{(t - 1)n^{\ell - \ell}}{t^\ell - 1} \]

and

\[ \Pi^+(b, a) = \Phi^+_l(\Pi^+_i(b, a))_{\ell \leq n} = \lambda_{i+1} \Pi^+_i(b, a) + \cdots + \lambda_n \Pi^+_i(b, a) \]

\[ = \frac{(t - 1)n^{\ell - 1} n - 2}{t^\ell - 1 n - 1} \frac{n^2 - 2}{n - 1} + \cdots + \frac{(t - 1)n^{\ell - n} n - 2}{t^\ell - 1 n - 1} \]

\[ = \frac{n^{\ell - 1} + n^{\ell - 2} + \cdots + n}{t^\ell - 1 n - 1} \frac{t - 1 n - 2}{n - 1} = \frac{t^{\ell - \ell} - 1}{n - 1}. \]

Now a joint application of Lemmas 4 and 3(iii) completes the proof. \( \square \)

We are ready to prove the main result of this section.

Theorem 3. For each \( a, b \in A \), we have \( b \prec_{\text{lex}} a \) if and only if \( aP_{\text{lex}} b \).

Proof. Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) two alternatives in \( A \), i.e., \( g_i(a) = a_i \) and \( g_i(b) = b_i \) for each \( i \leq n \). Note that since by hypothesis we are using a lexicographic implementation of PACMAN, the following chain of equivalences holds:

\[ aP_{\text{lex}} b \iff aY_{\text{lex}}^b \text{ and } bY_{\text{lex}}^a \]

\[ \iff \Pi(a, b) \in (\varepsilon_{\text{lex}}, 1] \quad \text{and} \quad \Pi(b, a) \in [-1, -\varepsilon_{\text{lex}}) \]

\[ \iff \Pi(a, b) > 0 \quad \text{and} \quad \Pi(b, a) < 0 \]

\[ \iff \Pi^+(a, b) > \Pi^-(b, a) \quad \text{and} \quad \Pi^+(b, a) < \Pi^-(a, b). \]

Next, we show that \( b \prec_{\text{lex}} a \) if and only if \( aP_{\text{lex}} b \).

\([\Rightarrow]\). Let \( b \prec_{\text{lex}} a \), with \( b_\ell < a_\ell \) for some \( \ell \leq n \), and \( a_i = b_i \) for each \( i < \ell \). By Lemma 4 and Theorem 1(i), it suffices to prove that \( aP_{\text{lex}} b \) holds in the case that \( a_i := \alpha_i = \min g_i \) and \( b_i := \beta_i = \max g_i \) for each \( i > \ell \). Note that for each \( t \geq 2 \), we have

\[ (t - 1)n^{\ell - \ell} > t^{\ell - \ell} - 1 > (t^{\ell - \ell} - 1) \frac{n - 2}{n - 1}. \]
Therefore, Lemma 5(iii) yields
\[ \Pi^+(a, b) > \Pi^-(b, a) \quad \text{and} \quad \Pi^+(b, a) < \Pi^-(a, b). \]

Now we can use the chain of equivalences given above to obtain \( a P a \).

We prove the contrapositive. Therefore, we assume that \( b \not< a \) and show that \( a \not< b \). If \( b \not< a \), then either \( a \not< b \) or \( a = b \). In the case that \( a \not< b \), the first part of the proof yields \( b P a \). On the other hand, if \( a = b \) then \( a P b \), using Example 5. In any case, we have \( a \not< b \), as claimed. \[ \square \]

In the Appendix we will give several examples of lexicographic implementations of PACMAN.

9. Conclusions

In this paper we have analyzed some theoretical aspects of PACMAN, a multiple criteria methodology based on a peculiar notion of intercriteria compensation. This notion, called compensability, is DM-oriented in the sense that it is not fixed \( a \) priori by the methodology, instead it is determined on the basis of information \( ad \) hoc provided by the DM. The first phase of PACMAN is indeed compensability analysis, which is the procedure that translates into numerical form this piece of intercriteria information.

The basic step of compensability analysis is the construction of a compensatory function for each ordered pair of distinct criteria. This construction is rather elaborate and lengthy, but it sheds light onto the DM’s scheme of preferences. In fact, a suitable elaboration/aggregation of the informative content of compensatory functions allows the DA to obtain additional insight into the DM’s scheme of preferences by

- associating to each criterion a pair of indices \((p^+, p^-)\), the active and passive compensatory powers of \( g \); these indices evaluate the two faces (active and passive) of a notion of compensatory weight/importance of a criterion;
- computing an index \( \gamma_{DM}(J) \) of the DM’s aptitude to compensate among criteria; note that this index quantifies the compensatoriness of the DM and not of the methodology.

PACMAN is a rather flexible multiple criteria methodology, since many parameters and aggregation functions of the model are to be determined by the DM according to his/her scheme of preferences. In this paper we have attempted to balance this flexibility and a concrete implementability of PACMAN, introducing some axioms and limiting the choices for the functions used in the modelization. We have formally defined a notion of implementation of PACMAN, and studied several types of implementations, \( e.g. \), regular, co-symmetric and lexicographic.

Regular implementations are those that satisfy some natural axioms. Therefore, all implementations examined in this paper have been assumed to be regular. We have shown that regular implementations satisfy some basic monotonicity properties with respect to dominance between alternatives.

Despite PACMAN’s approach to compensability is asymmetric in principle, there are regular implementations that satisfy some symmetry properties. These particular implementations, called co-symmetric, are characterized by the fact that all active functions used in the modelization are equal to the respective passive functions. The principal properties of a co-symmetric implementation are:

- the computations of binary indices are halved;
- the corresponding relational system of preferences is basic, being composed of preference and indifference only.

Finally, we have shown that there are co-symmetric implementations of PACMAN that allow one to represent the lexicographic ordering. These implementations, called lexicographic, employ weighted sums as the aggregation functions used to obtain global indices, with the weights being increasing functions of the passive and active compensatory power of each criterion. It is interesting to observe that once the criteria are ordered lexicographically, in PACMAN an \textit{universal selection} of the weights can restore the lexicographic ordering on an \textit{arbitrary} set of alternatives.

Future research will be concentrated on other operational aspects of PACMAN. The most relevant difficulty in applying this decision procedure is indeed the construction of compensatory functions. Each compensatory function requires a careful interaction between the DA and the DM, and there are a lot of these functions to be built. Thus the DM is asked to provide a great amount of data, which becomes excessive in case that there are many evaluation criteria. Furthermore, in some cases the DM might be unable to give even the very basic information needed for the construction of some compensatory functions.
A possible way to deal with these cases of incomplete information is to use the reference compensatory function and some variations of it to make up for lack of information. More specifically, one could first compute a partial DM compensability index \( \gamma_{\text{DM}}(J) \) on the basis of the existing information, and then employ (for the lacking compensatory functions) some suitable variations of the reference compensatory function, in a way such that \( \gamma_{\text{DM}}(J) = \gamma_{\text{DM}}(J) \). In other words, the DM compensability index is used as an invariant of the DM’s beliefs about compensability, so as to make up for incomplete information.

Appendix

In the sequel we give some numerical examples\(^1\) in order to illustrate a lexicographic implementation of PACMAN. These examples show how the implementation is sensitive to both the evaluation of alternatives and the choice of the parameter \( t \geq 2 \). (The parameter \( t \) appears in the weights used to obtain global indices from partial indices as a weighted sum, see property L3 in the definition of a lexicographic implementation.)

Let \( G = \{ g_1, g_2, \ldots, g_7 \} \) be a set of criteria, which are ordered lexicographically by the DM in a decreasing order of importance, i.e., \( g_1 > g_2 > \cdots > g_7 \). For simplicity, assume that each criterion evaluates feasible actions of \( A \) on a \([0,20]\) scale, i.e., \( \text{range}(g_i) = [0,20] \) for each \( i \in J = \{1,2,\ldots,7\} \).

Let \( a, b \in A \) be two alternatives evaluated by \( G \) as in Table 1. Note that \( a \succ_{\text{lex}} b \), since \( a_1 = b_1, a_2 = b_2 \) and \( a_3 > b_3 \), even if \( a_4 = 0 = \min g_i \) and \( b_4 = 20 = \max g_i \) for each \( i \in \{4,5,6,7\} \). The corresponding normalized differences are listed in Table 2, where the differences \( \Delta_i(a,b) = (a_i - b_i)/20 \) appear in the first row, and the differences \( \Delta_i(b,a) = (b_i - a_i)/20 \) in the second row.

In the sequel we give a detailed description of a lexicographic implementation of PACMAN such that \( t = 2 \) (see property L3 in Definition 9). According to Theorem 3, we will obtain \( aP_{\text{lex}} b \).

Phase I: Compensatory functions are defined according to a lexicographic implementation, i.e., \( \text{CF}_{a/b} = 1 \) if \( i < j \) and \( \text{CF}_{a/b} = 0 \) if \( i > j \).

Phase II: We compute binary indices using Lemma 5. We start with elementary indices. The evaluations of all active and passive elementary indices \( \Pi^+_i(a,b), \Pi^-_i(a,b), \Pi^+_j(a,b) \) and \( \Pi^-_j(b,a) \) are shown, respectively, in Tables 3–6. Each table contains a \( 7 \times 7 \) matrix, where active criteria are listed on rows and passive criteria on columns. (Note that elementary indices are not computed when \( i = j \).) For example, the framed element in Table 3 represents \( \Pi^+_i(a,b) \), whose value is 1 by Lemma 5(i). Furthermore, Lemma 5(ii) yields that the \( 7 \times 7 \) matrix contained in Table 3 (respectively, Table 5) is the transpose of the \( 7 \times 7 \) matrix contained in Table 4 (respectively, Table 6).

Next, we compute partial indices related to \( a, b \in A \). The values of all active and passive partial indices \( \Pi^+_i(a,b), \Pi^-_i(a,b), \Pi^+_j(b,a) \) and \( \Pi^-_j(b,a) \) are listed in Table 7. Note that, according to Lemma 5(ii), active and passive partial indices related to the same ordered pair of alternatives have the same value. Since by definition a lexicographic implementation is weighted, partial indices are calculated as averages of the corresponding elementary indices. (Note that the elementary indices to be averaged are all the elements of a row if we compute active indices, and all the elements of a column if we compute passive indices.) For example, \( \Pi^+_i(a,b) = 0 \) is obtained averaging the values of the second row of the matrix given in Table 3, whereas \( \Pi^-_i(b,a) = 5/6 \) is obtained averaging the values of the fifth column of the matrix given in Table 6.

To finish to second stage of the modelization, we compute the four global active and passive indices related to the couple \( \{a,b\} \subseteq A \). These global indices are listed in Table 8. As an example, Table 9 describes the computation of the active global index \( \Pi^+(b,a) \). Note that the weights \( \lambda^+_i \) are those already computed in Example 7.

Phase III: The two relations of compensated preference of \( a \) over \( b \) and of \( b \) over \( a \) are evaluated by computing, respectively, the net global index \( \Pi(a,b) = \Pi^+(a,b) - \Pi^-(b,a) \) and the net global index \( \Pi(b,a) = \Pi^+(b,a) - \Pi^-(a,b) \). Their (rounded-off) values are given in Table 10.

Since by definition a lexicographic implementation is crisp (i.e., \( e = 0 \)), we obtain \( a \mathcal{T}^\gamma b \) and \( b \mathcal{T}^\gamma a \). It follows that \( aP_{\text{lex}} b \), as claimed.

Observe that if as parameter \( t \geq 2 \) we use \( t = 3,4,5 \) in place of \( t = 2 \), then active, passive and net global indices have the (rounded-off) values given in Table 11.

Finally, for a better illustration of a lexicographic implementation of PACMAN, we consider other four examples where \( a \succ_{\text{lex}} b \). The evaluations of alternatives \( a, b \in A \) are given, respectively, in Tables 12–15. Note that in the first two examples we have \( aDb \). The corresponding (rounded-off) values of the global indices are given, respectively, in Tables 16–19 for several values of the parameter \( t \).

---

\(^1\) For all computations we use a MATLAB code.
Table 1
Evaluations of alternatives

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</tr>
<tr>
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Table 2
Normalized differences

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<th>$A_4$</th>
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<td>1/20</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
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<tr>
<td>$(b, a)$</td>
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<td>-1/20</td>
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<td>1</td>
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Table 3
Active elementary indices for the pair $(a, b)$

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<th>$\Pi^+_{a,b}(a, b)$</th>
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<th>7</th>
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<td>0</td>
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Table 4
Passive elementary indices for the pair $(a, b)$

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Table 5
Active elementary indices for the pair $(b, a)$

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<th>$\Pi^+_{b,a}(b, a)$</th>
<th>1</th>
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<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6
Passive elementary indices for the pair $(b, a)$

<table>
<thead>
<tr>
<th></th>
<th>$\Pi^-_{b,a}(b, a)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 7
Partial indices

<table>
<thead>
<tr>
<th>i</th>
<th>$H^+_i(a,b)$</th>
<th>$H^-_i(a,b)$</th>
<th>$H^+_i(b,a)$</th>
<th>$H^-_i(b,a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>5/6</td>
<td>5/6</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>5/6</td>
<td>5/6</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>5/6</td>
<td>5/6</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>5/6</td>
<td>5/6</td>
</tr>
</tbody>
</table>

Table 8
Global indices

<table>
<thead>
<tr>
<th>$H^+(a,b)$</th>
<th>$H^-(a,b)$</th>
<th>$H^+(b,a)$</th>
<th>$H^-(b,a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.126</td>
<td>0.126</td>
<td>0.098</td>
<td>0.098</td>
</tr>
</tbody>
</table>

Table 9
Computation of the active global index $H^+(b,a)$

<table>
<thead>
<tr>
<th>i</th>
<th>$H^+_i(b,a)$</th>
<th>$\lambda^+_i$</th>
<th>$\lambda^+ H^+_i(b,a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.504</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.252</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.126</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5/6</td>
<td>0.063</td>
<td>0.052</td>
</tr>
<tr>
<td>5</td>
<td>5/6</td>
<td>0.031</td>
<td>0.026</td>
</tr>
<tr>
<td>6</td>
<td>5/6</td>
<td>0.016</td>
<td>0.013</td>
</tr>
<tr>
<td>7</td>
<td>5/6</td>
<td>0.008</td>
<td>0.007</td>
</tr>
<tr>
<td>$\sum$</td>
<td>1</td>
<td></td>
<td>0.098</td>
</tr>
</tbody>
</table>

Table 10
Net global indices

<table>
<thead>
<tr>
<th>$\Pi(a,b)$</th>
<th>$\Pi(b,a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.028</td>
<td>-0.028</td>
</tr>
</tbody>
</table>

Table 11
Global indices (for $t = 3, 4, 5$)

<table>
<thead>
<tr>
<th>t</th>
<th>$\Pi^+(a,b)$</th>
<th>$\Pi^-(a,b)$</th>
<th>$\Pi^+(b,a)$</th>
<th>$\Pi^-(b,a)$</th>
<th>$\Pi(a,b)$</th>
<th>$\Pi(b,a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.074</td>
<td>0.074</td>
<td>0.031</td>
<td>0.031</td>
<td>0.043</td>
<td>-0.043</td>
</tr>
<tr>
<td>4</td>
<td>0.047</td>
<td>0.047</td>
<td>0.013</td>
<td>0.013</td>
<td>0.034</td>
<td>-0.034</td>
</tr>
<tr>
<td>5</td>
<td>0.032</td>
<td>0.032</td>
<td>0.007</td>
<td>0.007</td>
<td>0.025</td>
<td>-0.025</td>
</tr>
</tbody>
</table>

Table 12
Evaluations of alternatives (Example 2)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>b</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 13
Evaluations of alternatives (Example 3)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>b</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>
Table 14
Evaluations of alternatives (Example 4)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>9</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 15
Evaluations of alternatives (Example 5)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 16
Global indices (Example 2)

<table>
<thead>
<tr>
<th>t</th>
<th>(\Pi^+(a,b))</th>
<th>(\Pi^-(a,b))</th>
<th>(\Pi^+(b,a))</th>
<th>(\Pi^-(b,a))</th>
<th>(\Pi(a,b))</th>
<th>(\Pi(b,a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.008</td>
<td>0.008</td>
<td>0</td>
<td>0</td>
<td>0.008</td>
<td>-0.008</td>
</tr>
<tr>
<td>3</td>
<td>0.00092</td>
<td>0.00092</td>
<td>0</td>
<td>0</td>
<td>0.00092</td>
<td>-0.00092</td>
</tr>
<tr>
<td>4</td>
<td>0.00018</td>
<td>0.00018</td>
<td>0</td>
<td>0</td>
<td>0.00018</td>
<td>-0.00018</td>
</tr>
<tr>
<td>5</td>
<td>0.000051</td>
<td>0.000051</td>
<td>0</td>
<td>0</td>
<td>0.000051</td>
<td>-0.000051</td>
</tr>
</tbody>
</table>

Table 17
Global indices (Example 3)

<table>
<thead>
<tr>
<th>t</th>
<th>(\Pi^+(a,b))</th>
<th>(\Pi^-(a,b))</th>
<th>(\Pi^+(b,a))</th>
<th>(\Pi^-(b,a))</th>
<th>(\Pi(a,b))</th>
<th>(\Pi(b,a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.496</td>
<td>0.496</td>
<td>0</td>
<td>0</td>
<td>0.496</td>
<td>-0.496</td>
</tr>
<tr>
<td>3</td>
<td>0.333</td>
<td>0.333</td>
<td>0</td>
<td>0</td>
<td>0.333</td>
<td>-0.333</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>0.25</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
<td>-0.25</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>-0.2</td>
</tr>
</tbody>
</table>

Table 18
Global indices (Example 4)

<table>
<thead>
<tr>
<th>t</th>
<th>(\Pi^+(a,b))</th>
<th>(\Pi^-(a,b))</th>
<th>(\Pi^+(b,a))</th>
<th>(\Pi^-(b,a))</th>
<th>(\Pi(a,b))</th>
<th>(\Pi(b,a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.504</td>
<td>0.504</td>
<td>0.413</td>
<td>0.413</td>
<td>0.091</td>
<td>-0.091</td>
</tr>
<tr>
<td>3</td>
<td>0.667</td>
<td>0.667</td>
<td>0.278</td>
<td>0.278</td>
<td>0.389</td>
<td>-0.389</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>0.75</td>
<td>0.208</td>
<td>0.208</td>
<td>0.542</td>
<td>-0.542</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>0.8</td>
<td>0.167</td>
<td>0.167</td>
<td>0.633</td>
<td>-0.633</td>
</tr>
</tbody>
</table>

Table 19
Global indices (Example 5)

<table>
<thead>
<tr>
<th>t</th>
<th>(\Pi^+(a,b))</th>
<th>(\Pi^-(a,b))</th>
<th>(\Pi^+(b,a))</th>
<th>(\Pi^-(b,a))</th>
<th>(\Pi(a,b))</th>
<th>(\Pi(b,a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.976</td>
<td>0.976</td>
<td>0.004</td>
<td>0.004</td>
<td>0.972</td>
<td>-0.972</td>
</tr>
<tr>
<td>3</td>
<td>0.9963</td>
<td>0.9963</td>
<td>0.0006</td>
<td>0.0006</td>
<td>0.9957</td>
<td>-0.9957</td>
</tr>
<tr>
<td>4</td>
<td>0.9991</td>
<td>0.9991</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.9989</td>
<td>-0.9989</td>
</tr>
<tr>
<td>5</td>
<td>0.9997</td>
<td>0.9997</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.9996</td>
<td>-0.9996</td>
</tr>
</tbody>
</table>

References