A base function for generating contour traversal paths in stereolithography apparatus applications

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Abstract

The technique of layered manufacturing in rapid prototyping is to fabricate product prototype by scanning the cross-sectional contours of the product using a laser beam layer by layer. The outlines of geometrical objects on each layer are different, and each layer may contain several geometrical objects. In order to simplify the problem, each geometrical object on the same plane is approximating by its own minimum circumscribed circle. Therefore, the minimum traversal path of circles can be the based model of the minimum traversal path of scanning geometrical objects. Furthermore, the minimum traversal path of three circles is the degenerated case of the minimum traversal path of plural circles. And the problem of the minimum traversal path of three circles can be transferred to the problem of the minimum traversal path of one circle and two points in this paper. By using the concepts of reflection of light in physics and geometrical mathematics, the equation of the minimum traversal path of three circles is derived in this paper. This equation can be easily implemented in many areas of application, including robotic motion planning and path planning for submarine, ship, and airplane.

Keywords: Minimum traversal function; Stereolithography apparatus; Layered manufacturing; Path planning; Robotic motion planning; Analytic geometric function

1. Introduction

In order to provide fast and precise computation of the shortest path for the moving robot (Khatib, Craig, & Lozano-Perez, 1989; Nehmzow & Owen, 2000), circles are used to embrace obstacles in the robotic path planning. For example, there is an application on the mission planning software for the PHONIX autonomous underwater vehicle (AUV) (Brutzman, 1994; Leonhardt, 1999) of the US navy. It encloses the underwater obstacles by circles to avoid collisions that lead to the catastrophic loss of the vehicles. By using the tangents among circles, the PHONIX system derives the shortest path for the underwater vehicle rapidly as shown in Fig. 1. Quick deriving a precise shortest path is critical and crucial on the battlefield.

In the field of layered manufacturing (LM) of rapid prototyping, the well-known processes stereolithography apparatus (Jacobs, 1992) and laminated object manufacturing (LOM) (Chiu & Liao, 2003) fabricate the prototype of the product object with scanning the cross-sectional contours of the object layer by layer (Lan, Chou, Chen, & Gemmill, 1997). In order to minimize the total time for completing the fabrication, the minimization of the processing time for each scanning layer is thus essential. For each scanning layer, a laser beam scans along the contours in the layer to solidify the cross-section of the object. The path planning (Murty, 1995) for the scanning on each layer is therefore a traversal optimization problem with respect to the geometric entities in a two dimensional plane (Majhi, Janardan, Smid, & Gupta, 1999; Wah, Murty, Joneja, &
Fig. 2 illustrates the concepts of a LM example in rapid prototyping.

Since the shapes of objects on each scanning plane are different, it is very difficult to figure out the minimum traversal path among these objects. By using some smallest circumscribed circles to approximate these objects, the problem can be simplified as finding the minimum traversal path between circles (Tang & Pang, 2003). Therefore, finding the minimum traversal path between circles can be used in the path planning of LM. Finding the minimum traversal path of three circles is the fundamental case of finding the minimum traversal path of plural circles. The problem resolved in this paper is therefore can be defined as follows. Given three disjoint circles, find the path that traverses all three circles in a predefined sequence for which the sum of the circumferences of the three circles and the two connecting links is a minimum.

Fig. 3 shows three circles $C_1$, $C_2$, and $C_3$ with their centers $O_1$, $O_2$, $O_3$, and radii $r_1$, $r_2$, $r_3$, respectively. The starting point is at point $P_1$. Traverse the circumference of $C_1$ and then return back to the point $P_1$. Traverse the link $P_1P_2$ and then arrive the point $P_2$ on $C_2$. Traverse the circumference of $C_2$ and then return back to the point $P_2$. Then traverse across the link $P_2P_3$ to reach the point $P_3$ on $C_3$. Finally, traverse the circumference of $C_3$ and come back to the end point $P_3$ to finish up this traverse. To get the minimum length of this traversal path is the goal of this paper.

In this paper, the problem of the minimum traversal path of three circles can be further degenerated to the problem of the minimum traversal path of two points and one circle. With applying the light reflection (or refraction) phenomenon and geometrical mathematics, the minimum traversal function can be derived accordingly.

The fashion of using circles approximating objects is not only applying in the robotic motion planning (de Berg, van Kreveld, Overmars, & Schwarzkopf, 2004), the robotic arms’ rotation (O’Rourke, 1997), but also welcome in using disks to calculate the connectivity of objects (Guibas, Hershberger, Suri, & Zhang, 2001) of kinetic data structure.
in computational geometry. There are many studies concerning the connectivity of the static stationary disks (Preparata \& Shamos, 1985), such as the base stations of mobile phone communication. Also there are many complex studies in computing the connectivity of moving disks (Johnson, 1994; Toh, 1996), such as the Ad-Hoc mobile network. Due to the characteristics of radio wave, the communication of geodesy and the satellite constellation (Wood, 2001) are usually representing the radio coverage by disks, too. Besides, there are many related researches, such as circle packing (Graham, Lagarias, Mallows, Wilks, \& Yan, 2005) and the problem of circle visibility (Kaiser, 2002) that makes planning the shortest path which walks through circles and can be seen by one of the circles in anytime. Versatile studies have shown the convenience and practicability of the fashion of taking circles representing objects.

The rest of this paper is organized as follows. The problem transformation that converts three circles problem into a two points and one circle problem is given in the next section. The derivation of minimum traversal function is then presented in Section 3. The proposed algorithm of the problem rotation is presented in Section 4. Concluding remarks and potential applications are provided in Section 5.

2. Problem transformation

In this paper, we use $E_{i}^{0}$ to represent the problem of finding the minimum traversal path for $x$ distinct points and $y$ disjoint circles in a two dimensional plane. The goal of the proposed paper is to find the minimum traversal function of three disjoint circles $C_{1}$, $C_{2}$, and $C_{3}$. This problem is thus to be represented as $E_{i}^{0}$, i.e., finding the minimum traversal path for 0 distinct points and 3 disjoint circles in a two dimensional plane. The path of $E_{i}^{0}$ is an open route, i.e., the starting point of the traverse needs not to be coincided with the end point of the traverse.

Under such $E_{i}^{0}$ condition, we denote the length of the minimum traversal path by $L_{i}^{0}$. The function $\text{dist}(UV)$ represents the distance between point $U$ and point $V$. And the function $\text{cirf}(C)$ represents the length of the circumference of circle $C$. Thus, we can write

$$L_{i}^{0} = \min \{ \text{dist}(P_{1}P_{2}) + \text{dist}(P_{2}P_{3}) + \text{cirf}(C_{1}) + \text{cirf}(C_{2}) + \text{cirf}(C_{3}) \}$$

(1)

where $P_{1}$, $P_{2}$, and $P_{3}$ locate on the circumference of $C_{1}$, $C_{2}$, and $C_{3}$ respectively. Since the radii of the given three circles are constants, this property is to provide the following lemma.

Lemma 1. Given three disjoint circles $C_{1}$, $C_{2}$, and $C_{3}$ in the two dimensional plane. The problem $E_{i}^{3}$ can be reduced to the problem $E_{i}^{2}$ for one circle $C_{2}$ with two points $O_{1}$ and $O_{3}$ by adding a constant value.

Proof. Since the radii of the given three circles are constants, the traverses of the circumferences of these circles are also constants. Eq. (1) can be rewritten as

$$L_{i}^{0} = \min \{ \text{dist}(P_{1}P_{2}) + \text{dist}(P_{2}P_{3}) + \text{cirf}(C_{2}) + \text{cirf}(C_{1}) + \text{cirf}(C_{3}) \}$$

(2)

With adding and subtracting the two lengths of radii $r_{1}$ and $r_{3}$ into Eq. (2), we have

$$L_{i}^{0} = \min \{ \text{dist}(P_{1}P_{2}) + \text{dist}(P_{2}P_{3}) + \text{cirf}(C_{2}) + \text{cirf}(C_{1}) + \text{cirf}(C_{3}) - \text{dist}(O_{1}P_{1}) - \text{dist}(O_{3}P_{3}) \}$$

(3)

where $\text{dist}(O_{1}P_{1})$ and $\text{dist}(O_{3}P_{3})$ are the length of $r_{1}$ and $r_{3}$, respectively. Since a line with minimum path from a point to a circle must pass through the center of the circle, the three points $P_{2}$, $P_{1}$, and $O_{1}$ are collinear and the three points $P_{2}$, $P_{3}$, and $O_{3}$ are also collinear, too. The Eq. (3) can be rewritten as

$$L_{i}^{0} = \min \{ \text{dist}(O_{1}P_{2}) + \text{dist}(O_{2}P_{3}) + \text{cirf}(C_{2}) + \text{cirf}(C_{1}) + \text{cirf}(C_{3}) - \text{dist}(O_{1}P_{1}) - \text{dist}(O_{3}P_{3}) \}$$

(4)

Observing that the value of $\text{cirf}(C_{1}) + \text{cirf}(C_{3}) - \text{dist}(O_{1}P_{1}) - \text{dist}(O_{3}P_{3})$ in equation is constant, we set a constant variable $K$ to substitute it and then have

$$L_{i}^{0} = \min \{ \text{dist}(O_{1}P_{2}) + \text{dist}(O_{2}P_{3}) + \text{cirf}(C_{2}) \} + K$$

Since the term $\min \{ \text{dist}(O_{1}P_{2}) + \text{dist}(O_{2}P_{3}) + \text{cirf}(C_{2}) \}$ is the $E_{i}^{2}$ problem for one circle $C_{2}$ with two points $O_{1}$ and $O_{3}$, the problem $E_{i}^{3}$ can be reduced to the problem $E_{i}^{2}$ by adding a constant value. This constant value is the sum of the two circumferential lengths of circles $C_{1}$ and $C_{3}$, and is subtracted by the lengths of the two radii $r_{1}$ and $r_{3}$.

Since the term $\text{cirf}(C_{2})$ in $\min \{ \text{dist}(O_{1}P_{2}) + \text{dist}(O_{2}P_{3}) + \text{cirf}(C_{2}) \}$ is constant, the minimum traversal length $L_{i}^{0}$ of $E_{i}^{3}$ can be simplified as

$$L_{i}^{0} = \min \{ \text{dist}(O_{1}P_{2}) + \text{dist}(O_{2}P_{3}) \} + \text{cirf}(C_{2})$$

fig. 3. Traversal path of three circles.
length equal to 2 and two points A and B with their coordinates (−3, 4) and (4, 5), respectively. Observing that the shape of the discrete curve is approximately sinusoidal curve as shown in Fig. 6. The $E_1^2$ example in Fig. 6 contains one circle C with its radius length equal to 2 and two points A and B with their coordinates $(-3, -1)$ and $(4, -1)$, respectively. Since the points A and B are fixed and the variable point P lies on the circumference of circle C, the periods of Figs. 5 and 6 are $2\pi$ and $4\pi$. Although there is one minimum value in Fig. 5 when $\theta$ is equal to $\pi/2$, there are two minimum values appeared in Fig. 6 with $\theta$ equal to 1.4$\pi$ and 1.9$\pi$. Therefore, the derived solution for $E_1^2$ problem is not unique.

### 3. Derivation of equations

There are many approaches to pursue the extreme values. The best-known approach is using the first-order derivative for function $\ell(\theta)$ with respect to $\theta$ to be equal to zero. With applying the non-linear programming or numerical analysis methods, we also can find the extreme value or approximate extreme values. However, the general
root function of the $E^2_1$ problem cannot be solved by using all of these methods. In this paper, we propose a simpler method which combines the law of light reflection in physics and geometrical mathematics to find the optimal value of $P$ which results in the minimum traversal path of the $E^2_1$ problem. Let the refractive index of the medium with the incident ray be $n_1$ and that of the medium with the refractive ray be $n_2$. Fig. 7 illustrates the refraction of light between these two different mediums. The angles that the incident and refracted rays made with the line $N$ normal to the interface between the media are $\beta_1$ and $\beta_2$, respectively. Then

$$n_1 \sin \beta_1 = n_2 \sin \beta_2.$$  

This result, found by Willebrord Snell, is known as Snell’s law (Azadeh & Casperson, 1997; Dijksterhuis, 2004). If the mediums of the both sides are identical, i.e., $n_1 = n_2$, the incident angle $\beta_1$ is then equal to the refractive angle $\beta_2$.

Reflection is a special case of refraction. Fig. 8 shows an example of light reflection. The direction of dotted line $\beta$ to the interface between the media are $\beta_1$ and that of the medium with the refractive ray $\beta_2$. Let $\beta_1$ denote the incident angle and $\beta_2$ denote the reflection angle. According to the Snell’s law, we also have the equation $n_1 \sin \beta_1 = n_2 \sin \beta_2$ for the reflection. Since the light emission and reflection are at the same side, the mediums of both light directions are the same, i.e., $n_1 = n_2$. We get $\beta_1 = \beta_2$.

Based on the theory of Fermat’s Principle (Giannoni, Masiello, & Piccione, 2002), light traverses along the least time path. The light quickest traversal path from point $A$, via plane $M$, and to point $B$ is the set of line segments $\overline{AP}$ and $\overline{PB}$ with $\beta_1 = \beta_2$. Since the mediums of both sides are the same, the speed of light is constant and the light quickest traversal path is equal to the minimum traversal path. We then have

**Lemma 2.** With the incident angle $\beta_1$ equal to the reflection angle $\beta_2$, the length of traversal path along the line segments $\overline{AP}$ and $\overline{PB}$ is the shortest path.

**Proof.** In Fig. 9, the point $A^*$ is the mirrored point of $A$ against $M$. Point $D$ is the intersection of lines $\overline{AA^*}$ and $M$. Line $\overline{AA^*}$ is then perpendicular to $M$. Then the length of $\overline{AD}$ is equal to the length of $\overline{A^*D}$. Since the length of line segment $\overline{AP}$ is equal to that of the line segment $\overline{A^*P}$, the angle $\beta_4$ is equal to the angle $\beta_5$. When the incident angle $\beta_1$ is equal to the reflective angle $\beta_2$, the angle $\beta_3$ is equal to the angle $\beta_5$. Therefore, the angle $\beta_4$ is equal to the angle $\beta_1$, i.e., the three points $A^*, P$, and $B$ are collinear.

Assume there exists another point $p$ excluding $P$ on $M$, it results in the shorter path than $P$. Since the length of the line segment $\overline{Ap}$ is equal to that of the line segment $\overline{A^*p}$. The length of the path from $A$ via $p$ to $B$ is equal to the length of the path from $A^*$ via $p$ to $B$. According to the theory of trigonometric inequality, the length of the path from $A^*$ via $p$ to $B$ is greater than the length of the path from $A$ via $P$ to $B$. It contradicts the assumption. Therefore, the length of the path from $A$ via $P$ to $B$ is the shortest when the incident angle $\beta_1$ equal to the reflection angle $\beta_2$. \qed

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**Fig. 7.** The refraction of light between different mediums.

**Fig. 8.** Example of the light reflection.

**Fig. 9.** Geometrical analysis of the light reflection.
By applying the principle of light reflection to the $E_1^2$ problem, $M$ is the tangent line to circle $C$ at point $P$. Fig. 10, shows the relationship of the light reflection and the $E_1^2$ problem. In order to simplify derivation of solutions, let the center of circle $C$ be coincided with the origin. The traversal path can be found by any given point $P$ on circle $C$. Hence, the minimum traversal path can be derived from these traversal paths. Let $(x_A, y_A)$ be the coordinates of point $A^*$.

Since $P$ is a tangent point for line $M$ to circle $C$, the line segment $OP$ is perpendicular to $M$. Since the point $A^*$ is the mirrored point of $A$ against $M$, the line $AA^*$ is perpendicular to $M$, too. The slope of line $AA^*$ is equal to the slope of line segment $OP$, that is,

$$\frac{y_A - y_A^*}{x_A - x_A^*} = \frac{y_P}{x_P}.$$  

Both lengths of the line segments $AP$ and $A^*P$ are equal, we have

$$\sqrt{(x_P - x_A)^2 + (y_P - y_A)^2} = \sqrt{(x_P - x_A^*)^2 + (y_P - y_A^*)^2}.$$  

(7)

By solving the simultaneous Eqs. (7) and (8), the coordinate of $A^*$ is gained, that is,

$$x_A^* = \frac{x_A y_P^2 + 2xy_A y_P - x_A y_P^2}{x_P^2 + y_P^2},$$

$$y_A^* = \frac{y_P^2 + 2xy_A y_P - y_A y_P^2}{x_P^2 + y_P^2}.$$  

(9)

Since points $A^*$, $P$, and $B$ are collinear, the slopes of line segments $AP$ and $PB$ are collinear, that is,

$$\frac{y_P - y_A}{x_P - x_A} = \frac{y_B - y_A}{x_B - x_A}.$$  

(10)

If we introduce the polar coordinates

$$x_P = r \cos \theta,$$

$$y_P = r \sin \theta,$$

and rearrange terms, then equation becomes

$$r^3 \sin \theta - y_A r^2 \sin^2 \theta - 2x_A r^2 \cos \theta \sin \theta + y_A r^2 \cos^2 \theta = r \sin \theta - y_B,$$

$$r \cos \theta - x_B.$$  

(11)

Substituting the expression

$$\sin^2 \theta = 1 - \cos^2 \theta$$

into equation, we have

$$r^3 \sin \theta - 2x_A r^2 \cos \theta \sin \theta + 2y_A r^2 \cos^2 \theta - y_A r^2 \cos^2 \theta = r \sin \theta - y_B,$$

$$r \cos \theta - x_B.$$  

(12)

The variable $\sin \theta$ can be replaced by variable $\cos \theta$ with $\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$.

Since we can derive the same result with using $\sin \theta = \sqrt{1 - \cos^2 \theta}$ or $\sin \theta = -\sqrt{1 - \cos^2 \theta}$ to substitute into Eq. (11), we use the expression

$$\sin \theta = \sqrt{1 - \cos^2 \theta}$$

here for the derivation. We obtain

$$r \sqrt{1 - \cos^2 \theta} - 2x_A r^2 \cos \theta \sqrt{1 - \cos^2 \theta} + 2y_A r^2 \cos^2 \theta = r \cos \theta - x_B.$$  

By separating variables of $\cos \theta$ and $\sqrt{1 - \cos^2 \theta}$ and squaring, we obtain

$$4(x_A x_B^2 + x_A^2 y_B^2 + x_A^2 y_A^2 + y_A^2 y_B^2) \cos^4 \theta - 4r(x_A^2 x_B + x_A x_B^2 + x_A y_B^2 + x_A y_B^2) \cos^3 \theta + (r^2 x_A^2 + 2r x_A x_B + x_A^2 y_B^2 + x_A^2 y_A^2 + r^2 y_A^2 - 4x_A y_A y_B) + 2r^2 y_A y_B + 7y_A^2 - 4r^2 y_A y_B) \cos^2 \theta + 2r^2 y_A y_B + 7y_A^2 - 4r^2 y_A y_B) \cos \theta + 2(2x_A x_B + x_A^2 y_B - x_A y_B - x_A y_A) \times \cos \theta + (x_A^2 y_B^2 + 2x_A x_B y_A y_B + x_A y_A^2 - r^2 y_A - 2r^2 x_A x_B - 1) = 0.$$  

To evaluate the polynomial of degree 4 for variable $\cos \theta$, we generalize the four solutions as follows:
\[
\cos \theta = \begin{cases} 
\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \\
\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \\
\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \\
\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p}.
\end{cases}
\]

where \(k, n, p, q\) are as follows:

\[\begin{align*}
k &= r(ax_g + bx_d) \\
n &= \frac{1}{2} \sqrt{\frac{l}{j} + \frac{1}{2} \left( \frac{g}{h} + h \right)} \\
p &= \frac{m}{8a^2b^2n} \\
q &= l - \frac{1}{2} \left( \frac{g}{h} + h \right),
\end{align*}\]

where \(a, b, c, d, e, f, g, h, j, l, m\) are

\[\begin{align*}
a &= x_g^2 + y_g^2 \\
b &= x_g^2 + y_g^2 \\
c &= 2r^2 x_s x_g + r^2 y_g^2 + x_g^2 (r^2 - 4y_g^2) + 2r^2 y_s y_g + r^2 y_g^2 \\
d &= 2x_g x_s + x_s y_g (y_s - y_g) + x_s (2x_g - y_g (y_s - y_g)) \\
e &= x_g^2 (r^2 + y_g^2) - 2x_g y_s (r^2 - y_s y_g) + x_s^2 (-r^2 + y_g^2) \\
f &= 36r^2 (6e(ax_g + bx_d)^2 + cd(ax_g + bx_d) + 6ab^2) \\
g &= c^2 + 48abe + 24r^2 d(ax_g + bx_d) \\
h &= \frac{1}{2} \sqrt{2c^3 - 288abce + 2f + \sqrt{4g^2 + 4(c^3 - 144abce + f)^2}} \\
j &= \frac{1}{2} \sqrt{2c^3 - 288abce + 2f + \sqrt{4g^2 + 4(c^3 - 144abce + f)^2}} \\
l &= \frac{1}{2} \sqrt{2c^3 - 288abce + 2f + \sqrt{4g^2 + 4(c^3 - 144abce + f)^2}} \\
m &= r^2 (ax_g + bx_d)^3 - abc(ax_g + bx_d) - 4a^2b^2d).
\]

From Eq. (12), the solutions of angle \(\theta\) can be derived as follows

\[
\theta = \begin{cases} 
-\cos^{-1}\left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right) \\
-\cos^{-1}\left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right) \\
-\cos^{-1}\left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right) \\
-\cos^{-1}\left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right).
\end{cases}
\]

(13)

Let \(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \) and \(\theta_8\) be the eight solutions of angle \(\theta\). With combination of Eqs. (13) and (15), we have

\[
\theta = \begin{cases} 
\theta_1 &= -\cos^{-1}\left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right) \\
\theta_2 &= \cos^{-1}\left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right) \\
\theta_3 &= -\cos^{-1}\left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right) \\
\theta_4 &= \cos^{-1}\left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right) \\
\theta_5 &= \cos^{-1}\left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right) \\
\theta_6 &= -\cos^{-1}\left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right) \\
\theta_7 &= -\cos^{-1}\left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right) \\
\theta_8 &= \cos^{-1}\left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right).
\end{cases}
\]

(15)

(16)

From Eq. (16), the resolved eight angles can derive the corresponding locations of eight points \(P_1, P_2, P_3, P_4, P_5, P_6, P_7\), and \(P_8\) on the circle \(C\) with their coordinates \((x_{P_1}, y_{P_1}), (x_{P_2}, y_{P_2}), (x_{P_3}, y_{P_3}), (x_{P_4}, y_{P_4}), (x_{P_5}, y_{P_5}), (x_{P_6}, y_{P_6}), (x_{P_7}, y_{P_7})\), and \((x_{P_8}, y_{P_8})\), respectively. Substituting the resolved eight angles into the polar coordinates of \(P\) in Eq. (10), we obtain their eight coordinates, that is,

\[
P_1: \quad \begin{align*}
x_{P_1} &= r \left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right) \\
y_{P_1} &= -r \sqrt{1 - \left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right)^2} \\
x_{P_2} &= r \left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right) \\
y_{P_2} &= r \sqrt{1 - \left(\frac{1}{2} k - n - \frac{1}{2} \sqrt{q - p} \right)^2} \\
x_{P_3} &= r \left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right) \\
y_{P_3} &= r \sqrt{1 - \left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right)^2} \\
x_{P_4} &= r \left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right) \\
y_{P_4} &= r \sqrt{1 - \left(\frac{1}{2} k - n + \frac{1}{2} \sqrt{q - p} \right)^2} \\
x_{P_5} &= r \left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right) \\
y_{P_5} &= -r \sqrt{1 - \left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right)^2} \\
x_{P_6} &= r \left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right) \\
y_{P_6} &= r \sqrt{1 - \left(\frac{1}{2} k + n - \frac{1}{2} \sqrt{q + p} \right)^2} \\
x_{P_7} &= r \left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right) \\
y_{P_7} &= -r \sqrt{1 - \left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right)^2} \\
x_{P_8} &= r \left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right) \\
y_{P_8} &= r \sqrt{1 - \left(\frac{1}{2} k + n + \frac{1}{2} \sqrt{q + p} \right)^2}.
\end{align*}
\]

(17)

(18)

(19)

(20)

(21)

(22)

(23)

(24)

Eqs. (17) and (18) show that there exists reflection between points \(P_1\) and \(P_2\) about the \(x\) axis, that is, \(x\) values are kept and \(y\) values are flipped. The pairs of points \(P_3\) and \(P_4\), \(P_5\) and \(P_6\), \(P_7\) and \(P_8\) also have the reflection relationships.
Substituting the eight solutions of angle $\theta$ into Eq. (6) yields the eight lengths. With finding the shortest length of these eight lengths, we can get the solution of this problem. From Eq. (4), the minimum traversal length $L$ can be rewritten as

$$L = \min\{\ell(\theta_1), \ell(\theta_2), \ell(\theta_3), \ell(\theta_4), \ell(\theta_5), \ell(\theta_6), \ell(\theta_7), \ell(\theta_8)\}. \tag{25}$$

Fig. 11 shows the eight solutions for an $E_1^2$ problem with $x_A = -2.5$, $y_A = 2$, $x_B = 1.5$, $y_B = 2.5$, $r = 1$. The optimal point which results in the shortest length is the solution point $P_4$ with the coordinate $(-0.16,0.99)$. The minimum traversal length $L$, showed by the thicker line segment, is 4.8.

4. Algorithm of the problem rotation

In order to simplify the $E_1^2$ problem, the problem can be restricted in the first quadrant or the first and the second quadrants with a two-dimensional rotation of the $E_1^2$ problem. There are two points $A$ and $B$ in the $E_1^2$ problem, and each point can lie in any quadrant of the two-dimensional plane. The number of different combinations of quadrants of points $A$ and $B$ lay, with repetitions of quadrant, is $4^2 = 16$. Since the points $A$ and $B$ are exchangeable, without repetitions of quadrant, the number of different combinations can be reduced to 10. Moreover, there are two combinations, namely, $I$ and $II$ quadrants, $I$ and $II$ quadrants, which need not make rotation. Then, there are 8 different combinations left needed to be rotated. Consider an $E_1^2$ problem rotated to the first quadrant or the first and the second quadrants by an angle. Since a convention about the direction of rotation must be adopted, assume that counterclockwise (CCW) rotations are positive and clockwise (CW) are negative. Let $\theta_A$ and $\theta_B$ be the angles, CCW, from positive $x$ axis to the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively. Let $x$ be the angle from the vector $\overrightarrow{OB}$ to the vector $\overrightarrow{OA}$. Table 1 lists the eight different combinations that need to be rotated.

Less formally, we can summarize the descriptions of Table 1 as the $E_1^2$ rotation algorithm.

Algorithm. Rotation ($E_1^2$)

**Input.** An $E_1^2$ consists of two distinct points $A$ and $B$ and one circle $C$.

**Output.** A new $E_1^2$ with two points $A$ and $B$ lied in I and I or I and II quadrants.

1. If $B$ lies in II quadrant and $A$ lies in II or III quadrant then Rotate $E_1^2$ by an angle $-\pi/2$. 
2. If $B$ lies in IV quadrant and $A$ lies in I or IV quadrant then Rotate $E_1^2$ by an angle $\pi/2$. 
3. If both of $A$ and $B$ lie in III quadrant then Rotate $E_1^2$ by an angle $-$.$pi$. 
4. If $B$ lies in III quadrant and $A$ lies in IV quadrant then Rotate $E_1^2$ by an angle $pi$. 
5. If $B$ lies in I quadrant and $A$ lies in III quadrant then if $x < \pi$ 
6. then Rotate $E_1^2$ by an angle $-\theta_B$. 
7. else Rotate $E_1^2$ by an angle $2\pi - \theta_A$ and exchange $A$ and $B$. 
8. If $B$ lies in II quadrant and $A$ lies in IV quadrant then if $x < \pi$ 
9. then Rotate $E_1^2$ by an angle $-\theta_B$. 
10. else Rotate $E_1^2$ by an angle $2\pi - \theta_A$ and exchange $A$ and $B$. 
11. Stop.

Fig. 12 shows the two possible $E_1^2$ problems with $r = 1$ after using the proposed $E_1^2$ rotation algorithm, and their eight solutions. Their minimum traversal paths are represented by the thicker paths. The first possible case, as shown in Fig. 12a, is that two points $A$ (0.5, 2) and $B$ (2.5, 2.5) lie in the first quadrant. Fig. 12b shows the other possible case whose two points $A$ (-1.5, 2) and $B$ (2.2, 2.5) lie in the second and first quadrants, respectively.
5. Conclusions

In this paper, the minimum traversal path of three circles is defined as an $E_0^3$ problem, and is transformed into the $E_2^1$ problem. With applying the law of light reflection and geometrical mathematics, the eight roots of solution function of the minimum traversal path of three circles is derived. Using the proposed algorithm of the problem rotation, the problem can be transformed to the problem in first quadrant or in the first and the second quadrants. This solution can be quick effectively implemented in solving a variety of engineering applications, such as layered manufacturing, robotic motion planning, and path planning. There are several possible extensions of this work. For instance, the problem of finding the minimum traversal path of more than three circles is still essentially open. By taking advantage of this minimum traversal function of three circles, we are now exploring the traversal path problem of $n$ circles. Moreover, finding the minimum traversal path of geometrical object with polygonal shapes is also interesting to us.

References


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