Asymptotic Capacity of Large Fading Relay Networks under Random Attacks

Chuan Huang  
Dept. of ECE, Texas A&M University  
College Station, TX, 77843  
Email: huangch@tamu.edu

Jinhua Jiang  
Dept. of ECE, Stanford University  
Stanford, CA, 94305  
Email: jhjiang@stanford.edu

Shuguang Cui  
Dept. of ECE, Texas A&M University  
College Station, TX, 77843  
Email: cui@tamu.edu

Abstract—In this paper, we investigate the asymptotic ϵ-outage capacity of a half-duplex large fading relay network, which consists of one source node, one destination node, and N relay nodes. The relay nodes are assumed to be randomly deployed in a given area and under fatal independent random attacks with probability p. With a total power constraint on all the nodes, we examine the ϵ-outage rate of the amplify-and-forward (AF) strategy when N tends to infinity, assuming no channel state information at the relays. We further quantify the gap between the ϵ-outage rate and the ϵ-outage cut-set bound, which is determined by the attack probability p, the source vs. sum power allocation factor α, and the topology of the networks. Moreover, we examine the effect of random attacks on the ϵ-outage rate, and calculate the relative losses in low and high SNR regimes, respectively. Finally, for general SNR, we show that it is a quasiconcave problem to determine the optimal power allocation between the source and the relays, and we could obtain the optimal α efficiently.

I. INTRODUCTION

Node cooperation has been shown as an effective way to improve system capacity and to provide diversity in wireless networks. One of the promising cooperation schemes is to exploit the use of relays, while the capacity of general relay channels is still an open problem. Full-duplex relay channels, in which relay nodes can transmit and receive simultaneously, have been investigated in [1], [2], and [3], where various achievable rates and special-case capacity results have been obtained; on the other hand, half-duplex relay channels have been studied in [4] and [5], where the amplify-and-forward (AF) strategy in the Gaussian case is shown to asymptotically achieve the capacity when the number of relay nodes tends to infinity, and a joint source-channel coding scheme with power control among the relays was proposed, with the total power in the source and all the relays also scaling to infinity when the number of relays increases. In [6], the authors extended these results to fading Multiple-Input and Multiple-Output (MIMO) relay networks, where both the coherent and noncoherent relaying strategies are adopted, assuming perfect channel state information (CSI) and no CSI at the relays, respectively. For the coherent case, the authors showed that the achievable rate scales as O(\(\log(N)\)) in the high SNR regime; for the noncoherent case, the relay networks were shown to behave as a point-to-point MIMO channel in the high SNR regime.

In this paper, we impose a total power constraint on all nodes, i.e., the total power consumed among the source and all the relays is less than a finite value of P, which ensures the received power bounded even if the number of relays goes to infinity. We assume that the relays have no CSI. Hence, we do not deploy power control at the relay nodes, but simply set all the relay nodes to transmit with the same average power.

We focus on an unreliable networking scenario that takes into account the random failure of relays, which may appear in hostile environments, where each relay is prone to random attacks by enemies or random battery death. In such situations, we may know the total number of relays, but may not know how many of them are actually functioning. As such, we adopt a model where each of the relays is attacked with a given probability such that the number of survived relays is a random variable. We then study the outage performance of this network.

The remainder of this paper is organized as follows. In Section II, we present the network, channel, and signal models, and also introduce all the assumptions. In Section III, we derive an outage capacity upper bound for the discussed large relay network by using the multiple-access (MAC) cut-set. In Section IV, we analyze the achievable rate of the AF strategy, and examine the effect of random attacks on the ϵ-outage rate. In addition, we also derive the optimal power allocation scheme to maximize the ϵ-outage rate. Finally, the paper is concluded in Section V.

II. SYSTEM MODEL

A. General Assumptions

We consider a relay network with a pair of source node and destination node, which are assumed to be located at the fixed positions 0 and s3, respectively, and N relay nodes. Each of the relays is randomly located in a given area S with probability density function (PDF) \(p(s), s \in S\). For simplicity, we assume that the relays are uniformly distributed, i.e., \(p(s) = \frac{1}{m(s)}, s \in S\), where m(S) is the Lebesgue measure of area S. With a dead-zone assumption, we let \(\text{dist}(0,S) \geq 1\) and \(\text{dist}(s_3,S) \geq 1\), where \(\text{dist}(s,S)\) is the distance between a point s and the set S, which ensures that the received power is always bounded. The network topology is shown in Fig. 1. For the simplest 1-D case, i.e., all the nodes
are located along a straight line, we assume that the relays are located between two points: $s_1$ and $s_2$, with $0 < s_1 < s_2 < s_3$.

we assume that each node has only one antenna, the relays work in half-duplex modes, and the transmissions follow a time-slotted structure. Specifically, each time slot is split into two halves: In the first half, the source broadcasts the message to all the relays; in the second half, the relays transmit a certain message to the destination based on what they received in the first half. We assume no direct transmissions between the source and the destination nodes due to the relatively large distance.

Moreover, to model the effect of random attacks, we assume that each relay may die with probability $p$ independently. Thus, the number of surviving relay nodes (denoted as $L$) is a binomial random variable with parameter $(N, 1 - p)$. In this paper, we also put a constraint on the total power consumed by the source and all the relays. In particular, we set the total power spent in the source and the relays to be a finite value $P$, and denote the power allocation factor to the source node as $\alpha$. As such, the power in each relay is $(1 - \alpha)P/N$.

B. Channel and Signal Models

The channel input-output relationship between the source and the $i$-th relay is given as

$$r_i = \sqrt{\alpha P} h_{i1} x + n_i, \quad i = 1, 2, \ldots, N,$$

where $r_i$ is the received signal at the $i$-th relay, $x$ is the unit-power symbol transmitted by the source, $\alpha P$ is the power allocated to the source, fading coefficients $h_{i1}$'s are independently and identically distributed (i.i.d.) zero mean complex Gaussian random variables with unit variance, i.e., $h_{i1} \sim \mathcal{CN}(0, 1)$. $\rho_{i3}$ is a standard Gaussian random variable with parameter $(N, 1 - p)$, $h_{i3} \sim \mathcal{CN}(0, 1)$, $\rho_{i3}$ is the path-loss of the $i$-th relay-to-destination link with $\rho_{i3} = \frac{1}{\|s_i - s_3\|^\tau}$, and $w$ is AWGN with distribution $\mathcal{CN}(0, N_0)$.

III. CAPACITY UPPER BOUND OF A LARGE FADING RELAY NETWORK

In this section, we derive an upper bound on the $\epsilon$-outage capacity [8] of the large relay network defined in the previous section. For a given finite $N$, it is difficult to obtain an explicit expression for the outage capacity. As such, we focus on the case when $N$ tends to infinity, and obtain the corresponding asymptotic results.

Theorem 3.1: Assuming no CSI in the relays, when $N$ goes to infinity, the $\epsilon$-outage capacity upper bound of the large relay network is asymptotically given by

$$C_{\text{upper}} = \frac{1}{2} \log \left(1 + \gamma_{\text{upper}} \ln \left(\frac{1}{1 - \epsilon}\right)\right),$$

where $\epsilon$ is the target outage probability, and the upper bound of the average received SNR at the destination is defined as

$$\gamma_{\text{upper}} = (1 - p)(1 - \alpha) \text{SNR} \mathbb{E}(\rho_{i3}),$$

with $\text{SNR} = \frac{P}{N_0}$ and $\mathbb{E}(\cdot)$ denoting expectation.

Proof: Consider the MAC cut-set in the relay network, and assume that the surviving relays perfectly know the source message and transmit $t_i = \sqrt{(1 - \alpha)P/N} x$, $0 \leq i \leq L$. The received signal at the destination is given as

$$y = \sum_{i=1}^{L} \frac{(1 - \alpha)P \rho_{i3}}{N} h_{i3} x + w.$$

From [9], when $N \to \infty$, $M$ is asymptotically normal with $M \sim \mathcal{CN}(0, (1 - p)(1 - \alpha)P \mathbb{E}(\rho_{i3}))$. As such, the overall source-relays-destination transmission is over an equivalent Rayleigh fading channel. Correspondingly, the lower bound of the outage probability is

$$\epsilon = \Pr\left\{\frac{1}{2} \log \left(1 + \gamma_{\text{upper}} |h|^2\right) < C\right\},$$

where $C$ is the target rate, $h$ is a standard Gaussian random variable, and the average received SNR $\gamma_{\text{upper}}$ is defined as in (4). Then, based on the outage capacity definition given in Chapter 5 of [8], $C := \log (1 + \mathcal{F}^{-1}(1 - \epsilon) \text{SNR})$, where $\mathcal{F}^{-1}(x) = \text{Pr}( |h|^2 \leq x)$. By computing $\mathcal{F}(x)$, and subsequently $\mathcal{F}^{-1}(x)$, we can obtain the outage capacity upper bound given in (3).

Generally, for an arbitrarily given area $S$ and the PDF $p(s)$, it is hard to derive a closed-form expression for the received SNR defined in (4). Nevertheless, based on the results in [7], we derive the analytical result for the 1-D case:

$$\gamma_{\text{upper}} = \frac{(1 - p)(1 - \alpha) \text{SNR}}{(s_3 - s_2)(s_3 - s_1)}.$$
IV. AMPLIFY-AND-FORWARD RELAY STRATEGY

In the previous section, we derive an upper bound for the outage capacity of the large relay network. In this section, we discuss the ϵ-outage rate with the AF strategy, and also quantify the gap between the achievable rate and the upper bound.

A. Achievable Rate

We assume that the i-th relay node knows the average power of the received signal: \( E \left[ r_i^2 \right] = \alpha P \rho_{i1} + N_0 \), and performs the amplification according to

\[
t_i = \frac{\sqrt{(1-\alpha)P}}{N(\alpha P \rho_{i1} + N_0)} r_i,
\]

which ensures the sum power constraint across the relays: \( \sum_{i=1}^{N} |t_i|^2 = (1-\alpha)P \). Hence, from (1), (2), and (7), the received signal at the destination is given by

\[
y = \left( \sum_{i=1}^{L} \frac{\sqrt{(1-\alpha)P \rho_{i1} \rho_{i3}}}{N(\alpha P \rho_{i1} + N_0)} h_{i1} h_{i3} \right) x + \left( \sum_{i=1}^{L} \frac{(1-\alpha)P \rho_{i3}}{N(\alpha P \rho_{i1} + N_0)} h_{i3} n_i \right) + w.
\]

From [9], we know that when \( N \to \infty \), the distributions of \( A \) and \( B \) are asymptotically given by

\[
A \sim CN \left( 0, (1-p)E \left[ \frac{\alpha (1-\alpha)P \rho_{i1} \rho_{i3}}{N(\alpha P \rho_{i1} + N_0)} \right] \right) \quad (9)
\]

\[
B \sim CN \left( 0, (1-p)E \left[ \frac{(1-\alpha)P \rho_{i3}}{N(\alpha P \rho_{i1} + N_0)} \right] \right) \quad (10)
\]

Remark 4.1: Since \( h_{i1} \) and \( h_{i3} \) are independent and of zero mean, we have \( E(AB) = E(A)E(B) = 0 \), which means that \( A \) is uncorrelated with \( B \). Since \( A \) and \( B \) are asymptotically complex Gaussian random variables, they are independent of each other. Therefore, from (8), the large fading relay network can be modelled as a Rayleigh fading channel between the source and the destination asymptotically with fading coefficient \( A \) and AWGN \( B + w \).

Thus, the average received SNR at the destination can be written as

\[
\gamma_{AF} = \frac{(1-p)A}{1 + (1-p)B},
\]

where \( A \) and \( B \) are defined as

\[
A = E \left( \frac{\gamma_{i1} \gamma_{i3}}{1 + \gamma_{i1}} \right), \quad B = E \left( \frac{\gamma_{i3}}{1 + \gamma_{i1}} \right)
\]

with \( \gamma_{i1} \) and \( \gamma_{i3} \) the received SNRs of the i-th source-to-relay and relay-to-destination link, respectively: \( \gamma_{i1} = \alpha \text{SNR} \rho_{i1} \); and \( \gamma_{i3} = (1-\alpha)\text{SNR} \rho_{i3} \). Therefore, the ϵ-outage rate with the AF strategy is given by

\[
R_{AF} = \frac{1}{2} \log \left( 1 + \gamma_{AF} \ln \left( \frac{1}{1-\epsilon} \right) \right)
\]

where \( \epsilon \) is the target outage probability.

For the 1-D case, we have the following closed-form expression for \( A \) and \( B \):

\[
A = b \left\{ \frac{1}{s^2 + a} \ln \left( \frac{s^2 + a}{s^2 - a} \right) - \frac{\sqrt{a} \arctan \left( \frac{s}{\sqrt{a}} \right)}{s} \right\}
\]

\[
B = c \left\{ \frac{s^2}{s^2 + a} + \frac{\sqrt{a} \arctan \left( \frac{s}{\sqrt{a}} \right)}{s} \right\}
\]

where \( a = \alpha \text{SNR} \), and \( b = (1-\alpha)\text{SNR} \).

For a general SNR value, the AF strategy cannot achieve the MAC cut-set bound. Particularly, in the high SNR regime, we have the following results.

Theorem 4.1: When SNR goes to infinity, the ϵ-outage rate with the AF strategy and the MAC cut-set bound have the following asymptotic relationship:

\[
R_{AF} \sim C_{\text{upper}} - O(1).
\]

Proof: When \( SNR \to \infty \), we have \( 1 + \gamma_{i1} \sim \gamma_{i1} \). Hence, we have \( A \sim E(\gamma_{i3}) \), and \( B \sim E(\frac{\gamma_{i3}}{\gamma_{i1}}) \). Thus, the achievable rate with the AF strategy can be approximated as

\[
R_{AF} \sim \frac{1}{2} \log \left( 1 + (1-p)E(\gamma_{i3}) \right)
\]

\[
\approx \frac{1}{2} \log \left( 1 + (1-p)E(\frac{\gamma_{i3}}{\gamma_{i1}}) \right)
\]

\[
= \frac{1}{2} \log \left( (1-p)(1-\alpha)SNR E(\rho_{i3}) \ln \left( \frac{1}{1-\epsilon} \right) \right)
\]

\[
- \frac{1}{2} \log \left( (1-p) \frac{1-\alpha}{\alpha} E \left( \frac{\rho_{i3}}{\rho_{i1}} \right) \right)
\]

\[
= C_{\text{upper}} - O(1).
\]

Therefore, the theorem is proved.

Remark 4.2: The gap \( \frac{1}{2} \log \left( 1 + (1-p)\frac{1-\alpha}{\alpha} E \left( \frac{\rho_{i3}}{\rho_{i1}} \right) \right) \) is independent of SNR, and is determined by the power allocation factor \( \alpha \), the attack probability \( p \), and the network topology. Here, \( E \left( \frac{\rho_{i3}}{\rho_{i1}} \right) \) is a coefficient determined by the topology of the network. For the 1-D case, we have [7]

\[
E \left( \frac{\rho_{i3}}{\rho_{i1}} \right) = \left[ s + s^2 \frac{s^2 + 2s3 \ln(s3 - s)}{s3 - s} \right]_{s = s2}
\]

Next, we discuss the dependence of \( R_{AF} \) over the attack probability \( p \) in the low and high SNR regimes.
Proposition 4.1: In the low SNR regime, we have $1 + B \approx 1$. Hence, we have

$$R_{AF} \approx \frac{1}{2} (1 - p) A \ln \left( \frac{1}{1 - \epsilon} \right) = (1 - p) R_1,$$  

where $R_1 = \frac{1}{2} A \ln \left( \frac{1}{1 - \epsilon} \right)$ is the $\epsilon$-outage rate with the AF strategy in the low SNR regime when $p = 0$.

Remark 4.3: In the low SNR regime, random attack causes $p$-percent rate loss.

Proposition 4.2: In the high SNR regime, we have

$$R_{AF} \approx \frac{1}{2} \log \left( \frac{(1 - p) A}{1 + (1 - p) B} \ln \left( \frac{1}{1 - \epsilon} \right) \right),$$

where $R_2 = \frac{1}{2} \log \left( \frac{1}{1 + B} \ln \left( \frac{1}{1 - \epsilon} \right) \right)$ is the $\epsilon$-outage rate with the AF strategy in the high SNR regime when $p = 0$.

Remark 4.4: The second term is a constant determined by the attack probability $p$, the power allocation factor $\alpha$, and the network topology coefficient $E \left( \frac{\rho_3}{\rho_{13}} \right)$, and is independent of SNR.

B. Power Allocation

In this subsection, we mainly discuss the power allocation strategy to maximize the achievable rate with the AF strategy. In order to maximize (13), we only need to maximize (11). As such, we have the following results.

Theorem 4.2: The received SNR defined in (11) is a quasiconcave function over $\alpha$.

Proof: First, we have the following results for $A$ and $B$.

1) For $A$, we consider the following function

$$f(\alpha, s) = \frac{\alpha (1 - \alpha) SNR^2 \rho_{13} \rho_{33}}{1 + \alpha SNR \rho_{13}} = SNR^2 \rho_{13} \rho_{33} \left( C_1 \alpha + C_2 + \frac{C_3}{1 + \alpha SNR \rho_{13}} \right),$$

where $C_1, C_2, C_3$ are some constants, and $C_3 = -\left( \frac{1}{SNR \rho_{13}} + \frac{1}{SNR^2 \rho_{13}} \right) < 0$. Since $C_1 \alpha + C_2$ is affine and $\frac{C_3}{1 + \alpha SNR \rho_{13}}$ is concave, $f(\alpha, s)$ is concave in $\alpha$ for any given $s$. Moreover, (12) is the integral over $s$, which does not change the concavity of the original function.

Therefore, (12) is also concave in $\alpha$.

2) For $B$, we consider the following function

$$g(\alpha, s) = \frac{(1 - \alpha) SNR \rho_{33}}{1 + \alpha SNR \rho_{13}} = \frac{\rho_{33}}{\rho_{13}} \left( \frac{1}{1 + \alpha SNR \rho_{13}} - 1 \right).$$

Since $g(\alpha, s)$ is concave in $\alpha$, by similar argument as in 1), we have that $1 + B$ is convex in $\alpha$.

From 1) and 2), we know that $A > 0$ is concave and $1 + B > 0$ is convex. Based on Example 3.38 in [10], we know that $\frac{1 + B}{A}$ is a quasiconvex function. Therefore, we can conclude that $\gamma_{AF} = \frac{1 + B}{A} \alpha$ is a quasiconcave function in $\alpha$.

Since (11) is quasiconcave in $\alpha$, we can apply efficient convex optimization techniques to obtain the optimal $\alpha$: e.g., bisection search combined with the interior-point method [10].

In the low and high SNR regimes, we have the following results.

Proposition 4.3: When $SNR \rightarrow 0$, we have $1 + \gamma_{1i} \approx 1$, $1 + B \approx 1$, and from (11), the received SNR is asymptotically given by

$$\gamma_{AF} \approx (1 - \alpha) E(\rho_{13} \rho_{33}),$$

which implies $\alpha_{opt} = 0.5$.

Proposition 4.4: When $SNR \rightarrow \infty$, we have $1 + \gamma_{1i} \approx \gamma_{ii}$, and from (11), the received SNR is asymptotically given by

$$\gamma_{AF} \approx \frac{(1 - p)(1 - \alpha) SNR E(\rho_{13})}{1 + (1 - p) E(\rho_{13})}. \tag{20}$$

By letting the derivative of (20) be zero, we have that only one solution satisfying the condition $\alpha \in [0, 1]$, which is given as

$$\alpha_{opt} = \sqrt{\frac{(1 - p) E(\rho_{13})}{1 + (1 - p) E(\rho_{13})}}. \tag{21}$$

Taking the second-order derivative of (20), we have

$$\frac{d^2 \gamma_{AF}}{d \alpha^2} = \frac{-(1 - p)^2 E(\rho_{13})}{(1 - p) E(\rho_{13}) + (1 - (1 - p) E(\rho_{13})) \alpha^2}. \tag{22}$$

Since $\frac{d^2 \gamma_{AF}}{d \alpha^2} \leq 0$ such that $\gamma_{AF}$ in (20) is concave, we conclude that the power allocation factor given in (21) is the optimal solution.

Remark 4.5: From (21), we see that the optimal power allocation factor in the high SNR regime is determined by the attack probability $p$ and the network topology coefficient $E(\rho_{13})$. Moreover, for a given relay network, the optimal $\alpha$ is a decreasing function of $p$, i.e., when attacks are more likely, more power should be allocated to the relays.

V. NUMERICAL RESULTS

In this section, we present several numerical examples to validate our analysis. The following setup is considered:

1) The source, the destination, and the relays are on a straight line, i.e., we consider the 1-D case;
2) The locations of the source and the destination are 0 and $s_3 = 12$, respectively;
3) Relays are uniformly located on a line segment $[s_1 = 1, s_2 = 11]$.

Note that under these conditions, the dead zone assumption is satisfied, and the following figures are drawn over the analytical results derived in previous sections.

Fig. 2 shows the asymptotic $\epsilon$-outage rate of the AF strategy and the MAC cut-set bound of the large fading relay network.
From this figure, we observe that when SNR increases, the gap between the achievable rate and the upper bound turns to a constant. By Theorem 4.1, the gap can be computed approximately as 1.2 bits, when $\alpha = 0.5$, $p = 0.5$, and $\epsilon = 0.1$.

Fig. 3 shows the optimal power allocation factor under different values of the attack probability. When SNR is low, the optimal $\alpha$ is about 0.5 for all $p$ values; when SNR is high, the optimal $\alpha$ becomes smaller over large attack probabilities. From (11), we can compute the optimal $\alpha$ in the high SNR regime: for $p = 0.1$, $\alpha_{opt} = 0.73$; for $p = 0.3$, $\alpha_{opt} = 0.71$; and for $p = 0.5$, $\alpha_{opt} = 0.67$. Moreover, from these numerical results, we see that the optimal $\alpha$ is not a monotonic function of SNR.

Fig. 4 shows the loss of the AF $\epsilon$-outage rate due to random attacks. In the low SNR regime, the rate is approximately $(1 - p)$-percent of that for the case without random attacks. When SNR increases, the relative loss gets smaller.

VI. CONCLUSION

In this paper, we studied the outage performance of large fading relay networks under random attacks, and investigated the asymptotic MAC cut-set bound and the AF $\epsilon$-outage rate as the number of relays goes to infinity. With a total power constraint on all the nodes, we showed that the $\epsilon$-outage rate scales as $O(\log(SNR))$, which is of the same order as the MAC cut-set bound. In addition, the gap between the upper bound and the achievable rate was also computed. We also estimated the rate losses due to random attacks in the low and high SNR regimes, respectively. Moreover, in order to achieve the maximum rate, we examined the optimal power allocation between the source and the relays, and obtained the corresponding analytical results under low and high SNR assumptions.

REFERENCES