Null controllability of the linearized compressible Navier Stokes system in one dimension

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Abstract

In this paper we consider the one-dimensional compressible Navier–Stokes equations linearized around a constant steady state \((Q_0, V_0), Q_0 > 0, V_0 > 0,\) with periodic boundary conditions in the interval \(I_{2\pi} := (0, 2\pi)\). We explore the controllability of this linearized system using a control only for the velocity equation. We prove that the linearized system with homogeneous periodic boundary conditions is null controllable in \(H^1_{per}(I_{2\pi}) \times L^2(I_{2\pi})\) by a localized interior control when time is sufficiently large, where \(H^1_{per}(I_{2\pi})\) denotes the Sobolev space of periodic functions with mean value zero. We show null controllability of the system by proving an observability inequality with the help of two types of Ingham inequality.

We also consider the analogous problem with Dirichlet boundary conditions rather than periodicity. For this case, we show approximate controllability and null controllability in the case of creeping flow.

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1. Introduction and main results

The Navier–Stokes equations for a viscous compressible isentropic fluid in an interval are

\[
\begin{align*}
\rho_t(x, t) + (\rho u)_x(x, t) &= 0, \\
\rho(x, t)\left[u_t(x, t) + u(x, t)u_x(x, t)\right] + (p(\rho))_x(x, t) - \nu u_{xx}(x, t) &= 0,
\end{align*}
\]

where \(\rho(x, t)\) is the density of the fluid, \(u(x, t)\) is the velocity, \(\nu > 0\) is the fluid viscosity, and \(p\) denotes the pressure and is assumed to satisfy the constitutive law

\[p(\rho) = a\rho^\gamma \quad \text{for} \; a > 0, \; \gamma \geq 1.\]

In this paper we consider the compressible Navier–Stokes system in \(I_{2\pi} := (0, 2\pi)\), linearized around a constant steady state \((Q_0, V_0)\), with \(Q_0 > 0, V_0 > 0\),

\[
\begin{align*}
\rho_t(x, t) + V_0\rho_x(x, t) + Q_0u_x(x, t) &= 0, \\
u u_t(x, t) - \frac{\nu}{Q_0} u_{xx}(x, t) + V_0u_x(x, t) + a\gamma Q_0^{-2}\rho_x(x, t) &= f\chi_O,
\end{align*}
\]

where \(\chi_O\) is the characteristic function of an open subset \(O \subset I_{2\pi}\).

Initial conditions are:

\[\rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x), \quad \forall x \in I_{2\pi}.\]

Periodic boundary conditions are:

\[\rho(0, t) = \rho(2\pi, t); \quad u(0, t) = u(2\pi, t); \quad u_x(0, t) = u_x(2\pi, t), \quad \forall t \in (0, T).\]

**Definition 1.1.** The system \((1.1)–(1.3)\) is null controllable in the space \(Z\) at time \(T > 0\), if for any initial condition \(U_0 = (\rho_0, u_0) \in Z\), there exists a control \(f \in L^2(0, T; L^2(O))\) such that the solution \(U(\cdot) = (\rho, u)\) of \((1.1)–(1.3)\) with control \(f\) hits 0 at time \(T\), i.e.

\[U(T) = 0.\]

Note that integrating the first equation of \((1.1)\) in \((0, 2\pi)\) and using \(\rho(0, t) = \rho(2\pi, t)\) and \(u(0, t) = u(2\pi, t)\) we get

\[
\frac{d}{dt} \left(\int_0^{2\pi} \rho(x, t)dx\right) = 0.
\]

Therefore

\[
\int_0^{2\pi} \rho(x, T)dx = \int_0^{2\pi} \rho(x, 0)dx = \int_0^{2\pi} \rho_0(x)dx.
\]
This identity shows that for null-controllability of the system (1.1)–(1.3), \( \int_0^{2\pi} \rho_0(x) \, dx \) has to be equal to 0.

Observing this fact, we now introduce the following Sobolev space of periodic functions

\[
H^s_{\text{per}}(I_{2\pi}) = \left\{ \varphi : \varphi = \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \sum_{k \in \mathbb{Z}} |k|^{2s} |c_k|^2 < \infty \right\}
\]

with norm

\[
\|\varphi\|_{H^s_{\text{per}}(I_{2\pi})} = \left( \sum_{k \in \mathbb{Z}} (1 + |k|^{2s}) |c_k|^2 \right)^{\frac{1}{2}}.
\]

We also set

\[
\dot{H}^s_{\text{per}}(I_{2\pi}) = \left\{ \varphi \in H^s_{\text{per}}(I_{2\pi}) : \int_0^{2\pi} \varphi(x) \, dx = 0 \right\}
\]

(in terms of Fourier series this reduces simply to \( c_0 = 0 \) in the expansion), where the space \( \dot{H}^s_{\text{per}}(I_{2\pi}) \) will be equipped with the \( \|\cdot\|_{H^s_{\text{per}}(I_{2\pi})} \) defined above. Note that due to Poincaré’s inequality for the space \( \dot{H}^s_{\text{per}}(I_{2\pi}) \), the norm

\[
\|\varphi\|_{\dot{H}^s_{\text{per}}(I_{2\pi})} = \left( \sum_{k \in \mathbb{Z}} |k|^{2s} |c_k|^2 \right)^{\frac{1}{2}}
\]

will be equivalent to the norm \( \|\varphi\|_{H^s_{\text{per}}(I_{2\pi})} \) on \( \dot{H}^s_{\text{per}}(I_{2\pi}) \).

Our main positive result concerning null controllability of (1.1)–(1.3) is the following:

**Theorem 1.2.** For any \( T > \frac{2\pi}{\nu_0} \) and any initial condition \( (\rho_0, u_0) \in \dot{H}^1_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi}) \), the system (1.1)–(1.2) with periodic boundary condition (1.3) is null controllable at time \( T \) by a localized interior control \( f(\cdot) \in L^2(0, T; L^2(\mathcal{O})) \) acting only on the velocity equation, where \( \mathcal{O} \) is any nonempty open subset of \( I_{2\pi} \).

Our next negative result shows that if the initial density \( \rho_0 \) lies in a less regular space, then the linearized system is not null controllable with a square integrable control.

**Theorem 1.3.** For any \( s \) with \( 0 \leq s < 1 \), let us denote the space

\[
H_s = \dot{H}^s_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi}).
\]

The system (1.1)–(1.2) with periodic boundary condition (1.3) is not null controllable in the space \( H_s \) at any time \( T > 0 \) by interior controls \( f(\cdot) \in L^2(0, T; L^2(\mathcal{O})) \) acting only on the velocity equation, where \( \mathcal{O} \) is any nonempty open subset of \( I_{2\pi} \) or even the whole domain \( I_{2\pi} \).
Thus our null controllability result (Theorem 1.2) in the space $\hat{H}^1_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi})$ using $L^2$ localized interior control only for velocity is optimal in view of the above theorem.

In contrast, null controllability of the linearized system holds in a less regular space if we use two localized interior controls both for velocity and density. Let us consider the following system

$$
\begin{align*}
\rho_t(x,t) + V_0 \rho_x(x,t) + Q_0 u_x(x,t) &= g \chi_{\mathcal{O}_1}, \\
u u_t(x,t) - \frac{\nu}{Q_0} u_{xx}(x,t) + V_0 u_x(x,t) + a \gamma Q_0^{-2} \rho_x(x,t) &= f \chi_{\mathcal{O}_2}, \\
\rho(x,0) &= \rho_0(x), \\
u u(x,0) &= u_0(x), \\
\forall x \in I_{2\pi}, \\
\rho(0,t) &= \rho(2\pi,t); \\
u u(0,t) &= u(2\pi,t); \\
u u_x(0,t) &= u_x(2\pi,t), \\
\forall t \in (0,T),
\end{align*}
$$

where $\chi_{\mathcal{O}_i}$ is the characteristic function of an open subset $\mathcal{O}_i \subset I_{2\pi}$, for $i = 1, 2$.

**Theorem 1.4.** At any time $T$ with $T > \frac{2\pi}{\nu_0}$, the system (1.5) is null controllable in $L^2(I_{2\pi}) \times L^2(I_{2\pi})$ by two localized interior controls $g(\cdot) \in L^2(0,T; L^2(\mathcal{O}_1))$ and $f(\cdot) \in L^2(0,T; L^2(\mathcal{O}_2))$ acting for density and velocity both, where $\mathcal{O}_1$, $\mathcal{O}_2$ are any nonempty open subsets of $I_{2\pi}$.

The proof of this result can be accomplished following exactly the same lines as the proof of Theorem 1.2 and will be omitted. Remark 5.1 indicates why we get the null controllability result of system (1.5) in a space of lower regularity than the case of system (1.1)–(1.3).

The proofs of these null controllability results rely on an observability inequality (see Section 4) for the solutions of the adjoint system and the spectral analysis of the linearized operator. The spectrum of the linearized operator except zero lies on the left side of the complex plane. It consists of two infinite families of pairs of complex eigenvalues, with the real part of one pair of eigenvalues being convergent and the real part of the other pair divergent to $-\infty$. Moreover, there is no accumulation point in the spectrum and the absolute values of the eigenvalues in both families go to infinity. Thus the spectrum splits into one part of “hyperbolic” type and another part of “parabolic” type (see Lemma 2.6, Remark 2.7). Explicit expressions of eigenvalues and eigenfunctions in terms of a Fourier basis are obtained. This helps to split the space as well as the solution and observability inequality into hyperbolic and parabolic parts. It is also used to establish both positive and negative controllability results. We emphasize that Theorem 4.2 on splitting of the observability inequality is the key result in this work. The observability inequality will be established in Section 4 using two different types of Ingham inequality for complex frequencies and Theorem 4.2.

In [16] and [17], Mitra, Ramaswamy and Raymond have studied stabilizability of the same linearized system around $(Q_0, V_0)$, $Q_0 > 0$, $V_0 > 0$ as well as the nonlinear system using an $L^2$ interior control acting everywhere for velocity only. Using a similar spectral analysis, they construct a feedback control acting everywhere in an explicit way to stabilize the linear and nonlinear flows. In [16] they have proved that the same linearized system will not be stabilizable with arbitrary decay rate in any Hilbert subspace (where the system is well posed) of $L^2(I_{2\pi}) \times L^2(I_{2\pi})$ strictly containing $\hat{H}^1_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi})$ using an $L^2$ control in velocity only. The lack of null controllability of the linearized system in $H_s$, where $0 \leq s < 1$, can be viewed as a consequence of lack of stabilizability of the system with arbitrary rate of exponential decay.

Amosova has considered in [1] a compressible viscous fluid in one dimension in Lagrangian coordinates, with zero boundary condition for the velocity on the boundaries of the interval
(0, 1) and an interior control on the velocity equation. She proves local exact controllability to trajectories for the velocity, provided that the initial density is already on the targeted trajectory.

Ervedoza, Glass, Guerrero and Puel in [8] consider the compressible Navier–Stokes equation in one space dimension in a bounded domain \((0, L)\). They prove local exact controllability to constant states \((\bar{\rho}, \bar{v})\) with \(\bar{\rho} > 0, \bar{v} \neq 0\) using two boundary controls (both for density and velocity) when initial conditions are regular (namely both initial density and velocity lie in \(H^3(0, L)\)).

Chowdhury, Ramaswamy and Raymond establish in [4] that the linearized (around \((\bar{Q}_0, 0)\)) compressible Navier–Stokes system in one dimension is null controllable for regular initial data by a distributed control acting everywhere in the velocity equation. They also prove that this result is sharp by showing that the null controllability cannot be achieved by a localized interior control or by a boundary control. On the other hand, they obtain that the linearized system is approximately controllable. A similar result is also found for boundary control for the wave equation with structural damping in one dimension with periodic boundary conditions, by Rosier and Rouchon [19]. Their equation is equivalent to that of [4], after the system is reduced to a single equation for velocity. Martin, Rosier and Rouchon in [14] consider the same wave equation with structural damping in one dimension. Their transformed equation is equivalent to our system. Using spectral analysis and the method of moments, they obtain that their equation is null controllable with a moving distributed control for regular initial conditions (in \(H^{s+2} \times H^s, s > \frac{15}{2}\)) in sufficiently large time.

Chowdhury and Mitra consider the controllability of the linearized system around \((\bar{Q}_0, V_0), Q_0 > 0, V_0 > 0\) with periodic boundary conditions, using a control only for the velocity equation. Then, following the approach of Martin et al. in [14] using the method of moments, they establish in [5] that the linearized system with periodic boundary conditions is null controllable by a localized interior control when time is large enough, and for regular initial data (in \(\hat{H}^1_{\text{per}}(I_{2\pi}) \times H^s_{\text{per}}(I_{2\pi})\) with \(s > 6.5\)). They also show that the linearized system is approximately controllable in \(L^2(I_{2\pi}) \times L^2(I_{2\pi}).\) We note that the regularity assumptions on the data in both [14] and [5] are rather strong. Our result in this paper replaces these assumptions with natural ones \((\hat{H}^1_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi}))\).

Chaves-Silva, Rosier and Zuazua in [3] consider the wave equation with both viscous Kelvin–Voigt and frictional damping as a model of viscoelasticity in \(\mathbb{R}^N\). Their problem is not controllable if the control is confined to a stationary subset, but becomes controllable if the controlled region is moved around to reach every part of the domain. Our problem transforms to this if we go to a coordinate frame moving with velocity \(V_0\). They extend the results of [14] to the multidimensional case, but unlike [14], they require initial data only in \(H^1 \times L^2,\) just like our result below. They establish null controllability of the system with a moving internal control using the observability of the adjoint system and a new type of Carleman estimates. They consider a bounded domain with Dirichlet conditions and indicate how their proofs could be adapted to accommodate the periodic case. After such adaptation, our result would follow from theirs. However, our method of proof is entirely different.

The main novelty in our work is that we use spectral methods to prove null controllability with natural regularity assumptions on initial conditions (in \(\hat{H}^1_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi})\)). We also conclude that our result is sharp by showing that the null controllability cannot be achieved in less regular subspace of \(L^2(I_{2\pi}) \times L^2(I_{2\pi})\) by a \(L^2\) localized interior control for velocity. Only one interior control in the velocity equation is needed to get null controllability in \(\hat{H}^1_{\text{per}}(I_{2\pi}) \times L^2(I_{2\pi})\) and null controllability in \(L^2(I_{2\pi}) \times L^2(I_{2\pi})\) can be obtained if we use two localized \(L^2\) interior controls both for density and velocity. The essential step in our argument is a splitting argument
which divides the problem into a hyperbolic and parabolic part. This splitting is achieved by exploiting the Ingham inequality. Once the splitting is accomplished, the hyperbolic and parabolic parts can be controlled separately using any of a number of methods. We use Ingham inequalities, but we could, for instance, use Carleman estimates and the method of characteristics instead. Only the Carleman estimate for the heat equation would be required in this approach. The potential of generalizing to nonlinear and multidimensional situations depends primarily on finding other methods to accomplish the splitting.

Ervedoza et al. in [8] also use a splitting strategy to show controllability in $H^3(0, L) \times H^3(0, L)$ for the nonlinear system, using two boundary controls. Their splitting method is quite complicated, and it is not obvious if or how their method can be extended to distributed controls.

The nature of the problem becomes fundamentally different when the periodic boundary conditions are changed to Dirichlet boundary conditions $u(0, t) = u(2\pi, t) = \rho(0, t) = 0$. It is instructive to consider the first equation of (1.1) when the coupling to the velocity is removed, i.e.

$$\rho_t + V_0 \rho_x = 0. \quad (1.6)$$

Clearly any control on the velocity equation has no impact then. If we add a control to (1.6), then we obtain exact controllability as long as the controlled region extends to the upstream boundary. If it does not, however, then the control has no impact on the evolution of $\rho$ to the left of the controlled region. Hence the system is not even approximately controllable, although it is trivially null controllable in time $2\pi/V_0$ (no control is needed). Due to lack of backward uniqueness, null controllability does not imply approximate controllability. It is of interest to see whether the coupling with the velocity equation can change this. We shall answer this question for the case of creeping flow, i.e. the case where inertial terms are neglected in the Navier–Stokes equation. That is, we consider the problem

$$\rho_t + V_0 \rho_x + Q_0 u_x = 0,$$

$$-\frac{\nu}{Q_0} u_{xx} + a\gamma Q_0^{\gamma-2}\rho_x = f \chi_\Omega, \quad (1.7)$$

with Dirichlet boundary conditions. We shall show that this problem is indeed approximately controllable, for $T > 2\pi/V_0$ and an arbitrary open set $\Omega$. We note that we can solve for $u$ from the second equation of (1.7) and the boundary conditions. Using this, we shall reduce the control problem to a single equation for $\rho$:

$$\rho_t + V_0 \rho_x + a\gamma Q_0^{\gamma-1}(\rho - I(\rho)) = F \chi_\Omega, \quad (1.8)$$

where $I(\rho)$ denotes the average of $\rho$, and $F$ is a new control function assumed to have zero average. The original control function $f$ is given by the derivative of $F$ with respect to $x$. We shall consider this problem on the space $L^2$. We shall prove the following result.

**Theorem 1.5.** For $T > 2\pi/V_0$, and with a control $F \in L^2((0, T); L^2(\Omega))$, assumed to have zero average at all times, Eq. (1.8) with boundary condition $\rho(0, t) = 0$ and initial condition in $L^2(0, 2\pi)$ is approximately controllable and null controllable.
This paper is organized as follows. In Section 2, we study the existence and uniqueness of the linearized system (1.1)–(1.3) and the corresponding adjoint system. Then we analyze the behavior of the spectrum of the linearized operator. Section 3 is devoted to two different types of Ingham inequalities for complex frequencies. In Section 4, we establish the splitting of the observability inequality into two observability inequalities corresponding to hyperbolic and parabolic parts (see Theorem 4.2) of the adjoint system. Thereafter we give the completion of the proof of the observability inequality using two Ingham inequalities obtained before, and hence null controllability in $\dot{H}_{per}^1(I_{2\pi}) \times L^2(I_{2\pi})$ (Theorem 1.2) is proved. In Section 5, we show that system (1.1)–(1.3) is not null controllable in $H_s$ for $0 \leq s < 1$ (Theorem 1.3) by a square integrable interior control acting only in the velocity equation for any time $T$. The problem (1.7) with Dirichlet conditions is discussed in Section 6.

2. Linearized operator

We introduce the positive constants

$$b := a\gamma Q_0^{-2}, \quad \nu_0 := \frac{\nu}{Q_0}. \quad (2.1)$$

Let $Z = \dot{H}_{per}^1(I_{2\pi}) \times L^2(I_{2\pi})$ be the Hilbert space over the complex field $\mathbb{C}$ and endowed with the inner product

$$\left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_Z := b \int_0^{2\pi} \rho_x(x) \sigma_x(x) dx + \nu_0 \int_0^{2\pi} u(x) \sigma(x) dx.$$

We define the unbounded operator $(A, D(A))$ in $Z$ by

$$D(A) = \dot{H}_{per}^2(I_{2\pi}) \times H_{per}^2(I_{2\pi})$$

and

$$A = \begin{bmatrix} -V_0 \frac{d}{dx} & -Q_0 \frac{d}{dx} \\ -b \frac{d}{dx} & \nu_0 \frac{d^2}{dx^2} - V_0 \frac{d}{dx} \end{bmatrix}. \quad (2.2)$$

Setting $U(t) = (\rho(\cdot, t), u(\cdot, t))^T$ the system (1.1) with homogeneous boundary conditions (1.3) and $f = 0$ can be written as

$$U'(t) = AU(t), \quad U(0) = U_0 \in Z. \quad (2.3)$$

We have the following lemma.

**Lemma 2.1.** The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on $Z$, denoted by $(S(t))_{t \geq 0}$. For every $U_0 \in Z$, there is a unique solution $U$ of (2.3) in $C([0, T]; Z)$ and

$$\|U(t)\|_Z \leq M \|U_0\|_Z \quad \text{for all } t \in (0, T).$$
Proof. We can write the operator $A$ as $A = A_1 + A_2$, where

$$A_1 = \begin{bmatrix} -V_0 \frac{d}{dx} & -Q_0 \frac{d^2}{dx^2} \\ 0 & \nu_0 \frac{d^2}{dx^2} - V_0 \frac{d}{dx} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ -b \frac{d}{dx} & 0 \end{bmatrix}.$$  

Notice that $A_2$ is a bounded perturbation of the operator $A_1$ on the space $\mathbf{Z} = \dot{H}^1_{per}(I_{2\pi}) \times L^2(I_{2\pi})$. Therefore if we can show that $A_1$ generates a $C_0$ semigroup on $\mathbf{Z}$, then that will imply $A$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{Z}$. Now we will show that $A_1$ generates a $C_0$ semigroup on $\mathbf{Z}$. To do that we consider the following system

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = A_1 \begin{pmatrix} \rho \\ u \end{pmatrix},$$

that is

$$\rho_t + V_0 \rho_x + Q_0 u_x = 0,$$
$$u_t + V_0 u_x - \nu_0 u_{xx} = 0. \quad (2.4)$$

This system is easily solved by solving the second equation first and then plugging the result into the first equation. The requisite estimates are standard. \hfill \Box

Using semigroup theory (in fact Lemma 2.1) and parabolic regularity for the second equation of (1.1) we get

**Proposition 2.2.** If $f \in L^2(0, T; L^2(\Omega))$ and $(\rho_0, u_0) \in \dot{H}^1_{per}(I_{2\pi}) \times L^2(I_{2\pi})$, then there exists a unique solution of (1.1)–(1.3) such that $(\rho, u)$ is in $C([0, T]; \dot{H}^1_{per}(I_{2\pi}) \times L^2(I_{2\pi}))$ with $u$ in $L^2(0, T; H^1(I_{2\pi}))$.

**Remark 2.3.** Let us denote by $\mathbf{H} = \dot{L}^2(I_{2\pi}) \times L^2(I_{2\pi})$ the Hilbert space over the complex field $\mathbb{C}$ endowed with the inner product

$$\left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_{\mathbf{H}} := b \int_0^{2\pi} \rho(x)\bar{\sigma}(x) \, dx + Q_0 \int_0^{2\pi} u(x)\bar{v}(x) \, dx.$$  

$\mathbf{Z}' = \dot{H}^{-1}_{per}(I_{2\pi}) \times L^2(I_{2\pi})$ is the dual space of $\mathbf{Z}$ considering $\mathbf{H}$ as the pivot space and we denote the duality action $\langle \cdot, \cdot \rangle_{\mathbf{Z}'x}$ by $\langle \cdot, \cdot \rangle$. Note that

$$\left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_{\mathbf{H}} = \left\langle \begin{pmatrix} \rho \\ u \end{pmatrix}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \right\rangle_{\mathbf{Z}}, \quad \text{for } \begin{pmatrix} \rho \\ u \end{pmatrix} \in \mathbf{Z}, \begin{pmatrix} \sigma \\ v \end{pmatrix} \in \mathbf{H} \subset \mathbf{Z'}.$$  

**Lemma 2.4.** The adjoint operator $(A^*, \mathcal{D}(A^*))$ of $(A, \mathcal{D}(A))$ is defined by $\mathcal{D}(A^*) = \{(\sigma, v) \in \dot{L}^2(I_{2\pi}) \times H^1_{per}(I_{2\pi}) \mid b \sigma + \nu_0 v_x \in H^1_{pers}(I_{2\pi}) \}$, $\mathbf{Z}' = \dot{H}_{per}^{-1}(I_{2\pi}) \times L^2(I_{2\pi})$ and

$$A^* = \begin{bmatrix} V_0 \frac{d}{dx} & Q_0 \frac{d^2}{dx^2} \\ b \frac{d}{dx} & \nu_0 \frac{d^2}{dx^2} + V_0 \frac{d}{dx} \end{bmatrix}.$$  

Moreover $(A^*, \mathcal{D}(A^*))$ is the infinitesimal generator of a strongly continuous semigroup $(S^*(t))_{t \geq 0}$ on $\mathbf{Z}'$.  

The adjoint system of (1.1)–(1.2) is

\[-\sigma_t(x, t) - V_0 \sigma_x(x, t) - Q_0 v(x, t) = 0,\]
\[-v_t(x, t) - v_0 v_{xx}(x, t) - V_0 v_x(x, t) - b \sigma_x(x, t) = 0,\]

\[\sigma(0, t) = \sigma(2\pi, t); \quad v(0, t) = v(2\pi, t); \quad v_x(0, t) = v_x(2\pi, t),\]

\[\sigma(x, T) = \sigma_T(x); \quad v(x, T) = v_T(x).\]  

(2.5)

Setting \( V = (\sigma, v)^T \), the adjoint system (2.5) can be written as

\[-V'(t) = A^*V(t), \quad V(T) = (\sigma_T, v_T)^T,\]  

(2.6)

where \( A^* \) is given in Lemma 2.4. Using semigroup theory (Lemma 2.4) we get

**Proposition 2.5.** If \((\sigma_T, v_T)\) in \( Z' \) then there exists a unique solution of the adjoint problem (2.6) such that \((\sigma, v)\) is in \( C([0, T]; Z') \).

### 2.1. Spectral analysis

Here we study the behavior of the spectrum of the linearized operator on \( Z \). Notice that the space \( Z \) has a natural Fourier basis, which we denote by \( \{\phi_0\} \cup \{\phi_k^{(l)}; \ l = 1, 2, k \in Z^*\} \). Here we use the notation \( Z^* = Z \setminus \{0\} \). Let us define the following spaces

\[ V_0 = \text{span}\{\phi_0\}, \quad V_k = \text{span}\{\phi_k^{(l)}; \text{ for } l = 1, 2\}, \quad \text{for } k \in Z^*. \]

So \( Z \) is the orthogonal sum of the two-dimensional spaces \( \{V_k\}_{k \in Z^*} \) and \( V_0 = \ker(A) \). We see that each of these \( V_k \) is invariant under \( A \) and the restriction of the operator \( A \) on \( V_k \) has a \( 2 \times 2 \) matrix representation \( A_k \) in terms of a basis of \( V_k \). For every \( k \in Z^* \), we find out the two eigenvalues of \( A_k \) and the spectrum of \( A \) is the union of zero and the spectrum of \( A_k \) for every \( k \in Z^* \). Similar details can be found in [5].

**Lemma 2.6.** The spectrum of \( A \) consists of 0 and two sequences of pairs of complex eigenvalues \( \{-\lambda_k^h, -\lambda_k^p, \forall k \in Z^*\} \) where \( -\lambda_k^h = -\overline{\lambda_{-k}^h}, -\lambda_k^p = -\overline{\lambda_{-k}^p}, \forall k \in Z^* \). One pair of eigenvalues has convergent real part and other pair of eigenvalues has divergent real part.

For \( k = 0 \), we denote \(-\lambda_0^h = 0\). For \( k \in Z^* \) with \( k^2 < \frac{4bQ_0}{v_0^2} \),

\[ \lambda_k^h = \frac{[k^2v_0 - ik(\sqrt{4bQ_0 - k^2v_0^2} + 2V_0)]}{2}, \quad \lambda_k^p = \frac{[k^2v_0 + ik(\sqrt{4bQ_0 - k^2v_0^2} - 2V_0)]}{2}, \]

and for \( k \in Z^* \) with \( k^2 \geq \frac{4bQ_0}{v_0^2} \),

\[ \lambda_k^h = \frac{[(k^2v_0 - |k|\sqrt{k^2v_0^2 - 4bQ_0}) - 2ikV_0]}{2}, \]
\[ \lambda_k^p = \frac{[(k^2v_0 + |k|\sqrt{k^2v_0^2 - 4bQ_0}) - 2ikV_0]}{2}. \]
Defining \( \omega_0 = \frac{bQ_0}{v_0} \), we get

\[
\operatorname{Re} \lambda^h_k \to \omega_0 \quad \text{as} \quad |k| \to \infty, \\
\operatorname{Re} \lambda^p_k \to v_0 \quad \text{as} \quad |k| \to \infty, \\
\left| \frac{\operatorname{Im} \lambda^h_k}{k} \right| \to V_0, \quad \left| \frac{\operatorname{Im} \lambda^p_k}{k} \right| \to V_0, \quad \text{as} \quad |k| \to \infty.
\]

**Remark 2.7.** From the expression of eigenvalues it is clear that all eigenvalues are simple at least if \( k \) is large enough. So if there are multiple eigenvalues, there would only be a finite number of them. Note that the spectrum of \( A \) can have repeated, in fact double eigenvalues for some special value of \( k \) under some condition on the constants \( bQ_0, V_0, v_0 \). In fact, if \( \frac{2\sqrt{bQ_0}}{v_0} \in \mathbb{N} \), then

\[
\lambda^h_k = \lambda^p_k, \quad \lambda^h_{-k} = \lambda^p_{-k}, \quad \text{for} \quad k = \frac{2\sqrt{bQ_0}}{v_0}.
\]

Also for \( k^2 < \frac{4bQ_0}{v_0^2} \), if \( \frac{2\sqrt{bQ_0 - V_0^2}}{v_0} \in \mathbb{N} \) then

\[
\lambda^p_k = \lambda^p_{-k} \in \mathbb{R}, \quad \text{for} \quad k = \frac{2\sqrt{bQ_0 - V_0^2}}{v_0}.
\]

In these cases, where two eigenvalues for the same \( k \) coincide, there may be a generalized eigenvector. It may also be possible that two eigenvalues corresponding to different values of \( k \) are the same.

**Remark 2.8.** For future reference, we define

\[
N_0 = 2 \left\lfloor \frac{\sqrt{bQ_0}}{v_0} \right\rfloor + 1, \quad (2.7)
\]

where the symbol \( [\ . \ ] \) denotes the integer part function.

**Remark 2.9.** We have \( \operatorname{Re} \lambda^h_k \to \omega_0 \) as \( |k| \to \infty \), but \( \left| \frac{\operatorname{Im} \lambda^h_k}{k} \right| \to V_0 \) as \( |k| \to \infty \), and this is a property of hyperbolic operators. The other pair of eigenvalues \( \{ \lambda^p_k, k \in \mathbb{Z}^+ \} \) has the sectorial property that is \( \frac{\operatorname{Re} \lambda^p_k}{k^2} \to v_0 \) as \( |k| \to \infty \), and \( \left| \frac{\operatorname{Im} \lambda^p_k}{k} \right| \to V_0 \) as \( |k| \to \infty \). This is a typical property of parabolic operators. In contrast, the linearized operator around \( (Q_0, V_0 = 0) \) has only parabolic property and generates an analytic semigroup. The corresponding spectral analysis for the operator linearized around \( (Q_0, V_0 = 0) \) is studied in [4].

In the next lemmas, we assume that \( A \) has simple eigenvalues. For the case of multiple eigenvalues, the analogous lemmas follow by introducing generalized eigenfunctions suitably. The multiple eigenvalues of \( A \), if at all they occur, are only finitely many. Since only the asymptotic...
behavior of the coefficients of eigenfunctions is needed for our later analysis, we avoid the explicit calculations of the generalized eigenfunctions. Thus we give the following lemmas under the assumption of simple eigenvalues.

**Lemma 2.10.** Suppose that the spectrum of $A$ has simple eigenvalues. An eigenfunction of $A$ for eigenvalue $-\lambda^h_0 = 0$ is

$$
\xi^h_0 = \frac{1}{\sqrt{2\pi Q_0}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

For $k \in \mathbb{Z}^*$, $A$ has an eigenfunction $\xi^h_k$ corresponding to eigenvalue $-\lambda^h_k$ with the expression

$$
\xi^h_k = \frac{1}{\theta^h_k} \begin{pmatrix} e^{-ikx} \\ \hat{N}^h_k e^{-ikx} \end{pmatrix}, \quad k \in \mathbb{Z}^*,
$$

where $\theta^h_k = \sqrt{2\pi (b + Q_0 |\hat{N}^h_k|^2)}$ and

$$
\hat{N}^h_k = -\frac{k^2 v_0 - k^2 \sqrt{v_0^2 - 4bQ_0}}{2ik Q_0}, \quad k \in \mathbb{Z}^*.
$$

For $k \in \mathbb{Z}^*$, an eigenfunction of $A$ corresponding to eigenvalue $-\lambda^p_k$ is

$$
\xi^p_k = \frac{1}{\theta^p_k} \begin{pmatrix} e^{-ikx} \\ \hat{N}^p_k e^{-ikx} \end{pmatrix}, \quad k \in \mathbb{Z}^*,
$$

where $\theta^p_k = \sqrt{2\pi (b + Q_0 |\hat{N}^p_k|^2)}$ and

$$
\hat{N}^p_k = -\frac{k^2 v_0 + k^2 \sqrt{v_0^2 - 4bQ_0}}{2ik Q_0}, \quad k \in \mathbb{Z}^*.
$$

Moreover, $\|\xi^h_k\|_H = 1 = \|\xi^p_k\|_H$. $\forall k \in \mathbb{Z}^*$.

**Lemma 2.11.** $A^*$ has the same spectrum as $A$. Under the assumption that the spectrum of $A$ has simple eigenvalues, a family of eigenfunctions $\{(\xi^h_0)^*\} \cup \{(\xi^h_n)^* : n \in \mathbb{Z}^*\}$, of $A^*$ is defined as follows. The eigenfunction of $A^*$ for eigenvalue $-\lambda^h_0 = 0$ is

$$
(\xi^h_0)^* = \frac{1}{\sqrt{2\pi Q_0}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

For $k \in \mathbb{Z}^*$ the eigenfunction of $A^*$ corresponding to $-\lambda^h_k$ is

$$
(\xi^h_k)^* = \frac{1}{\psi^k_0} \begin{pmatrix} e^{-ikx} \\ N^h_k e^{-ikx} \end{pmatrix}, \quad k \in \mathbb{Z}^*,
$$

where $\psi^h_k = \sqrt{2\pi (b + Q_0 |\hat{N}^h_k|^2)}$. $\forall k \in \mathbb{Z}^*$.
where $\psi_k^h = \frac{\sqrt{2\pi(b - Q_0|N_k^h|^2)}}{\sqrt{(b + Q_0|N_k^h|^2)}},$ and

$$N_k^h = \frac{k^2v_0 - k^2\sqrt{v_0^2 - 4bQ_0}}{2ikQ_0}, \quad k \in \mathbb{Z}^*.$$

For $k \in \mathbb{Z}^*$, the eigenfunction of $A^*$ for eigenvalue $-\lambda_k^h$ is

$$(\xi_k^p)^* = \frac{1}{\psi_k^p} \left( e^{-ikx} N_k^p e^{-ikx} \right), \quad k \in \mathbb{Z}^*,$$

where $\psi_k^p = \frac{\sqrt{2\pi(b - Q_0|N_k^p|^2)}}{\sqrt{(b + Q_0|N_k^p|^2)}},$ and

$$N_k^p = \frac{k^2v_0 + k^2\sqrt{v_0^2 - 4bQ_0}}{2ikQ_0}, \quad k \in \mathbb{Z}^*.$$

The families of eigenfunctions $\{\xi_k^h \cup \xi_n^h : n \in \mathbb{Z}^*\}$ and $\{(\xi_k^h)^* \cup (\xi_n^h)^* : n \in \mathbb{Z}^*\}$ satisfy the following bi-orthogonality relation

$$\langle \xi_k^h, (\xi_n^h)^* \rangle = \delta_n^k, \quad k, n \in \mathbb{Z}, \quad \langle \xi_k^p, (\xi_n^p)^* \rangle = \delta_n^k, \quad k, n \in \mathbb{Z}^*,$$

and

$$\langle \xi_k^h, (\xi_n^p)^* \rangle = 0 = \langle \xi_n^h, (\xi_k^p)^* \rangle, \quad k \in \mathbb{Z}, \ n \in \mathbb{Z}^*,$$

where $\delta_n^k$ is the Kronecker symbol.

Moreover,

$$|\psi_k^h| \to \sqrt{2\pi b}, \quad \frac{|\psi_k^p|}{|k|} \to \sqrt{2\pi/Q_0v_0}, \quad as \ |k| \to \infty,$$

$$|N_k^h| \frac{c}{|k|} \to 0, \quad \frac{|N_k^p|}{Q_0} \to \frac{v_0}{Q_0}, \quad as \ |k| \to \infty. \quad (2.8)$$

**Remark 2.12.** In Lemma 2.11, $\psi_k^h$ and $\psi_k^p$ can be zero if and only if $k = \frac{2\sqrt{bQ_0}}{v_0}$. We have excluded this case by assuming that the spectrum of $A$ has simple eigenvalues. For $k = \frac{2\sqrt{bQ_0}}{v_0}$, using generalized eigenfunctions of $A$ and $A^*$ corresponding to each eigenvalues $-\lambda_k^h$ and $-\lambda_k^h$, the normalizing factors $\psi_k^h$ and $\psi_k^p$ will be different. If the spectrum of $A$ has multiple eigenvalues other than the case of $k = \frac{2\sqrt{bQ_0}}{v_0}$, the same analysis as in Lemma 2.11 will be applicable.

Finally we have the following lemma under the assumption that $A$ has simple eigenvalues. In the case of multiple eigenvalues, we have a similar analysis considering the generalized eigenfunction suitably.
Lemma 2.13. Under the assumption that $A$ has simple eigenvalues, we have the following results:

(a) The eigenfunctions \( \{ \xi_0^h \} \cup \{ \xi_n^h, \xi_n^p : n \in \mathbb{Z}^* \} \) form a Riesz basis on $Z$. Every $z \in Z$ can be represented uniquely by

\[
 z = \sum_{n \in \mathbb{Z}} \langle z, (\xi_n^h)^* \rangle \xi_n^h + \sum_{n \in \mathbb{Z}^*} \langle z, (\xi_n^p)^* \rangle \xi_n^p, 
\]

and there exist positive numbers $m$, $M$ such that

\[
m \left( \sum_{n \in \mathbb{Z}} |\langle z, (\xi_n^h)^* \rangle|^2 + \sum_{n \in \mathbb{Z}^*} |\langle z, (\xi_n^p)^* \rangle|^2 \right) \leq \|z\|_{Z}^2 
\leq M \left( \sum_{n \in \mathbb{Z}} |\langle z, (\xi_n^h)^* \rangle|^2 + \sum_{n \in \mathbb{Z}^*} |\langle z, (\xi_n^p)^* \rangle|^2 \right).
\]

(b) The eigenfunctions \( \{ (\xi_0^h)^* \} \cup \{ (\xi_n^h)^*, (\xi_n^p)^* : n \in \mathbb{Z}^* \} \), of $A^*$ corresponding to eigenvalues $-\lambda_0^h = 0$ and $-\overline{\lambda_n^h}, -\lambda_n^p : n \in \mathbb{Z}^*$, also form a Riesz basis on $Z'$. Every $z' \in Z'$ can be represented uniquely by

\[
 z' = \sum_{n \in \mathbb{Z}} \langle z', \xi_n^h \rangle (\xi_n^h)^* + \sum_{n \in \mathbb{Z}^*} \langle z', \xi_n^p \rangle (\xi_n^p)^*,
\]

and there exist positive numbers $m'$, $M'$ such that

\[
m' \left( \sum_{n \in \mathbb{Z}} |\langle z', \xi_n^h \rangle|^2 + \sum_{n \in \mathbb{Z}^*} |\langle z', \xi_n^p \rangle|^2 \right) \leq \|z'\|_{Z'}^2 \leq M' \left( \sum_{n \in \mathbb{Z}} |\langle z', \xi_n^h \rangle|^2 + \sum_{n \in \mathbb{Z}^*} |\langle z', \xi_n^p \rangle|^2 \right).
\]

Similar details in the space $L^2(I_{2\pi}) \times L^2(I_{2\pi})$, corresponding to Lemma 2.13 can be found in [16]. Details regarding Riesz basis can be found in [6], for example.

Now we define the following subspaces of $Z'$

\[
 (Z')^h = \text{span}\{(\xi_k^h)^* ; k \in \mathbb{Z}\}, \quad (Z')^p = \text{span}\{(\xi_k^p)^* ; k \in \mathbb{Z}^*\}.
\]

Using the fact that the eigenfunctions of $A^*$ form a Riesz basis on $Z'$, we have

\[
 Z' = (Z')^h + (Z')^p.
\]

Since any $z' \in Z'$ has the expansion as in (2.10), we define the hyperbolic projection $(\pi^h)^* : Z' \rightarrow Z'$ with $\text{Range}(\pi^h)^* \subset (Z')^h$ as

\[
 (\pi^h)^*(z') = \sum_{n \in \mathbb{Z}} \langle z', \xi_n^h \rangle (\xi_n^h)^*.
\]
and the parabolic projection \( (\pi^p)^* : \mathbb{Z}' \rightarrow \mathbb{Z}' \) with \( \text{Range}( (\pi^p)^*) \subset (\mathbb{Z}')^p \) by
\[
(\pi^p)^*(z') = \sum_{n \in \mathbb{Z}^*} \langle z', \xi_n \rangle (\xi_n)^*.
\] (2.12)

The restriction of the operator \( A^* \) on to \((\mathbb{Z}')^p\) is \( A^*_p \) defined by \( A^*_p = A^*(\pi^p)^* \).

3. Two Ingham type inequalities for complex frequencies

In order to prove the observability inequality we need Ingham types inequalities for the function

\[
g(t) = \sum_{n \in \mathbb{Z}} \beta_n e^{-\lambda_n t},
\]
where \( \sum_{n \in \mathbb{Z}} |\beta_n|^2 < \infty \) and \( \{-\lambda_n\}_{n \in \mathbb{Z}} \) is a sequence of complex numbers in one case with convergent real part and in another case divergent real part, as follows:

(i) When \( \lambda_n = \lambda_n^h \) and we have \( \lambda_n^h = \omega_0 + \varepsilon_n - inV_0 \) with \( \varepsilon_n \rightarrow 0 \), as \( |n| \rightarrow \infty \); that is, the real part of \( -\lambda_n \) is converging.

(ii) When \( \lambda_n = \lambda_n^p \) and we have \( \Re \lambda_n^p \rightarrow v_0 \), as \( |n| \rightarrow \infty \); that is, the real part of \( \lambda_n \) is diverging.

For case (i) we have the following hyperbolic Ingham inequality. The proof is a minor modification of Ingham’s original argument [10]. A version of this inequality was used in [18].

**Proposition 3.1.** Let \( T > \frac{2\pi}{V_0} \). There exist \( N \geq N_0 \), as defined in (2.7) and positive constants \( C \) and \( C_1 \) depending on \( T \) such that for \( g(t) = \sum_{|n|>N} \beta_n e^{-\lambda_n t} \) with \( \sum_{|n|>N} |\beta_n|^2 < \infty \) and

\[
-\lambda_n = -\lambda_n^h, \quad -\lambda_n^h = -\omega_0 + \varepsilon_n + inV_0
\]

with \( \varepsilon_n \rightarrow 0 \), as \( |n| \rightarrow \infty \), we have

\[
C \sum_{|n|>N} |\beta_n|^2 \leq \int_0^T |g(t)|^2 dt \leq C_1 \sum_{|n|>N} |\beta_n|^2.
\] (3.1)

Now in the next proposition we state the Ingham type inequality for the case of complex frequencies with real part diverging as \( n^2 \) when \( n \rightarrow \infty \) and from that we derive our required parabolic Ingham type inequality for case (ii) in Proposition 3.3.

**Proposition 3.2.** Let \( g(t) = \sum_{n>N} \beta_n e^{i\omega_n t} \) with \( \sum_{n>N} |\beta_n|^2 < \infty \), for any \( N \in \mathbb{N} \). Let \( \omega_n = a_n + ib_n \) and \( a_n, b_n \) satisfy the following properties,

(i) \( b_n > 0 \) and there exists a \( \beta > 0 \), such that \( \frac{b_n}{a_n} > \beta \) together with the following separation condition:
(ii) there exist \( r > 1 \) and \( \tilde{\delta} > 0 \), such that

\[
|\omega_n - \omega_k| \geq \tilde{\delta} |n^r - k^r|,
\]

and there exist \( \epsilon > 0, A \geq 0, B \geq \tilde{\delta} \) such that

\[
\epsilon (A + Bn^r) \leq |\omega_n| < A + Bn^r.
\]

Then for any \( T > 0 \) there exists a constant \( C > 0 \) depending on \( T \) such that

\[
\int_0^T \left| \sum_{|n| > N} \beta_ne^{i\omega_nt} \right|^2 dt \geq C \sum_{|n| > N} \frac{|\beta_n|^2}{|\omega_n|} e^{-2b_nT}.
\] (3.2)

The proof of the inequality (3.2) is originally due to Hansen [9]. It is a consequence of Muntz Szasz Theorems and a similar inequality for real exponentials has been studied in Proposition 3.2 in [13]. In the form used here, the inequality is stated as Proposition 4.2 in Chapter 2 of [12], and quoted in [7] (see Eq. (5) and the inequalities following it on p. 943). Proposition 3.2 is also related to the recent work of Seidman, Avdonin and Ivanov in [21].

**Proposition 3.3 (Applicability of Proposition 3.2).** For any \( T > 0 \) there exists a constant \( D > 0 \) depending on \( T \) such that

\[
\int_0^T \left| \sum_{|n| > N} \beta_n e^{-\lambda_n^p t} \right|^2 dt \geq D \sum_{|n| > N} |\beta_n|^2 e^{-2Re(\lambda_n^p)T},
\] (3.3)

where \( \sum_{|n| > N} |\beta_n|^2 < \infty \) for \( N \geq N_0 \).

**Proof.** In order to apply Proposition 3.2, let us define for \( n > N \geq N_0 \)

\[
i \omega_{2n-1} = -\lambda_n^p, \quad i \omega_{2n} = -\lambda_{-n}^p
\]

i.e. for \( k > 2N \)

\[
i \omega_k = \begin{cases} 
-\lambda_{-k}^p, & k \text{ is even} \\
-\lambda_{k+1}^p, & k \text{ is odd}
\end{cases}
\]

Let us also define for \( k > 2N \)

\[
\tilde{\beta}_k = \begin{cases} 
\beta_{-k}^p, & k \text{ is even} \\
\beta_{k+1}^p, & k \text{ is odd}
\end{cases}
\]
From Lemma 2.6, for \( |n| > N_0 \), we have

\[
\begin{align*}
Re \lambda_n^p &= \frac{n^2 v_0 + |n|\sqrt{n^2 v_0^2 - 4bQ_0}}{2} > 0, \\
Im \lambda_n^p &= -nV_0.
\end{align*}
\]

Calling \( \omega_k = a_k + ib_k \), we have

\[
\begin{align*}
a_k &= \begin{cases} 
Im \lambda_{\frac{k}{2}}^p &= \frac{kV_0}{2}, & \text{if } k \text{ is even,} \\
Im \lambda_{\frac{k+1}{2}}^p &= -\frac{(k+1)V_0}{2}, & \text{if } k \text{ is odd}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
b_k &= \begin{cases} 
Re \lambda_{\frac{k}{2}}^p &= \frac{(\frac{k}{2})^2 v_0 + \frac{k}{2} \sqrt{(\frac{k}{2})^2 v_0^2 - 4bQ_0}}{2}, & \text{if } k \text{ is even} \\
Re \lambda_{\frac{k+1}{2}}^p &= \frac{(\frac{k+1}{2})^2 v_0 + \frac{k+1}{2} \sqrt{(\frac{k+1}{2})^2 v_0^2 - 4bQ_0}}{2}, & \text{if } k \text{ is odd}
\end{cases}
\end{align*}
\]

Observe that for \( \forall k > 2N \geq 2 \), \( b_k > 0 \) and we get from above expression

\[
\frac{b_k}{|a_k|} > \frac{v_0}{2V_0} := \beta.
\]

Further, the separation condition

\[
|\omega_k - \omega_l| > \delta |k^2 - l^2|
\]

holds for an appropriate \( \delta > 0 \). Thus \( r = 2 > 1 \). Indeed, if \( k \) and \( l \) are not consecutive, then

\[
|\omega_k - \omega_l| \geq |b_k - b_l| \geq C|k^2 - l^2|
\]

for some constant \( C \). If \( k = 2m - 1, l = 2m, m \in \mathbb{N} \), then \( b_k = b_l \). However

\[
|\omega_k - \omega_l| = |a_k - a_l| = 2mV_0 > C[2m^2 - (2m - 1)^2] = C|k^2 - l^2|
\]

for a suitable \( C \). By calculation we see

\[
|\omega_k| < (\sqrt{\frac{V_0^2 + v_0^2}{2}})k^2 \quad \text{and} \quad |\omega_k| > \frac{v_0}{8} k^2 = \epsilon B k^2,
\]

where \( B = \sqrt{\frac{V_0^2 + v_0^2}{2}} \) and \( \epsilon = \frac{v_0}{8B} \). Finally we have

\[
\epsilon B k^2 < |\omega_k| < B k^2.
\]
Therefore the conditions (i) and (ii) stated in Proposition 3.2 are satisfied and hence from (3.2), we get

\[
\int_0^T \left| \sum_{n>2N} \tilde{\beta}_n e^{i\omega_n t} \right|^2 dt \geq \frac{C}{\sum_{n>2N} \frac{1}{|\omega_n|} \sum_{n>2N} |\tilde{\beta}_n|^2} \frac{1}{e^{-2b_n T}}.
\]

(3.4)

Since

\[
\int_0^T \left| \sum_{n>2N} \tilde{\beta}_n e^{i\omega_n t} \right|^2 dt \geq \int_0^T \left| \sum_{n>2N} \tilde{\beta}_n e^{i\omega_n t} \right|^2 dt,
\]

we get

\[
\int_0^T \left| \sum_{n>2N} \tilde{\beta}_n e^{i\omega_n t} \right|^2 dt \geq \frac{C}{\sum_{n>2N} \frac{1}{|\omega_n|} \sum_{n>2N} |\tilde{\beta}_n|^2} \frac{1}{e^{-2b_n T}}
\]

\[
\geq \frac{C}{\sum_{n>2N} \frac{1}{|\omega_n|} \sum_{n>2N} |\tilde{\beta}_n|^2} \frac{e^{b_n T}}{e^{-2b_n T}}
\]

\[
\geq \frac{C}{\sum_{n>2N} \frac{1}{|\omega_n|} \sum_{n>2N} |\tilde{\beta}_n|^2} \frac{e^{b_n T}}{e^{-2b_n T}},
\]

since \(e^{b_n T} > \frac{1}{\sqrt{1 + \frac{1}{b_n^2}}} b_n > \frac{T}{\sqrt{1 + \frac{1}{b_n^2}}}\).

Thus (3.3) is proved. \(\square\)

4. Observability inequality

We first state the standard equivalent condition for null controllability of our system (1.1)–(1.3), i.e. the observability inequality for the adjoint system. Then we will prove the observability inequality and this will complete the proof of Theorem 1.2. Our aim is to split the observability inequality into a hyperbolic part and a parabolic part. Once we have that splitting, then, by the two Ingham type inequalities (3.1) and (3.3), we can prove the observability inequalities for these parts separately. Finally, combining these two inequalities we prove our required observability inequality.

**Proposition 4.1.** The system (1.1)–(1.3) is null controllable in the space \(Z\) if there exists a positive constant \(c\) such that for any terminal condition \((\sigma_T, v_T) \in Z'\), the solution \((\sigma, v)\) of the adjoint problem (2.6) satisfies

\[
\int_0^T \int_\Omega |v(x, t)|^2 dx dt \geq c \| (\sigma(\cdot, 0), v(\cdot, 0)) \|_{Z'}^2.
\]

(4.1)
The proof is standard (see, for example, [15]) and will be omitted.

To use Ingham inequalities we need that the exponentials should satisfy a certain gap condition. This leads us to assume that the eigenvalues of $A$ are simple. The detailed analysis under this assumption is given in Section 4.1. Since the spectrum of $A$ may have at most finite number of multiple eigenvalues, we are able to prove the observability inequality in this situation too, after a slight modification of the analysis given in Section 4.1. This case is studied in Section 4.2.

4.1. Case of $A$ with simple eigenvalues

Here we consider the case when the spectrum of $A$ has no repeated eigenvalue. We establish the inequality for high-frequency components corresponding to the eigenvalues $-\lambda_n^h, -\lambda_n^p$ for $|n|$ sufficiently large in Section 4.1.1 and then include the remaining finite number of terms in the inequality in Section 4.1.2.

4.1.1. Splitting into hyperbolic and parabolic parts

If $V_T = (\sigma_T, v_T)$ is in $Z'$, then we have the following expansion

$$V_T = \begin{pmatrix} \sigma_T \\ v_T \end{pmatrix} = \sum_{n \in \mathbb{Z}} d_n^h (\xi_n^h)^* + \sum_{n \in \mathbb{Z}^*^} d_n^p (\xi_n^p)^*.$$  \hfill (4.2)

We denote the hyperbolic and parabolic projection of $(\sigma_T, v_T)$ as follows

$$\begin{pmatrix} \sigma_T^h \\ v_T^h \end{pmatrix} = (\pi^h)^* \begin{pmatrix} \sigma_T \\ v_T \end{pmatrix}; \quad \begin{pmatrix} \sigma_T^p \\ v_T^p \end{pmatrix} = (\pi^p)^* \begin{pmatrix} \sigma_T \\ v_T \end{pmatrix}.$$  

From (2.11) and (2.12)

$$\begin{pmatrix} \sigma_T^h \\ v_T^h \end{pmatrix} = \sum_{n \in \mathbb{Z}} d_n^h (\xi_n^h)^*; \quad \begin{pmatrix} \sigma_T^p \\ v_T^p \end{pmatrix} = \sum_{n \in \mathbb{Z}^*^} d_n^p (\xi_n^p)^*.$$  

We denote the tail of the series on the right hand side of expansion (4.2) from index $N$ (will be chosen suitably large later) by $(\sigma_T, v_T, \infty)$ and define by

$$V_T, \infty = \begin{pmatrix} \sigma_T, \infty \\ v_T, \infty \end{pmatrix} = \sum_{|n| > N} d_n^h (\xi_n^h)^* + d_n^p (\xi_n^p)^*.$$  \hfill (4.3)

Further we denote the solution of the adjoint problem (2.6) with terminal condition $(\sigma_T, v_T, \infty)$ by

$$V_\infty (\cdot, t) = \begin{pmatrix} \sigma_\infty (\cdot, t) \\ v_\infty (\cdot, t) \end{pmatrix} = \sum_{|n| > N} d_n^h e^{-\lambda_n^h (T-t)} (\xi_n^h)^* + d_n^p e^{-\lambda_n^p (T-t)} (\xi_n^p)^*.$$  \hfill (4.4)

Observe that using the expressions for the eigenfunctions,

$$v_\infty (x, t) = \sum_{|n| > N} d_n^h \frac{N_n^h}{\psi_n^h} e^{-inx} e^{-\lambda_n^h (T-t)} + d_n^p \frac{N_n^p}{\psi_n^p} e^{-inx} e^{-\lambda_n^p (T-t)}.$$  \hfill (4.5)
Since these expressions will appear repeatedly in the sequel, we set

\[ \alpha_n = d_n^h N_n^h \psi_n^h, \quad \beta_n = d_n^p N_n^p \psi_n^p. \]  

(4.6)

The corresponding hyperbolic and parabolic parts of the velocity \( v_\infty \) are defined by

\[ v_h^\infty(x, t) = \sum_{|n| > N} \alpha_n e^{-inx} e^{-\gamma_n(T-t)} \quad \text{and} \quad v_p^\infty(x, t) = \sum_{|n| > N} \beta_n e^{-inx} e^{-\gamma_n(T-t)}, \]  

(4.7)

and we have

\[ v_\infty(x, t) = v_h^\infty(x, t) + v_p^\infty(x, t). \]

**Theorem 4.2.** Let \((\sigma, v)\) be the solution of the adjoint problem (2.6) with terminal condition \((\sigma_T, v_T)\). Then there exist \( N \in \mathbb{N}, N \geq N_0 \) and a positive integer \( c \) depending on \( T, \mathcal{O}, I_{2\pi} \) such that the following inequality is true for \( T > \frac{2\pi}{V_0} \).

\[
\int_0^T \int_{\mathcal{O}} |v_\infty(x, t)|^2 \, dx \, dt \geq c \left( \int_0^T \int_{\mathcal{O}} |v_h^\infty(x, t)|^2 \, dx \, dt + \int_0^T \int_{\mathcal{O}} |v_p^\infty(x, t)|^2 \, dx \, dt \right). \]  

(4.8)

**Proof.** Let us define

\[ M_T := \int_0^T \int_{\mathcal{O}} |v_\infty(x, t)|^2 \, dx \, dt, \]

\[ M_h^T := \int_0^T \int_{\mathcal{O}} |v_h^\infty(x, t)|^2 \, dx \, dt, \]

\[ M_p^T := \int_0^T \int_{\mathcal{O}} |v_p^\infty(x, t)|^2 \, dx \, dt. \]

Our aim is to show

\[ M_T \geq c(M_h^T + M_p^T), \quad \text{for } T > \frac{2\pi}{V_0}, \]  

(4.9)

where here and in the following \( c \) will denote a generic constant.

As a function of time \( \tau \), the quantities defined above satisfy the following.

\[ M_h^\tau \leq 2(M_\tau + M_p^\tau), \quad M_p^\tau \leq 2(M_\tau + M_h^\tau), \quad \forall \tau > 0, \]

\[ M_{\tau_1} < M_{\tau_2}, \quad 0 < \tau_1 < \tau_2. \]  

(4.10)
Then for establishing (4.9), it suffices to prove

\[ c M^h_T \leq M_T. \]  \hspace{1cm} (4.11)

To accomplish this, we shall need to consider the integration over a reduced time interval \((0, T - 2\tau)\), where \(\tau\) is small enough so that we still have \(T - 2\tau > \frac{2\pi}{V_0}\).

Applying Proposition 3.1 to \(v^h\) for \(x\) fixed in \(\mathcal{O}\) and then integrating over \(\mathcal{O}\), we get positive constants \(C, C_1\) depending on \(T, \tau, \mathcal{O}\) such that

\[ M^h_T \leq C_1 \sum_{|n| > N} |a_n|^2 \leq C M^h_{T-2\tau}. \]  \hspace{1cm} (4.12)

Thus

\[ M^h_T \leq c M^h_{T-2\tau}. \]  \hspace{1cm} (4.13)

Now our claim is that, for any arbitrary small \(\epsilon > 0\), there exist \(N_\epsilon > N_0\) and a constant \(c\) independent of \(\epsilon, N_\epsilon\) such that

\[ M^p_{T-2\tau} < c \left( M_T + \frac{\epsilon^2}{N^2} M^h_T \right), \hspace{0.5cm} \forall N > N_\epsilon. \]  \hspace{1cm} (4.14)

Suppose that we have established (4.14). Then (4.13), (4.10) and (4.14) give

\[ M^h_T \leq c M^h_{T-2\tau} \leq c \left( M_{T-2\tau} + M^p_{T-2\tau} \right) \leq c \left( M_T + \frac{\epsilon^2}{N^2} M^h_T \right). \]

Thus

\[ \left( 1 - \frac{c \epsilon^2}{N^2} \right) M^h_T \leq c M_T. \]

By choosing \(\epsilon > 0\) suitably small (in fact choosing \(\epsilon^2 < \frac{N_0^2}{2c}\)), we get (4.11). Hence (4.9) is proved. Now we are going to give the proof of the claim.

In order to get a sufficiently small \(\epsilon > 0\) as the coefficient of \(M^h_T\) in the right hand side of the inequality (4.14), we multiply \(v^h\) by \(e^\omega(T-s)\) and integrate. We have \(T > \frac{2\pi}{V_0}\), and then for \(t \in \left( \frac{2\pi}{V_0}, T \right)\) and \(x \in I_{2\pi}\), we define

\[ \hat{v}^h(x, t) = \int_{t-\frac{2\pi}{V_0}}^{t} e^\omega(T-s) v^h_\infty(x, s) ds, \hspace{0.5cm} \hat{v}^p(x, t) = \int_{t-\frac{2\pi}{V_0}}^{t} e^\omega(T-s) v^p_\infty(x, s) ds. \]
Now we want to establish

\[ M_T^{P} - 2τ ≤ c \int \int_{\frac{2\pi}{V_0}} |\hat{v}^p(x, t)|^2 \, dx \, dt ≤ c \left( M_T + \frac{\epsilon^2}{N^2} M_T^h \right), \tag{4.15} \]

and this will prove (4.14).

By the definition of \( \hat{v}^h \) and \( \hat{v}^p \) we get

\[ \hat{v}^h(x, t) + \hat{v}^p(x, t) = \int_{t - \frac{2\pi}{V_0}}^{t} e^{\epsilon_0(T - s)} v_\infty(x, s) \, ds. \]

Thus there exists a positive \( c \) depending on \( T, \omega_0 \) and \( O \) such that

\[ \int \int_{\frac{2\pi}{V_0}} |\hat{v}^h(x, t) + \hat{v}^p(x, t)|^2 \, dx \, dt = \int \int_{\frac{2\pi}{V_0}} \int_{t - \frac{2\pi}{V_0}}^{t} e^{\epsilon_0(T - s)} v_\infty(x, s) \, ds \, dx \, dt \]

\[ ≤ c \int \int_{\frac{2\pi}{V_0}} \int_{0}^{T} |v_\infty(x, s)|^2 \, ds \, dx \, dt ≤ cM_T. \tag{4.16} \]

Using (4.7) and the fact \( \lambda_n^h = \epsilon_n - inV_0 \) with \( \epsilon_n \to 0 \) as \( |n| \to \infty \), we have

\[ \hat{v}^h(x, t) = \sum_{|n| > N} \alpha_n e^{-inx} \frac{e^{\frac{2\pi \epsilon_n}{V_0}} - 1}{\epsilon_n - inV_0} e^{(\epsilon_n - inV_0)(T - t)}, \quad t \in \left( \frac{2\pi}{V_0}, T \right), \quad x \in I_{2\pi}. \tag{4.17} \]

Notice that for any \( \epsilon > 0 \), we can choose a large enough \( N_1 \geq N_0 \) such that

\[ |e^{\frac{2\pi \epsilon_n}{V_0}} - 1| < \epsilon, \quad \forall |n| \geq N_1. \tag{4.18} \]

Now applying the right hand side inequality of (3.1) to the function \( \hat{v}^h(x, t) \) for \( t \) extended to \( (0, T) \) and for \( x \) fixed in \( O \), we get after integrating over \( O \)

\[ \int \int_{\frac{2\pi}{V_0}} |\hat{v}^h(x, t)|^2 \, dx \, dt ≤ \int \int_{0}^{T} |\hat{v}^h(x, t)|^2 \, dx \, dt ≤ c \frac{\epsilon^2}{N^2} \sum_{|n| > N} |\alpha_n|^2, \tag{4.19} \]

where \( \epsilon \) can be chosen as mentioned in (4.18) and \( N > N_1 \).
Applying the left side inequality of (3.1) of Proposition 3.1 to $v_{h}^\infty(x, t)$ for $x$ fixed in $\mathcal{O}$ and then integrating over $\mathcal{O}$, we get

$$
\sum_{|n| > N} |\alpha_n|^2 \leq c M_T^h. \tag{4.20}
$$

Now (4.19) and (4.20) imply

$$
\int_{\frac{2\pi}{T}}^T \int_{\mathcal{O}} \left| \hat{v}^h(x, t) \right|^2 dx dt \leq c \frac{\epsilon^2}{N^2} M_T^h. \tag{4.21}
$$

Here the constant $c$ depends only on $T$, $\mathcal{O}$ and $\omega_0$. The triangle inequality together with (4.16) and (4.21) yields

$$
\int_{\frac{2\pi}{T}}^T \int_{\mathcal{O}} \left| \hat{v}^p(x, t) \right|^2 dx dt \leq c \left( M_T + \frac{\epsilon^2}{N^2} M_T^h \right). \tag{4.22}
$$

So the right hand side of the inequality (4.15) is proved.

Using the expression for $v_T^p$ from (4.7),

$$
\int_0^{T-2\pi} \int_{I_2\pi} \left| v_{\infty}^p(x, t) \right|^2 dx dt = 2\pi \int_0^{T-2\pi} \sum_{|n| > N} \left| \beta_n \right|^2 e^{-2 \text{Re} \lambda_n^p (T-t)} \, dt
\leq 2\pi \sum_{|n| > N} \left| \beta_n \right|^2 \left[ 1 - e^{-2 \text{Re} \lambda_n^p (T-2\pi)} \right] e^{-2 \text{Re} \lambda_n^p (2\tau)} \leq c \sum_{|n| > N} \left| \beta_n \right|^2 e^{-2 \text{Re} \lambda_n^p (2\tau)}. \tag{4.23}
$$

Hence

$$
M_T^p = \int_0^{T-2\pi} \int_{I_2\pi} \left| v_{\infty}^p(x, t) \right|^2 dx dt \leq c \sum_{|n| > N} \left| \beta_n \right|^2 e^{-2 \text{Re} \lambda_n^p (2\tau)}. \tag{4.24}
$$

We have from (4.7)

$$
\hat{v}^p(x, t) = \sum_{|n| > N} \beta_n e^{-inx} e^{(\frac{\omega}{\lambda_n^p} + \omega_0)(T-t)}, \tag{4.25}
$$
where

\[
\tilde{\beta}_n = \frac{\beta_n}{\lambda_n^p} \left[ 1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}} \right].
\]  

(4.26)

Note that for all \( t \in \left(\frac{2\pi}{V_0}, T\right) \), \( e^{\omega_0(T-t)} > 1 \) as \( \omega_0 > 0 \). Thus

\[
\int_{\Omega} |\hat{v}^p(x, t)|^2 \, dx = \int_{\Omega} \left| \sum_{|n| > N} \tilde{\beta}_n e^{-inx} e^{-\lambda_n^p (T-t)} \right|^2 e^{2\omega_0(T-t)} \, dx \\
\geq \int_{\Omega} \left| \sum_{|n| > N} \tilde{\beta}_n e^{-inx} e^{-\lambda_n^p (T-t)} \right|^2 \, dx.
\]  

(4.27)

We apply Ingham’s inequality (3.3) to the function

\[
g(t) = \sum_{|n| > N} \tilde{\beta}_n e^{-inx} e^{-\lambda_n^p (T-t)}, \quad t \in (T - \tau_1, T).
\]

for \( x \) fixed in \( \Omega \). After integration of the resulting inequality over \( \Omega \) and using (4.27), we get

\[
\int_{T - \tau_1}^{T} \int_{\Omega} |\hat{v}^p(x, t)|^2 \, dx \, dt \geq D \sum_{|n| > N} |\tilde{\beta}_n|^2 e^{-2 \text{Re} \lambda_n^p \tau_1}.
\]  

(4.28)

For any \( 0 < \tau_1 < T - \frac{2\pi}{V_0} \) we have from (4.28)

\[
\int_{\frac{2\pi}{V_0}}^{T} \int_{\Omega} |\hat{v}^p(x, t)|^2 \, dx \, dt \geq \int_{T - \tau_1}^{T} \int_{\Omega} |\hat{v}^p(x, t)|^2 \, dx \, dt \geq D \sum_{|n| > N} |\tilde{\beta}_n|^2 e^{-2 \text{Re} \lambda_n^p \tau_1}.
\]  

(4.29)

Now we claim that if we choose \( 0 < \tau_1 < \tau \) and large enough \(|n| \in \mathbb{N}\), then there exists a positive constant \( c \) depending only on \( T, \Omega \) and \( \omega_0 \) such that

\[
\frac{|\beta_n|^2}{2 \text{Re} \lambda_n^p} e^{-2(\text{Re} \lambda_n^p)(2\tau)} \leq c |\tilde{\beta}_n|^2 e^{-2 \text{Re} \lambda_n^p \tau_1}.
\]  

(4.30)

Then (4.24), (4.29), (4.30) will imply the left side inequality of (4.15).

To compare \( \beta_n \) and \( \tilde{\beta}_n \), we note that for \(|n| > N_0, |\text{Im} \lambda_n^p| \leq \frac{1}{\beta} \text{Re} \lambda_n^p \) and \( 0 < \text{Re} \lambda_n^p - \omega_0 < \text{Re} \lambda_n^p \). Thus there exists a positive constant \( C_1 > 0 \) such that

\[
C_1 |\lambda_n^p - \omega_0|^2 < (\text{Re} \lambda_n^p)^2.
\]  

(4.31)
From (4.26) we have

\[ |\tilde{\beta}_n|^2 e^{-2 Re \lambda_n^p \tau_1} = \frac{|\beta_n|^2 |1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}}|^2}{|\lambda_n^p - \omega_0|^2} e^{-2 Re \lambda_n^p \tau_1} \]

\[ = \frac{|\beta_n|^2 |1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}}|^2}{|\lambda_n^p - \omega_0|} |1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}}|^2 e^{-2 Re \lambda_n^p \tau_1} \]

By (4.31) for \(|n| > N_0\),

\[ |\tilde{\beta}_n|^2 e^{-2 Re \lambda_n^p \tau_1} \geq C_1 \frac{|\beta_n|^2 |1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}}|^2}{2 Re \lambda_n^p} e^{-2 Re \lambda_n^p (2\tau_1)}. \]

Since \(e^{\frac{2 Re \lambda_n^p \tau_1}{2 Re \lambda_n^p}} > \tau_1\) and \(0 < \tau_1 < \tau\), we get

\[ |\tilde{\beta}_n|^2 e^{-2 Re \lambda_n^p \tau_1} \geq C_1 \frac{|\beta_n|^2 |1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}}|^2}{2 Re \lambda_n^p} e^{-2(Re \lambda_n^p)(2\tau)}. \]

Now \(|1 - e^{-\left(\lambda_n^p - \omega_0\right) \frac{2\pi}{V_0}}| \to 1\) as \(|n| \to \infty\) gives that there exists \(N_2 \geq N_0\) such that for \(|n| \geq N_2 \geq N_0\)

\[ |\tilde{\beta}_n|^2 e^{-2 Re \lambda_n^p \tau_1} \geq C \frac{|\beta_n|^2}{2 Re \lambda_n^p} e^{-2(Re \lambda_n^p)(2\tau)}. \]

Thus we establish (4.30) for \(|n| > N_2\). Let us define

\[ N_\epsilon = \max\{N_1, N_2\} > N_0. \]

Then for this \(N_\epsilon\), (4.30), (4.15) and hence (4.14) hold. This completes the proof of Theorem 4.2.

4.1.2. Completion of the proof of the observability inequality

Using the hyperbolic Ingham inequality (3.1), it is now straightforward to show that

\[ \int_0^T \int_0^T |v_{h\infty}^h(x, t)|^2 dx dt \geq C \sum_{|n| > N} |\alpha_n|^2. \]  (4.32)

Moreover, it follows from the asymptotics of the eigenfunctions and (2.8) in Lemma 2.11 that

\[ \left\| (\sigma_{h\infty}^h(\cdot, 0), v_{h\infty}^h(\cdot, 0)) \right\|^2_{Z} \leq C \sum_{|n| > N} |\alpha_n|^2. \]
Hence by (4.32)

\[
\int_0^T \int_\mathcal{O} |v^h_\infty(x,t)|^2 \, dx \, dt > C \| (\sigma^h_\infty(\cdot, 0), v^h_\infty(\cdot, 0)) \|_{Z'}^2.
\]

Similarly, for the parabolic part, by inequality (3.3) we find

\[
\int_0^T \int_\mathcal{O} |v^p_\infty(x,t)|^2 \, dx \, dt \geq C \sum_{|n| > N} |\beta_n|^2 e^{-2 \text{Re} \lambda_n^p T},
\]

(4.33)

and by (2.8) in Lemma 2.11 we get

\[
\| (\sigma^p_\infty(\cdot, 0), v^p_\infty(\cdot, 0)) \|_{Z'}^2 \leq C \sum_{|n| > N} |\beta_n|^2 e^{-2 \text{Re} \lambda_n^p T}.
\]

Hence

\[
\int_0^T \int_\mathcal{O} |v^p_\infty(x,t)|^2 \, dx \, dt > C \| (\sigma^p_\infty(\cdot, 0), v^p_\infty(\cdot, 0)) \|_{Z'}^2.
\]

This leaves only finitely many terms for $|n| \leq N$. If all the eigenvalues are distinct, then the missing finitely many exponentials can be added one by one in the inequality (4.1) as the required gap condition for Ingham inequality holds. For similar details see for example [15] (Chapter 4, Theorem 4.3). Then the observability inequality (4.1) follows and by Proposition 4.1 we conclude Theorem 1.2.

4.2. Case of A with multiple eigenvalues

Let us assume that the operator $A^*$ has a finite number of multiple eigenvalues with finite multiplicity. Without loss of generality we shall order the eigenvalues of $A^*$ in such a way that the degenerate eigenvalues come first. Thus, we assume that for $n = 1, 2, ..., m$, $A^*$ has eigenvalues $-\tilde{\lambda}_n$ with multiplicity $N_n$.

\{\xi^*_n,j : j = 1, 2, 3, ..., N_n\} are the linearly independent generalized eigenfunctions of $A^*$ for $-\tilde{\lambda}_n$. For $|n| > m$, $A^*$ has all distinct eigenvalues and the element of $Z'$ has a unique series representation by the eigenfunctions of $A^*$.

We decompose any element $(\sigma_T, v_T) \in Z'$ in the form

\[
(\sigma_T, v_T) = (\sigma_{T,1}, v_{T,1}) + (\sigma_{T,2}, v_{T,2}),
\]

where the first part $(\sigma_{T,1}, v_{T,1})$ represents the finite dimensional contribution resulting from degenerate eigenvalues and defined by
(σ_{T,1}, v_{T,1}) = \sum_{n=1}^{m} \sum_{j=1}^{N_{n}} d_{n,j} \xi_{n,j}^{\sigma_{n}}

and the second part (σ_{T,2}, v_{T,2}) represents the contribution from the remaining eigenmodes. Let (σ_{1}, v_{1}) be the solution of the adjoint problem (2.6) with terminal condition (σ_{T,1}, v_{T,1}) and (σ_{2}, v_{2}) be the solution of the adjoint problem (2.6) with terminal condition (σ_{T,2}, v_{T,2}).

To derive the observability inequality when A has multiple eigenvalues, we need a slight modification of the analysis in the case of A with simple eigenvalues. Using Ingham inequalities, we get bounds for the coefficients of (temporal) exponentials. This works regardless of whether eigenvalues are degenerate or not. For degenerate eigenvalues the main difficulty is that we do not get bounds for the coefficients of each eigenfunction, but only for some linear combination of these coefficients. We handle this situation mainly by Lemma 4.3.

**Lemma 4.3.** There exists a constant \( c > 0 \) such that

\[
\int_{0}^{\epsilon} \int_{\mathcal{O}} \left| v_{1}(x,t) \right|^2 \, dx \, dt \geq c \int_{0}^{\epsilon} \int_{I_{2\pi}} \left| v_{1}(x,t) \right|^2 \, dx \, dt \geq c \left\| (\sigma_{1}(\cdot,0), v_{1}(.0)) \right\|^2_{Z'}, \tag{4.34}
\]

**Proof.** First we prove there exists a constant \( c > 0 \) (independent of (σ_{1}, v_{1})) such that

\[
\int_{0}^{\epsilon} \int_{I_{2\pi}} \left| v_{1}(x,t) \right|^2 \, dx \, dt \geq c \left\| (\sigma_{1}(\cdot,0), v_{1}(.0)) \right\|^2_{Z'}. \tag{4.35}
\]

We denote the finite dimensional space of solutions to the adjoint equation which is generated by the generalized eigenfunctions of \( A^* \) corresponding to multiple eigenvalues by \( Z' \). We note that if \( v_{1} = 0 \) for \( t \in (0, \epsilon) \), then so is \( \sigma_{1} \) by virtue of the adjoint equation. But this implies that \( (\sigma_{1}, v_{1}) = (0,0) \) for all \( t \). Consequently, both sides of (4.35) define norms on \( Z' \), and the inequality follows from the equivalence of norms on a finite dimensional space.

Next we prove there exists a constant \( c > 0 \) (independent of (σ_{1}, v_{1})) such that

\[
\int_{0}^{\epsilon} \int_{\mathcal{O}} \left| v_{1}(x,t) \right|^2 \, dx \, dt \geq c \int_{0}^{\epsilon} \int_{I_{2\pi}} \left| v_{1}(x,t) \right|^2 \, dx \, dt. \tag{4.36}
\]

We note that all generalized eigenfunctions are analytic in \( x \) (in fact given by trigonometric functions). Hence \( v_{1} \) on \( I_{2\pi} \) is uniquely determined by its restriction to \( \mathcal{O} \). The inequality now follows from this and the equivalence of norms on a finite dimensional space. \( \square \)

**Proof of Theorem 1.2 in the case of multiple eigenvalues.** As in the case where A has simple eigenvalues, we have

\[
\int_{0}^{T} \int_{\mathcal{O}} \left| v_{2}(x,t) \right|^2 \, dx \, dt \geq c \left\| (\sigma_{2}(\cdot,0), v_{2}(\cdot,0)) \right\|^2_{Z'}. \tag{4.37}
\]
Now we note that $v_1$ is of the form
\begin{equation}
\sum_{n=1}^{m} a_n(x) \exp(-\tilde{\lambda}_n(T-t)) + b_n(x)t \exp(-\tilde{\lambda}_n(T-t)),
\end{equation}

where $a_n$ and $b_n$ are some linear combination of the second component of the generalized eigenfunctions with the eigenvalue $-\tilde{\lambda}_n$. With coefficient $(a_n(x) + b_n(x)t)\exp(-\tilde{\lambda}_n(T-t))$ occurs only once in (4.38) and the required gap condition to apply Ingham inequality is satisfied. Thus by an argument similar to the one, for example [15] (Chapter 4, Theorem 4.3) and [11] (Chapter 4, Theorem 4.5), we can add these terms one by one for $n = 1, 2, \ldots, m$ in to (4.37). In this fashion, we get
\begin{equation}
\int_0^T \int_\mathcal{O} \left| v_2(x,t) + \sum_{n=1}^{m} a_n(x)e^{-\tilde{\lambda}_n(T-t)} + b_n(x)t e^{-\tilde{\lambda}_n(T-t)} \right|^2 \, dx \, dt
\geq c \left( \| \sigma_2(\cdot, 0), v_2(\cdot, 0) \|^2_{Z} + \sum_{n=1}^{m} \| a_n \|^2_{L^2(I_{2\pi})} + \| b_n \|^2_{L^2(I_{2\pi})} \right),
\end{equation}

and consequently
\begin{equation}
\int_0^T \int_\mathcal{O} \left| v(x,t) \right|^2 dx \, dt \geq c \left( \| \sigma_2(\cdot, 0), v_2(\cdot, 0) \|^2_{Z} \right).
\end{equation}

If $T > 2\pi/V_0$, then also $T - \epsilon > 2\pi/V_0$ if $\epsilon$ is small enough. For such $\epsilon$, we also have a constant $c > 0$ such that
\begin{equation}
\int_\epsilon^T \int_\mathcal{O} \left| v(x,t) \right|^2 dx \, dt \geq c \left( \| \sigma_2(\cdot, \epsilon), v_2(\cdot, \epsilon) \|^2_{Z} \right).
\end{equation}

For $\epsilon$ small enough, by (4.41) we find
\begin{equation}
\int_0^T \int_\mathcal{O} \left| v(x,t) \right|^2 dx \, dt \geq \int_\epsilon^T \int_\mathcal{O} \left| v(x,t) \right|^2 dx \, dt \geq c \left( \| \sigma_2(\cdot, \epsilon), v_2(\cdot, \epsilon) \|^2_{Z} \right).
\end{equation}

It is an immediate consequence of well-posedness that there exists a constant $c > 0$ such that
\begin{equation}
\int_0^{\epsilon} \int_\mathcal{O} \left| v_2(x,t) \right|^2 dx \, dt \leq c \left( \| \sigma_2(\cdot, \epsilon), v_2(\cdot, \epsilon) \|^2_{Z} \right).
\end{equation}
Thus (4.42) and (4.43) give the inequality
\[
\int_0^T \int_\mathcal{O} |v(x,t)|^2 \, dx \, dt \geq c \int_0^\epsilon \int_\mathcal{O} |v_2(x,t)|^2 \, dx \, dt.
\] (4.44)

Consequently,
\[
\int_0^\epsilon \int_\mathcal{O} |v_1(x,t)|^2 \, dx \, dt \leq c \left( \int_0^\epsilon \int_\mathcal{O} |v(x,t)|^2 \, dx \, dt + \int_0^\epsilon \int_\mathcal{O} |v_2(x,t)|^2 \, dx \, dt \right)
\]
\[
\leq c \int_0^T \int_\mathcal{O} |v(x,t)|^2 \, dx \, dt.
\] (4.45)

Using (4.34) and (4.45) we get
\[
\int_0^T \int_\mathcal{O} |v(x,t)|^2 \, dx \, dt \geq c \left( \|\sigma_1(\cdot, 0), v_1(\cdot, 0)\|^2_{L^2} \right). \tag{4.46}
\]

Then by (4.40) and (4.46) we finally get
\[
\int_0^T \int_\mathcal{O} |v(x,t)|^2 \, dx \, dt \geq c \left( \|\sigma_1(\cdot, 0), v_1(\cdot, 0)\|^2_{L^2} + \|\sigma_2(\cdot, 0), v_2(\cdot, 0)\|^2_{L^2} \right)
\]
\[
\geq c \left( \|\sigma(\cdot, 0), v(\cdot, 0)\|^2_{L^2} \right). \tag{4.47}
\]

Hence Theorem 1.2 is proved in the case when \( A \) has multiple eigenvalues. \( \square \)

5. Lack of null controllability in \( H_s \), for \( 0 \leq s < 1 \)

The system (1.1) with periodic condition (1.3) and initial condition (1.2) can be written in the following form on \( H_s \):
\[
U'(t) = AU(t) + Bf(t), \quad U(0) = U_0 \in H_s, \tag{5.1}
\]
where \( U(t) = (\rho(\cdot, t), u(\cdot, t)) \) and \( A \) is an unbounded operator on \( H_s \) defined as (2.2) with
\[
\mathcal{D}(A) = \mathcal{H}_{per}^{s+1}(I_{2\pi}) \times \mathcal{H}_{per}^2(I_{2\pi}),
\]
and the control operator \( B \in \mathcal{L}(L^2(\mathcal{O}), H_s) \) is defined by
\[
Bf(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.
\]
Let $V = (\sigma, v)$ and consider the corresponding adjoint problem on $H^s$: 

$$-V'(t) = A^*V(t)$$  \hfill (5.2) 

with the boundary conditions 

$$\sigma(0, t) = \sigma(2\pi, t); \quad v(0, t) = v(2\pi, t); \quad v_x(0, t) = v_x(2\pi, t)$$ 

and terminal condition $\sigma(x, T) = \sigma_T(x); \quad v(x, T) = v_T(x)$; where $A^*$ has the expression given in Lemma 2.4 and 

$$D(A^*) = \{(\sigma, v) \in \dot{H}_{per}^{-s+1}(I_{2\pi}) \times H_{per}^1(I_{2\pi}) \mid b\sigma + v_0v_x \in H_{per}^1(I_{2\pi})\}.$$

The unbounded operator $A$ generates a $C_0$ semigroup on $H^s$ (for $s = 0$ this can be proved using energy estimates and the Lumer–Phillips theorem, and the case of general $s$ follows by interpolation between $s = 0$ and $s = 1$). Moreover, $A$ and $A^*$ have the same spectrum and eigenvalues on $H^s$ as stated in Section 2.1 for the space $Z$.

**Proof of Theorem 1.3.** The null controllability of the system (5.1) on $H^s$ at time $T > 0$ is equivalent to the following observability inequality (cf. Proposition 4.1): There exists $C > 0$ depending on $T$ and $\mathcal{O}$ such that 

$$\int_0^T \int_{\mathcal{O}} |v(x, t)|^2 \, dx \, dt \geq C \left\| (\sigma(\cdot, 0), v(\cdot, 0)) \right\|^2_{H^s}$$ \hfill (5.3) 

for any terminal condition $(\sigma_T, v_T) \in H^s$, where $(\sigma, v)$ is the solution of the adjoint problem (5.2). We will show by contradiction that for any $T > 0$ the inequality (5.3) cannot be true. Assume for some $T > 0$ the observability inequality (5.3) is true. Consider the sequence of terminal conditions $(\sigma_{T,n}, v_{T,n}) = (\xi^h_n)^*$ for all $n \in \mathbb{Z}^*$. Then $(\sigma_n, v_n)$, the solution of the system (5.2) with terminal condition $(\sigma_{T,n}, v_{T,n}) = (\xi^h_n)^*$, has the expression 

$$\begin{pmatrix} \sigma_n(x, t) \\ v_n(x, t) \end{pmatrix} = e^{-\lambda^h_n(T-t)}(\xi^h_n)^*.$$ \hfill (5.4) 

So $v_n(x, t) = \frac{N_n}{\psi_n} e^{-inx} e^{-\lambda^h_n(T-t)}$ and 

$$\int_0^T \int_{\mathcal{O}} |v_n(x, t)|^2 \, dx \, dt = \frac{|\mathcal{O}||N^h_n|^2}{|\psi_n^h|^2} e^{-2\text{Re} \lambda^h_n T} \left[ 1 - e^{-2\text{Re} \lambda^h_n T} \frac{\text{Re} \lambda^h_n}{\text{Re} \lambda^h_n T} \right].$$
Since $|\mathcal{O}| \leq 2\pi$ and $Re \lambda_n^h \rightarrow \omega_0 > 0$, $|\psi_n^h|^2 \rightarrow 2\pi b$ as $|n| \rightarrow \infty$ and for large enough $|n| \in \mathbb{N}$ $|N_n^h|^2 < \frac{9\omega_0}{4Q_0^{n^*}}$, there exists a constant $M > 0$ and $N \in \mathbb{N}$ such that

$$\int_0^T \int_\mathcal{O} |v_n(x, t)|^2 \, dx \, dt \leq \frac{M}{n^2}, \quad \forall |n| > N. \quad (5.5)$$

Now for every $n \in \mathbb{Z}^*$, $\| (\sigma_n(\cdot, 0), v_n(\cdot, 0)) \|^2_{H^s_x} = e^{-2Re \lambda_n^h T} \| (\xi_n^h)^* \|^2_{H^s_x}$. We can check that $|n|^{2s} \| (\xi_n^h)^* \|^2_{H^s_x} \rightarrow 1$ and $e^{-2Re \lambda_n^h T} \rightarrow e^{-2\omega_0 T}$ as $|n| \rightarrow \infty$. So for large enough $|n| \in \mathbb{N}$, say $|n| > N$,

$$\| (\xi_n^h)^* \|^2_{H^s_x} > \frac{1}{2|n|^{2s}} \quad \text{and} \quad e^{-2Re \lambda_n^h T} > \frac{e^{-2\omega_0 T}}{2}, \quad \forall |n| > N. \quad (5.6)$$

Thus

$$\| (\sigma_n(\cdot, 0), v_n(\cdot, 0)) \|^2_{H^s_x} > \frac{e^{-2\omega_0 T}}{4|n|^{2s}}, \quad \forall |n| > N. \quad (5.6)$$

Since we have assumed the observability inequality (5.3) is true, for all $n \in \mathbb{Z}^*$

$$\int_0^T \int_\mathcal{O} |v_n(x, t)|^2 \, dx \, dt \geq C \| (\sigma_n(\cdot, 0), v_n(\cdot, 0)) \|^2_{H^s_x}. \quad (5.5)$$

Then by (5.5) and (5.6), for all $|n| > N$

$$\frac{e^{-2\omega_0 T}}{4} < \frac{M}{Cn^{2-2s}}. \quad (5.7)$$

That leads to a contradiction. Hence the theorem follows. \qed

**Remark 5.1.** If we use two controls in both density and velocity, for system (1.5) the observability inequality (for $s = 0$) is

$$\int_0^T \int_{\mathcal{O}_1} |\sigma(x, t)|^2 \, dx \, dt + \int_0^T \int_{\mathcal{O}_2} |v(x, t)|^2 \, dx \, dt \geq c \| (\sigma(\cdot, 0), v(\cdot, 0)) \|^2_{L^2 \times L^2}. \quad (5.8)$$

Now by using the splitting argument as in Theorem 4.2 for $\int_0^T \int_{\mathcal{O}_1} |\sigma(x, t)|^2 \, dx \, dt$, we get that for any $T$ with $T > \frac{2\pi}{V_0}$ there exists a positive constant $c_1$ such that

$$\int_0^T \int_{\mathcal{O}_1} |\sigma(x, t)|^2 \, dx \, dt > c_1 \| (\sigma^h(\cdot, 0)) \|^2_{L^2} + \| (\sigma^p(\cdot, 0)) \|^2_{L^2}. \quad (5.9)$$
Together with a similar inequality for \( v \), (5.9) gives (5.8) in \( L^2(I_{2\pi}) \times L^2(I_{2\pi}) \). In contrast, in the case of one control the square integral of \( \sigma^h_\infty(\cdot, 0) \) is estimated by \( \int_0^T \int_\Omega |v^h_\infty(x, t)|^2 \, dx \, dt \) requiring \( \sigma \in \dot{H}^{-1}_{per}(I_{2\pi}) \).

6. The case of creeping flow with Dirichlet boundary conditions

We have stated the theorem in the introduction for the interval \((0, 2\pi)\) in order to correspond with our results on the periodic case. For the proof, we shall change the interval to \((0, 1)\); this avoids introducing factors of \(2\pi\) in a number of formulae below. Clearly, the problem is equivalent; the only difference is that instead of \( T > 2\pi/V_0 \) we require \( T > 1/V_0 \).

Taking \( V_0 = 1, \frac{v}{Q_0} = 1 \) and denoting \( b := \alpha \gamma Q_0^{-2} \) in the system (1.7) we consider the problem

\[
\begin{align*}
\rho_t + \rho_x &= -Q_0u_x, \\
u_{xx} &= b\rho_x + f,
\end{align*}
\]

for \( t > 0, x \in (0, 1) \), with the boundary conditions

\[
\begin{align*}
u(0, t) &= \nu(1, t) = \rho(0, t) = 0,
\end{align*}
\]

and an initial condition for \( \rho \). The control \( f \) has support confined to \( x \in [a, c] \).

By integrating the second equation with respect to \( x \), we find

\[
\rho_t + \rho_x = -Q_0b\rho + Q_0bI(\rho) - Q_0F + Q_0I(F),
\]

where

\[
F(x, t) = \int_0^x f(y, t) \, dy,
\]

and

\[
I(\rho) = \int_0^1 \rho(x, t) \, dx.
\]

Actually, it will suffice to consider \( F \) such that \( F \) is supported on \([a, c]\) and \( I(F) = 0 \). We shall limit our attention to such controls in the following. We introduce the operator

\[
A\rho = -\rho_x - Q_0b\rho + Q_0bI(\rho),
\]

in \( L^2(0, 1) \) with domain \( D(A) = \{ \rho \in H^1(0, 1) \mid \rho(0) = 0 \} \). It is clear that \( A \) generates a contraction semigroup.

We want to show approximate controllability on the time interval \([0, T]\). In the usual fashion, this amounts to showing that any solution of the adjoint problem
\[ \dot{\sigma} = -A^*\sigma, \]  
(6.7)

which is orthogonal to the space of available controls is identically zero. Orthogonality to available controls means that \( \sigma \) is independent of \( x \) on \([a, c]\).

6.1. The spectrum

In this subsection, we calculate the eigenvalues and eigenfunctions of \( A \). Clearly, \( A^* \) is exactly the same, only with the direction of \( x \) reversed. Assume \( A\rho = \lambda \rho \). Then \( \rho \) is of the form \( \rho(x) = \exp(\alpha x) - 1 \), and by inserting this into the equation we find

\[ \lambda = -\alpha - Q_0 b = -\frac{Q_0 b}{\alpha} (e^\alpha - 1). \]  
(6.8)

We thus need to find the nonzero roots of

\[ e^\alpha = 1 + \alpha + \frac{\alpha^2}{Q_0 b}. \]  
(6.9)

All nonzero roots are easily seen to be simple. For roots of large modulus, we find the asymptotics

\[ \alpha = 2\pi n + 2 \log |n| + 2 \log(2\pi/Q_0) + i\pi \text{ sgn}(n) + O\left(\frac{\log |n|}{n}\right), \]  
(6.10)

where \( n \) is an integer.

The error term in the last formula does not change the completeness interval or the excess of the system \( \exp(\alpha x) \), where \( \alpha \) varies over the roots of \( (6.9) \) [2]. To determine the excess, we need to find out how many roots there are with imaginary parts between \(-2n\pi \) and \( 2n\pi \) for \( n \) sufficiently large. We claim that this number is \( 2n + 1 \). A simple homotopy argument shows that, instead of \( (6.9) \), we can consider the simpler equation

\[ e^\alpha = \alpha^2. \]  
(6.11)

This equation has a single real root, which is negative. For nonreal roots, let \( \log(\alpha) \) denote the principal branch of the logarithm function (with the convention that the argument is in \((-\pi, \pi))\). Clearly \( (6.11) \) is equivalent to

\[ \alpha = 2\log(\alpha) + 2\pi i k, \]  
(6.12)

for some integer \( k \). An application of the argument principle shows that there is a single root for every \( k \) with \( |k| \geq 2 \). Namely, consider the equation

\[ \alpha = 2\epsilon \log(\alpha) + 2\pi i k, \]  
(6.13)

for \( \epsilon \in [0, 1] \) and a contour \( \Gamma \) which consists of the following four pieces:

1. \( \text{arg}(z) = -\pi, \ \delta < |z| < M \),
2. \( -\pi < \text{arg}(z) < \pi, \ |z| = M \),
3. \( \text{arg}(z) = \pi, \ \delta < |z| < M \),
4. \( \text{arg}(z) = -\pi, \ |z| = M \).

With \( \Gamma \) as in the diagram, it follows from the argument principle that the number of roots of \( e^\alpha = \alpha^2 \) with \( |\alpha| < M \) and \( |\text{arg}(\alpha)| < 2\pi \) is equal to the number of zeros of the function

\[ \frac{1}{2\pi} \log \left( \frac{R}{R^2 - \epsilon^2 \log^2(\alpha)} \right) \]  

inside the contour \( \Gamma \), which is 2n + 1. Thus, the number of roots of \( e^\alpha = \alpha^2 \) is 2n + 1, and the number of roots of \( e^\alpha = 1 + \alpha + \frac{\alpha^2}{Q_0 b} \) is \( 2n + 1 \) as well.
3. \( \arg(z) = \pi, \delta < |z| < M, \)
4. \(-\pi < \arg(z) < \pi, |z| = \delta.\)

Here \( \delta \) is chosen small and \( M \) is large. For \( |k| \geq 2 \) and \( 0 \leq \epsilon \leq 1 \), there are no roots on the contour \( \Gamma' \), hence the number of roots enclosed by \( \Gamma' \) is independent of \( \epsilon \), i.e. equal to 1. This leaves the roots of (6.12) for \( k = 0, \pm 1 \) to be investigated. A numerical application of the argument principle finds two roots for \( k = 0 \) and one each for \( k = \pm 1 \). This confirms the count claimed above. According to [20], the system \( \{\exp((2n + 1)\pi i + 2\log |n|)x\}, n \in \mathbb{Z} \) has excess 1. Consequently, the system \( \{\exp(\alpha x)\} \), where \( \alpha \) varies over the roots of (6.9) has excess 2. However, there is the double root at zero, which produces no eigenfunctions. We conclude that the eigenfunctions form a complete minimal set.

We state this as a theorem:

**Theorem 6.1.** The eigenfunctions of \( A \) or of \( A^* \) form a complete minimal set.

In the following, let \( \phi_k, k = 1, 2, \ldots \) be an enumeration of the eigenfunctions of \( A \), with eigenvalues \( \lambda_k \), and let \( \psi_k \) be an enumeration of the eigenfunctions of \( A^* \), with eigenvalues \( \tilde{\lambda}_k \). We can choose these eigenfunctions in such a way that they are bi-orthogonal:

\[
\int_0^1 \overline{\phi}_k(x)\psi_j(x)\,dx = \delta_{jk}. \quad (6.14)
\]

### 6.2. Proof of observability

We shall need the following lemma:

**Lemma 6.2.** There exists a function \( \chi(x) \), with support in \((a, c)\), such that \( \chi \) is not orthogonal to any eigenfunction of \( A^* \), and has zero integral.

The construction proceeds recursively. We begin with a function \( \chi_1 \), which has zero integral and is not orthogonal to \( \psi_1 \). Then having constructed \( \chi_n \), we find a function \( q_n \) such that \( q_n \) has zero integral and is orthogonal to \( \psi_1, \psi_2, \ldots, \psi_n \), but not orthogonal to \( \psi_{n+1} \) (note that the eigenfunctions \( \psi_k \) are analytic functions, so they are linearly independent not only on the full interval \((0, 1)\), but also on any subinterval). Then we set \( \chi_{n+1} = \chi_n + \gamma_n q_n \), where \( \gamma_n \) is at our disposal as long as \( \chi_{n+1} \) is not orthogonal to \( \psi_{n+1} \). By choosing \( \gamma_n \) sufficiently small as \( n \to \infty \), we can make this process converge in any Sobolev norm in which we want it to converge.

Now consider any solution of \( \dot{\sigma} = -A^*\sigma \), with terminal condition \( \sigma(T) = s \). Let

\[
s_n = \int_0^1 \overline{\phi}_n(x)s(x)\,dx. \quad (6.15)
\]

We then find that

\[
\int_a^b \chi(x)\sigma(x, t)\,dx = \sum_{n=1}^{\infty} s_n \exp(\overline{\lambda}_n(T - t)) \int_a^b \chi(x)\psi_n(x)\,dx. \quad (6.16)
\]
If $\sigma$ is orthogonal to all admissible controls, the left hand side of the equation is zero. If $T > 1$, then the system $\{\exp(\tilde{\lambda}_n(T - t))\}$ is minimal over the interval $(0, T)$. Consequently $s_n = 0$ for every $n$, which implies $s = 0$.

We thus obtain

**Theorem 6.3.** If $T > 1$, then the system (6.3) is approximately controllable with a distributed control that is localized on the interval $[a, c]$.

We remark that if the term $Q_0 I(\rho)$ in (6.3) is omitted, then approximate controllability does not hold. Thus this term (which is of rank one!) fundamentally changes everything. Indeed, this term is responsible for changing an operator with no eigenfunctions to one with a complete set.

### 6.3. Null controllability

As before, we consider the problem

\[
\rho_t = A\rho = -\rho_x - Q_0 b\rho + Q_0 b I(\rho) + \hat{f}, \quad \rho(0, t) = 0, \quad (6.17)
\]

where $\hat{f} \in L^2((0, T); L^2(O))$ with the constraint that

\[
\int_O \hat{f}(x, t) \, dx = 0 \quad (6.18)
\]

for every $t$. We shall prove the following theorem for the system (6.17).

**Theorem 6.4.** The system (6.17) is null controllable by a control $\hat{f} \in L^2((0, T); L^2(O))$ with the constraint stated in (6.18) for any open interval $O = (a, c) \subset (0, 1)$, any $T > 1$ and initial data in $L^2(0, 1)$.

To show this, we consider solutions of the adjoint problem

\[
\sigma_t = -A^*\sigma = -\sigma_x + Q_0 b\sigma - Q_0 b I(\sigma), \quad \sigma(1, t) = 0. \quad (6.19)
\]

By setting $\sigma = e^{Q_0 b t} \tau$, we can eliminate the term $Q_0 b\sigma$, i.e. we have

\[
\tau_t = -A^*\tau = -\tau_x - Q_0 b I(\tau), \quad \tau(1, t) = 0. \quad (6.20)
\]

For any interval $I$, and any time $t$, we define the seminorm

\[
N_I(t)^2 = \min_{\gamma \in \mathbb{R}} \int_I |\tau(x, t) - \gamma|^2 \, dx. \quad (6.21)
\]

The observability inequality we need to prove is

\[
\|\tau(\cdot, 0)\|_{L^2(0, 1)}^2 \leq C \int_0^T N_O(t)^2 \, dt = CM^2. \quad (6.22)
\]
Since the nonlocal term \( Q_0 b I(\tau) \) only adds a constant to \( \tau \), we can use the method of characteristics to show \( N_I(t) \leq CM \), where \( I \) is any interval \((\alpha, \beta)\) such that \( \alpha > a - T + t \) and \( \beta < c + t \). Since we assumed \( T > 1 \), we can set \( \alpha = 0 \) and \( \beta = 1 \) for \( t \in (1 - c, T - a) \). Let \([t_0, t_1]\) be a subinterval of \((1 - c, T - a)\). For any \( t \in [t_0, t_1] \), we have

\[
\int_0^1 (\tau - I(\tau))^2 \, dx \leq CM^2. \tag{6.23}
\]

It remains to get a bound on \( I(\tau) \). For this, we need to use the boundary condition at \( x = 1 \). By integrating (6.20) along characteristics, we find

\[
|\tau(x, t)| \leq Q_0 b \int_t^{t_1} |I(\tau)(s)| \, ds \tag{6.24}
\]

for \( t \in [t_0, t_1] \) and \( x \geq 1 - (t_1 - t) \). Let us introduce the notation

\[
U(t) = \int_t^{t_1} |I(\tau)(s)| \, ds. \tag{6.25}
\]

Clearly, we have

\[
\int_{1-(t_1-t)}^1 |\tau(x, t)|^2 \, dx \leq C(t_1 - t)U(t)^2. \tag{6.26}
\]

On the other hand,

\[
\int_{1-(t_1-t)}^1 |\tau(x, t)|^2 \, dx \geq \frac{1}{2} (t_1 - t) I(\tau)(t)^2 - \int_{1-(t_1-t)}^1 |\tau(x, t) - I(\tau)(t)|^2 \, dx. \tag{6.27}
\]

We can combine the last two inequalities in the form

\[
U'(t)^2 \leq C \left( U(t)^2 + \frac{M^2}{t_1 - t} \right). \tag{6.28}
\]

Using this differential inequality, and the fact that \( U(t_1) = 0 \), we easily get bounds for \( U \) and \( U' \) of order \( M \) when \( t < t_1 \).

We have thus proved that for \( t \in [t_0, t_2] \), where \( t_0 < t_2 < t_1 \), we have

\[
\int_0^1 \tau(x, t)^2 \, dx \leq CM^2. \tag{6.29}
\]
By well-posedness of the adjoint problem, the same inequality holds for $t = 0$. Thus the observability inequality (6.22) is proved and hence Theorem 6.4 follows.

By Theorem 6.3 and Theorem 6.4 we finally conclude Theorem 1.5 when $V_0 = 1$.

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