A New Axiomatization for Involutive Monoidal T-norm-based Logic

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Abstract

On the real unit interval, the notion of a Girard monoid coincides with the notion of a t-norm-based residuated lattice with strong induced negation. A geometrical approach toward these Girard monoids, based on the notion of rotation invariance, is turned in an adequate axiomatization for the Involutive Monoidal T-norm-based residuated Logic (IMTL).

1 Introduction

A system $S_T$ of many-valued logic is called $t$-norm-based residuated (upon some particular t-norm $T$) iff $S_T$ has a conjunction connective $\&_T$ with truth degree function $T$, and all the other connectives of $S_T$ have associated truth degree functions which are defined from $T$ (using possibly some truth degree constants). It is important here that one considers together with the conjunction $\&_T$ an implication connective $\rightarrow_T$ with truth degree function $I_T$ characterized by the adjointness condition$^1, 2$

$$T(x, y) \leq z \iff x \leq I_T(y, z), \quad (1)$$

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$^1$In another terminology, $I_T$ and $T$ form a residual pair.

$^2$Throughout this introduction $x$ and $y$ are arbitrary elements in $[0,1]$. 
and also a standard negation connective $-T$ with truth degree function $N_T$, given as

$$N_T(x) = I_T(x, 0).$$

(2)

The t-norm $T$ determines a reasonable\(^3\) implication function via (1). As is well known, the adjointness condition forces the t-norm to be left-continuous. That is, a t-norm $T$ determines a function $I_T$ via (1) just in the case that $T$ is left-continuous.

Obviously the properties of the negation essentially depend on properties of the t-norm $T$. So one has for each left-continuous t-norm $T$, as shown e.g. in [4, 9], in the case that $T$ does not have zero divisors a very simple negation function $N_T$: $N_T(1) = 0$ and $N_T(u) = 1$ for all $u \neq 0$. In all these cases, hence, $-T$ is not an involutive negation.

Sometimes one adds to such a system $S_T$ a separate negation connective $\neg$ with a truth degree function $N$ such that $\neg$ becomes an involutive negation, i.e. a negation with a truth degree function over $[0,1]$ which is a decreasing bijection, and hence continuous, and satisfies $N(N(x)) = x$ as well. This has e.g. been done in [2]. The situation, however, that $N_T$ is itself involutive is the case we are interested here.

The problem of adequate axiomatizations of such logics is a current research topic. Mainly one is interested to axiomatize the core systems for whole classes of such t-norm-based residuated logics. To reach this goal one determines such logics by algebraic semantics, constituted by classes of truth degree structures\(^4\) which are taken as equational classes such that the structures which form these classes reflect basic (algebraic) properties of the t-norms which constitute the logics. Because of the implication connective $\rightarrow_T$ these classes of algebraic structures are classes of residuated lattices, attached with a commutative product. Main recent results are surveyed e.g. in [3], and explained in more detail e.g. in [4, 5].

As shown in [8] the graph of a t-norm is rotation invariant in a geometric sense if and only if this t-norm is residuated and its induced negation is strong. Motivated by this result, two algebraic constructions (called rotation construction and rotation-annihilation construction) are introduced. The corresponding rotation invariance is the property

$$T(x, y) \leq z \iff T(y, N(z)) \leq N(x).$$

(3)

To visualize it, when the negation $N$ is the standard negation $1-x$, see Figure 1 where rotations of certain ordinal sums\(^5\) are shown. On the right-hand side the ordinal sum has two summands: the Łukasiewicz t-norm and the product t-norm. On the left-hand side the summand is the rotation of the product. Observe the invariance of their graphs under the rotation of $[0,1]^3$ with angle $\frac{2\pi}{3}$ which leaves the points $(0,0,1)$ and $(1,1,0)$ fixed.

This rotation construction seems to be of basic importance for t-norms which has an involutive internal negation. And the present paper indicates that also within the realm of

\(^3\)Here “reasonable” essentially means that $\rightarrow_T$ satisfies a suitable version of the rule of detachment.

\(^4\)This is the same style of characterization of a logic as e.g. classical logic may be determined by the class of all Boolean algebras, or as infinite valued Łukasiewicz logic may be determined by the class of all MV-algebras.

\(^5\)For an overview on ordinal sums see, e.g., [9].
the corresponding t-norm-based residuated logics with involutive (standard) negation the rotation invariance property (3) has a similar fundamental character as the adjointness condition (1).

Finally, a comment concerning notation has to be made. Chains of implications of the form $\varphi \to \psi \to \chi$ (without parentheses) will be used instead of: $\varphi \to \psi$ and $\psi \to \chi$.

## 2 Two Equivalent Axiomatizations

The monoidal t-norm-based logic ($\text{MTL}$) was adequately axiomatized in [1], also for the case that one has an involutive (standard) negation. We shall use here the fact that the t-norm-based residuated lattices in $[0,1]$ with a strong induced negation, i.e. the t-norm-based residuated lattices in $[0,1]$ which have as their standard negation an involutive one, can also be characterized as determined by t-norms which have the rotation invariance property (3). This leads us to an axiomatization different from the one given in [1], but equivalent to it, and using in the axiom schemata the reference to the rotation invariance property instead of the reference to the adjointness condition (1).

Both axiomatizations, i.e. the one of [1] and our present one, will have some common core axioms, determined by the schemata:

- **Ax1** $((\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
- **Ax2** $\varphi \& \psi \to \psi \& \varphi$,
- **Ax3** $\varphi \land \psi \to \varphi$,
- **Ax4** $\varphi \land \psi \to \psi \land \varphi$,
And both axiomatizations will have the rule of detachment (w.r.t. the implication $\rightarrow$) as their only inference rule.

From these core axioms, $\text{Ax}1$ formalizes the transitivity of the implication connective, and $\text{Ax}5$ the pre-linearity condition.

For the monoidal t-norm logic with an (internal) involutive negation $\text{IMTL}$ the corresponding logical calculus was introduced by Esteva/Godo [1]. The system $\text{IMTL}$ is the most general one for logics based upon left-continuous t-norms and having an internal involutive negation. It expands the above mentioned core system of axioms by the schemata

\begin{align*}
\text{Ax}_{\text{EG}7} & \quad \varphi \land \psi \rightarrow \varphi, \\
\text{Ax}_{\text{EG}8} & \quad \varphi \land (\varphi \rightarrow \psi) \rightarrow \varphi \land \psi, \\
\text{Ax}_{\text{EG}9} & \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi \rightarrow \chi), \\
\text{Ax}_{\text{EG}10} & \quad (\varphi \land \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)), \\
\text{Ax}_{\text{EG}11} & \quad \overline{0} \rightarrow \varphi, \\
\text{Ax}_{\text{EG}12} & \quad (\varphi \rightarrow \overline{0}) \rightarrow \neg \varphi, \\
\text{Ax}_{\text{EG}13} & \quad \neg \varphi \rightarrow (\varphi \rightarrow \overline{0}).
\end{align*}

Together with other interesting results, Esteva/Godo [1] gave a completeness proof for this logic $\text{IMTL}$.

We shall give a syntactical proof that the following logical calculus $\text{RMTL}$, our “Rotation invariant Monoidal T-norm-based Logic”, is an equivalent axiomatization of the same logic $\text{IMTL}$. This calculus $\text{RMTL}$ is determined by the above mentioned core system together with the axiom schemata:

\begin{align*}
\text{Ax}_{\text{GJ}7} & \quad \varphi \land \psi \rightarrow \varphi \land \psi, \\
\text{Ax}_{\text{GJ}8} & \quad (\varphi \land \psi \rightarrow \chi) \rightarrow (\psi \land \neg \chi \rightarrow \neg \varphi), \\
\text{Ax}_{\text{GJ}9} & \quad (\psi \land \neg \chi \rightarrow \neg \varphi) \rightarrow (\varphi \land \psi \rightarrow \chi), \\
\text{Ax}_{\text{GJ}10} & \quad (\varphi \rightarrow \psi) \rightarrow \neg (\varphi \land \neg \psi), \\
\text{Ax}_{\text{GJ}11} & \quad \neg (\varphi \land \neg \psi) \rightarrow (\varphi \rightarrow \psi), \\
\text{Ax}_{\text{GJ}12} & \quad \varphi \rightarrow \varphi \land \overline{1}, \\
\text{Ax}_{\text{GJ}13} & \quad \varphi \rightarrow \neg \neg \varphi.
\end{align*}
Obviously, the scheme $\text{Ax}_{\text{EG}}7$ could be replaced within the system IMTL by the scheme $\text{Ax}_{\text{GJ}}7$, i.e. the list of core axioms could be extended by that scheme.

**Proposition 1** All the axiom schemata $\text{Ax}_{\text{GJ}}6$ to $\text{Ax}_{\text{GJ}}13$ of the logical calculus RMTL can be derived within the system IMTL.

**Proof:** The schemata $\text{Ax}_{\text{GJ}}8$ and $\text{Ax}_{\text{GJ}}9$ are immediate corollaries of the IMTL-derived formula (38) of [1]. And the schemata $\text{Ax}_{\text{GJ}}10$ and $\text{Ax}_{\text{GJ}}11$ follow similarly from the IMTL-derived formula (39) of that paper. Furthermore $\text{Ax}_{\text{GJ}}12$ is easily derived from formula (20) of [1] via the core axioms $\text{Ax}1$ and $\text{Ax}2$, using the definition $\bar{1} =_{\text{def}} 0 \rightarrow 0$ as in [1].

For axiom schema $\text{Ax}_{\text{GJ}}13$ it is sufficient to derive $\varphi \rightarrow ((\varphi \rightarrow 0) \rightarrow 0)$, and for this formula it is sufficient to derive $\varphi \& (\varphi \rightarrow 0) \rightarrow 0$. And this is obviously a derivable formula.

So it remains to look at $\text{Ax}_{\text{GJ}}7$. But this is mentioned as IMTL-derivable in [1], formula (9) of Proposition 1. \qed

**Proposition 2** All the axiom schemata $\text{Ax}_{\text{EG}}6$ to $\text{Ax}_{\text{EG}}13$ of IMTL can be derived within the logical calculus RMTL.

**Proof:** To simplify the following derivations inside RMTL, let us first remark that one has

$$(\varphi \& \psi \rightarrow \chi) \rightarrow (\psi \& \varphi \rightarrow \chi) \quad (4)$$

from $\text{Ax}2$ and $\text{Ax}1$, here specified as

$$(\psi \& \varphi \rightarrow \varphi \& \psi) \rightarrow ((\varphi \& \psi \rightarrow \chi) \rightarrow (\psi \& \varphi \rightarrow \chi)).$$

And furthermore one gets scheme $\text{Ax}_{\text{EG}}7$

$$\varphi \& \psi \rightarrow \varphi \quad (5)$$

from $\text{Ax}_{\text{GJ}}7$, $\text{Ax}3$ via $\text{Ax}1$. So one has $\varphi \& \bar{1} \rightarrow \varphi$ besides $\text{Ax}_{\text{GJ}}12$, and hence also

$$\varphi \rightarrow \varphi \quad (6)$$

from

$$(\varphi \rightarrow \varphi \& \bar{1}) \rightarrow ((\varphi \& \bar{1} \rightarrow \varphi) \rightarrow (\varphi \rightarrow \varphi)).$$

Using rotation invariance $\text{Ax}_{\text{GJ}}8$ with $\bar{1}$ for $\psi$, one gets

$$(\varphi \& \bar{1} \rightarrow \chi) \rightarrow (\bar{1} \& \neg \chi \rightarrow \neg \varphi) .$$
and hence from $\varphi \& \bar{1} \rightarrow \varphi$ and $\text{Ax}1$ in the form
\[(\varphi \& \bar{1} \rightarrow \varphi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \& \bar{1} \rightarrow \chi))\]
the formula
\[(\varphi \rightarrow \chi) \rightarrow (\varphi \& \bar{1} \rightarrow \chi).\]

So one arrives at the chain of implications
\[(\varphi \rightarrow \chi) \rightarrow (\bar{1} \& \neg \chi \rightarrow \neg \varphi) \rightarrow (\neg \chi \& \bar{1} \rightarrow \neg \varphi).\]

But in general one has
\[(\psi \& \bar{1} \rightarrow \varrho) \rightarrow (\psi \rightarrow \varrho)\]
because of
\[(\psi \rightarrow \psi \& \bar{1}) \rightarrow ((\psi \& \bar{1} \rightarrow \varrho) \rightarrow (\psi \rightarrow \varrho))\]
and $\psi \rightarrow \psi \& \bar{1}$. Hence one finally has a first form of the contraposition law:
\[(\varphi \rightarrow \chi) \rightarrow (\neg \chi \rightarrow \neg \varphi).\] (7)

Another form of the contraposition law comes from the chain of implications
\[(\varphi \rightarrow \neg \chi) \rightarrow (\varphi \& \bar{1} \rightarrow \neg \chi) \rightarrow (\bar{1} \& \neg \neg \chi \rightarrow \neg \varphi) \rightarrow (\neg \neg \chi \rightarrow \neg \varphi) \rightarrow (\chi \rightarrow \neg \varphi)\]
which uses (5), (4), $\text{Ax}_{\text{GJ}}8$ and $\text{Ax}_{\text{GJ}}13$, and which gives
\[(\varphi \rightarrow \neg \chi) \rightarrow (\chi \rightarrow \neg \varphi).\] (8)

Furthermore one has the implication chain
\[(\varphi \& \psi \rightarrow \chi) \rightarrow (\psi \& \neg \chi \rightarrow \neg \varphi) \rightarrow (\neg \neg \varphi \rightarrow \neg (\psi \& \neg \chi))\]
by rotation invariance $\text{Ax}_{\text{GJ}}8$ and contraposition (7). With $\text{Ax}_{\text{GJ}}11$ this gives also
\[(\neg \neg \varphi \rightarrow \neg (\psi \& \neg \chi)) \rightarrow (\neg \neg \varphi \rightarrow (\psi \rightarrow \chi)),\]
and hence at all
\[(\varphi \& \psi \rightarrow \chi) \rightarrow (\neg \neg \varphi \rightarrow (\psi \rightarrow \chi)).\]
Now $\text{Ax}1$ together with $\text{Ax}_{\text{GJ}}13$ immediately yield the exportation law $\text{Ax}_{\text{EG}}10$:

$$((\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)).$$

(9)

Having also in mind that one gets from $\text{Ax}_{\text{GJ}}10$ and (8) the chain of implications

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \neg (\psi \& \neg \chi)) \rightarrow (\psi \& \neg \chi \rightarrow \neg \varphi),$$

and that one can continue it via two applications of the rotation scheme $\text{Ax}_{\text{GJ}}8$ as

$$(\psi \& \neg \chi \rightarrow \neg \varphi) \rightarrow (\neg \chi \& \neg \neg \varphi \rightarrow \neg \psi) \rightarrow (\neg \neg \varphi \& \neg \neg \psi \rightarrow \neg \neg \chi).$$

(10)

Now the second form of the rotation scheme $\text{Ax}_{\text{GJ}}9$ gives the chain of implications

$$(\neg \neg \varphi \& \neg \neg \psi \rightarrow \neg \neg \chi) \rightarrow (\neg \chi \& \neg \neg \varphi \rightarrow \neg \psi) \rightarrow$$

$$(\psi \& \neg \chi \rightarrow \neg \varphi) \rightarrow (\varphi \& \psi \rightarrow \chi),$$

and hence together with (10) also the importation law $\text{Ax}_{\text{EG}}9$: Now let us define

$$\bar{0} =_{\text{def}} \neg \bar{1}.$$  

(11)

Then one has immediately

$$\neg \bar{0} \rightarrow \bar{1} \quad \text{and} \quad \bar{1} \rightarrow \neg \bar{0}.$$  

On the other hand one has $\neg \varphi \rightarrow \bar{1}$ from $\text{Ax}_{\text{GJ}}12$, $\text{Ax}2$, and (5), and therefore scheme $\text{Ax}_{\text{EG}}11$

$$\bar{0} \rightarrow \varphi$$  

(12)

via (8) and $\text{Ax}1$.

Also the following implications now are easily derived:

$$(\varphi \rightarrow \bar{0}) \rightarrow (\varphi \rightarrow \neg \bar{1}) \rightarrow (\bar{1} \rightarrow \neg \varphi).$$

Because one has via $\text{Ax}_{\text{GJ}}10$, $\text{Ax}_{\text{GJ}}12$, and $\text{Ax}2$ together with (7) the chain of implications

$$(\bar{1} \rightarrow \psi) \rightarrow \neg (\bar{1} \& \neg \psi) \rightarrow \neg \neg \psi \rightarrow \psi,$$

this gives together scheme $\text{Ax}_{\text{EG}}12$

$$(\varphi \rightarrow \bar{0}) \rightarrow \neg \varphi.$$  

(13)

Arguing as in [5], Lemma 2.2.7, one gets

$$\varphi \rightarrow (\psi \rightarrow \varphi).$$
from (5) together with (9), (10). So one arrives via the implications
\[ \varphi \to (\neg \bar{0} \to \varphi) \to (\varphi \& \neg \bar{0} \to \varphi) \]
and contraposition (7) at the formula \( \varphi \& \neg \bar{0} \to \varphi \), which immediately gives scheme \( \text{Ax}_{\text{EC}13} \)
\[ \neg \varphi \to (\varphi \to \bar{0}) \).

(14)

At all we have now derived all those formulas which act in [5] as axiom schemata for the logical calculus \( \text{BL} \) of basic t-norm logic, with the only exception of the schema (A4). So we have all those formulas derivable which have in [5] a derivation which does not refer to axiom schema (A4). In particular this means that we can derive in \( \text{RMTL} \) the formula
\[ ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to \psi \wedge \chi) , \]
which is formula (12) in Lemma 2.2.9 of [5]. Therefore we have via \( \text{Ax}_{\text{GJ}7}, \text{Ax}1, \) and (10) also
\[ (\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)) , \]
which means that we get scheme \( \text{Ax}_{\text{EC}8} \)
\[ \varphi \& (\varphi \to \psi) \to \varphi \wedge \psi \]
(15)
from \( \varphi \& (\varphi \to \psi) \to \varphi \), i.e. from (5), and from \( \varphi \& (\varphi \to \psi) \to \psi \). This last formula again can be derived as in [5].

All together now the proposition is proved. \( \square \)

3 Concluding Remarks

This paper shows that the geometrical understanding of Girard monoids [6] is useful not only in introducing interesting new algebraic tools like the rotation construction [7], and the rotation-annihilation construction for t-norms [8].

The main idea of rotation invariance can also be transformed into axiom schemata for logical calculi, and then serves as a substitute for the adjointness condition which usually is used to combine conjunction and implication connectives in t-norm-based residuated logics, as discussed, e.g., in [1, 3, 4, 5].

References


