Maximal Domain for Strategy-Proof Rules with One Public Good

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In the context of the provision of one pure public good, we study how large a preference domain can be to allow for the existence of strategy-proof rules satisfying the no vetoer condition. This question is qualified by the additional requirement that a domain should include “a minimally rich domain.” We first characterize generalized median voter schemes as the unique class of strategy-proof rules on minimally rich domains. Then we establish that the unique maximal domain, including a minimally rich one which allows for the existence of strategy-proof rules satisfying the no vetoer condition, is the domain of convex preferences. Journal of Economic Literature Classification Number: D71.

1. INTRODUCTION

In the context of the provision of one pure public good, we study domains of preferences which allow for strategy-proofness. Strategy-proofness assures us that no agent has an incentive to misrepresent his preferences in order to obtain some benefit regardless of whether the other agents misrepresent or not. It is well known (Gibbard-Satterthwaite theorem; see [11, 16])

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that there does not exist any nondictatorial strategy-proof rule on unrestricted domains when there are at least three alternatives. There are situations where the structure of the set of alternatives suggests natural restrictions, and one may hope to get positive results for strategy-proofness on such restricted domains. A reasonable assumption on preferences is “single-peakedness” (see [7]). On the domain of single-peaked preferences, Moulin [13] shows that generalized median voter schemes are the class of strategy-proof rules. Later on, many authors succeed in the same type of characterization on similar but various domains. Then, one might ask how robust these results are. If domains are enlarged excessively from the single-peaked one, we will encounter a Gibbard-Satterthwaite type negative result: no rule is strategy-proof except for trivial ones. Conversely, if domains are restricted too much, the above characterization results will be lost. We pay special attention to the possibility of designing rules satisfying the following additional “no vetoer condition”: no individual should be able to avoid any alternative to be the outcome by declaring some preference. In this paper, we first consider subdomains of the single-peaked domain which retain the characterization results; and then we investigate how much we can enlarge such domains while keeping the existence of strategy-proof rules satisfying the no vetoer condition.

After the seminal work of Moulin [13], Border and Jordan [9] accomplished a similar characterization on the domain of quadratic preferences, which is much smaller than that of single-peaked preferences. Rules on larger domains have also been studied. Moulin [14] studies a class of strategy-proof rules on the domain of “single-plateaued preferences.” This domain contains the single-peaked one and differs from it by allowing preferences with closed interval as maximal sets. For these preferences, a characterization of strategy-proof and uncompromising rules also exists (see Berga [6]). Barbera and Jackson [2] deal with “weakly single-peaked preferences,” that is, preferences whose maximal set on the range of the rule can contain two alternatives. From all these results, two natural questions arise, which are discussed in the following two paragraphs.

The first question is how much can we weaken the condition on the domain of preferences while preserving generalized median voter schemes as the unique class of strategy-proof rules? Strategy-proofness on a domain

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2 In this paper, we assume that a rule is a function from the set of preference profiles onto the set of alternatives.

3 This assumption says that more is strictly preferred to less up to the unique maximal element (peak), and less to more beyond that point.

4 He refers to these rules as “variants of the Condorcet winner.” Since then, this class has received different names. For example, “phantom voter schemes” in Border and Jordan [9] and “voting by committees” in Barbera, Sonnenschein, and Zhou [5] and Serizawa [17].

5 It is the natural extension of uncompromisingness in Border and Jordan [9].
implies strategy-proofness on its subdomains. Thus, the smaller the domain, the weaker the condition. In this respect, strategy-proofness on the domain of quadratic preferences is the weakest condition in the literature that characterizes generalized median voter schemes. However, it is still a strong assumption for quadratic preference that the attitude before and after the peak is symmetric. Yet, it is often the case that a person is eager for the public good before reaching his peak, but does not care so much after that point. Therefore, in this paper, we propose the less demanding condition of “minimally rich” domains defined in Section 3. We establish that a social choice function on a minimally rich domain is strategy-proof if and only if it is a generalized median voter scheme.

The second question, which is the main one addressed in the paper, is how much can we enlarge the domain of single-peaked preferences without giving up strategy-proofness? This question is qualified in several ways. Note that dictatorships are strategy-proof on the universal domain. Thus, to eliminate such trivial rules, we employ the no vetoer condition. Furthermore, we additionally require domains to include some minimally rich domain. Since the conditions of minimal richness are satisfied by a variety of small domains as we will discuss in Section 3, this additional requirement is weak. Then, we establish that the unique maximal domain including a minimally rich domain for strategy-proofness and the no vetoer conditions is the domain of convex preferences.

In the literature, there are some results of domain maximality for strategy-proofness. Blin and Satterthwaite [8] study a domain on which the majority rule with the Borda completion is strategy-proof. The maximal domains where the much wider class of generalized median voter schemes are strategy-proof are also studied by Barberá, Sonnenschein, and Zhou [5] and Serizawa [17]. Although any strategy-proof rule on the domain of single-peaked preferences must be a generalized median voter scheme, a strategy-proof rule may not belong to this class when the rule is defined outside that domain. Accordingly, this literature may exclude interesting rules by considering only generalized median voter schemes. Thus, the novelty of our paper is that we do not restrict a priori the class of rules to be considered and we establish a domain maximality result by only imposing properties on rules.

In different frameworks, some authors have also studied domains for strategy-proofness without assuming a prespecified structure of rules. In the framework of abstract social alternatives, Kalai and Müller [12] study rules that are generated by an underlying social welfare function and that therefore are defined not only on the set of preferences but also on the class of feasible sets. Their result is closely related in intent to our result, but

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6 See Example 4 in Section 3.
their restriction that the social choice function is rationalized by a social welfare function implies that their result does not hold when the feasible set is fixed (see Footnote 10). In the framework of one private good, Ching and Serizawa [10] have recently given a domain maximality result for rules satisfying strategy-proofness along with Pareto-efficiency and distributional requirements, again without assuming any prespecified structure for the rules.

The paper is organized as follows. Section 2 contains notation and the basic definitions. In Section 3, we characterize generalized median voter schemes as the unique class of strategy-proof rules on a minimally rich domain, and then we establish the main result which refers to domain maximality. In Section 4, we expose a conjecture for further research. Finally, the proofs and some complementary results are found in the Appendix.

2. NOTATION AND DEFINITIONS

Let \( N = \{1, \ldots, n\} \) be the set of agents, where \( n \geq 2 \). The set of alternatives is a closed interval \( Z = [a, b] \) in the real line \( \mathbb{R} \). Preferences are binary relations on \( Z \). We assume that they are representable by continuous utility functions on \( Z \), and we denote by \( \mathcal{U} \) the set of all preferences. Let \( u = (u_1, \ldots, u_n) \in \mathcal{U}^n \) be a preference profile. When we want to emphasize the role of coalition \( C \subseteq N \), we write \( (u^C, u^{-C}) \) to represent the preference profile, where \( u^C = (u^C)_i \in \mathcal{U}^C \), \( u^{-C} = (u^C)_i \in \mathcal{U}^{-C} \), and \( \mathcal{U}^T \) denotes the set of preference profiles of coalition \( T \subseteq N \). Given \( u' \in \mathcal{U} \), the maximal set of \( u' \) in \( Z \) is \( \mathcal{F}(u') \equiv \text{Arg max}_{z \in Z} u(z) \). When it is a singleton, we denote its unique element by \( \tau(u') \). Let \( (\mathcal{F}(u^1), \ldots, \mathcal{F}(u^n)) \) be a profile of maximal sets. A domain is a subset \( \mathcal{P} \) of \( \mathcal{U} \). A social choice function, or simply a rule, is a function \( f \) from \( \mathcal{P} \) to \( Z \). Throughout the paper, we assume that \( f \) is onto \( Z \). When we want to emphasize the domain of definition of a rule, we call it a social choice function on \( \mathcal{P} \).

**Definition 1.** A preference \( u' \in \mathcal{U} \) is **convex** if, for any \( z \), its upper contour set of \( z \) (that is, \( \{z' \in Z : u'(z') \geq u'(z)\} \)) is convex.

Let \( \mathcal{P}_c \subseteq \mathcal{U} \) be the set of all convex preferences.

**Definition 2.** A preference \( u' \in \mathcal{U} \) is **single-peaked** if it has a unique maximal element \( \tau(u') \in Z \) (the peak of \( u' \)) and for any \( z, z' \in Z \), if either \( z < z' \leq \tau(u') \) or \( z > z' \geq \tau(u') \) then \( u'(z') > u'(z) \).

Let \( \mathcal{P}_s \subseteq \mathcal{U} \) be the set of all single-peaked preferences.
Definition 3. A preference $u' \in \mathcal{U}$ is quadratic if it has a unique maximal element $z(u') \in Z$ (the peak of $u'$) and for any $z, z' \in Z$, $u'(z') > u'(z)$ if and only if $|z' - z(u')| < |z - z(u')|$.

Let $\mathcal{P}_o \subseteq \mathcal{U}$ be the set of all quadratic preferences. Note that $\mathcal{P}_o \subseteq \mathcal{P} \subseteq \mathcal{P}_c \subseteq \mathcal{U}$.

Now we introduce some properties for social choice functions. The most relevant one refers to the strategic behavior of agents. We say that $f$ is "strategy-proof" if the best strategy for agents is to tell the truth.

Definition 4. A social choice function $f: \mathcal{P}^n \rightarrow Z$ is strategy-proof if for any $u = (u_1, \ldots, u_n) \in \mathcal{P}^n$, $i \in N$, and $u' \in \mathcal{P}$, $u'(f(u)) \geq u'(f(u_i, u'_{-i}))$. Otherwise, $f$ is said to be manipulable. Moreover, we say that agent $i$ manipulates $f$ at profile $u$ via $v_i$ if $u'(f(v_i, u'_{-i})) > u'(f(u))$.

We also define the notion of an option set, a useful tool to deal with strategy-proofness which was introduced by Barberà and Peleg [4]. It is the set of alternatives attainable by a coalition when the preferences of agents in the complementary coalition are fixed.

Definition 5. Given $f: \mathcal{P}^n \rightarrow Z$ and $C \subseteq N$, define the option set for $-C$ given $u^C$, denoted by $\sigma_f(u^C)$, as the range of $f(u^C, \cdot)$; that is,

$$\sigma_f(u^C) = \{ z \in Z : \exists u^{-C} \in \mathcal{P}^{-C} \text{ such that } f(u^C, u^{-C}) = z \}.$$

The next two definitions refer to the power that some rules give to some agents. The first one, dictatorship, assigns "complete" power to some agent, the dictator. The second one allows for agents to have some "partial" power by allowing them to rule out some alternatives as possible outcomes.

Since these properties are unattractive, our definitions identify (by negation) those functions that would not allow for them.

Definition 6. A social choice function $f: \mathcal{P}^n \rightarrow Z$ is dictatorial if there exists an agent $i \in N$ (the dictator) such that for any $u \in \mathcal{P}^n$, $f(u) \in \mathcal{F}(u_i)$. Otherwise, $f$ is said to be nondictatorial.

Definition 7. A social choice function $f: \mathcal{P}^n \rightarrow Z$ has a vetoer if there exist $i \in N$, $z \in Z$, and $u' \in \mathcal{P}$, such that for any $u^{-i} \in \mathcal{P}^{-1}$, $f(u) \neq z$. Otherwise, $f$ is said to satisfy the no vetoer condition.

Note that the no vetoer condition is equivalent to saying that for any $i \in N$ and $u' \in \mathcal{P}$, $\sigma_f(u') = Z$.

The next condition requires that if all the agents agree on a best alternative, the outcome must be chosen from the common best alternatives.
Definition 8. A social choice function \( f : \mathcal{P}^n \rightarrow Z \) respects unanimity if for any \( u \in \mathcal{P}^n \) such that \( \bigcap_{i \in N} \mathcal{F}(u') \neq \emptyset \), \( f(u) \in \bigcap_{i \in N} \mathcal{F}(u') \).

The following requirement refers to the information given by the maximal set of a preference. This condition is satisfied by rules for which no essential loss of information occurs when considering only the profile of maximal sets instead of the whole preference profile.

Definition 9. A social choice function \( f : \mathcal{P}^n \rightarrow Z \) is tops-only if for any \( u, u' \in \mathcal{P}^n \) such that for any \( i \in N, \mathcal{F}(u_i) = \mathcal{F}(u'_i) \), \( f(u) = f(u') \).

We now define generalized median voter schemes and we follow the formulation of Barberá, Massó, and Serizawa [3]. These social choice functions are a particular class of tops-only rules which provide a natural extension of the basic idea of the median voter scheme. To define them, we need to introduce a certain structure among the agents which, in some sense, will stand for the power of coalitions over the alternatives.

Definition 10. A right (left) coalition system is a correspondence \( \mathcal{W} \) that assigns to every \( z \in Z \) a collection \( \mathcal{W}(z) \) of coalitions satisfying the following conditions:

1. Voter sovereignty: \( \mathcal{W}(z) = 2^N \setminus \emptyset \) (\( \mathcal{W}(\beta) = 2^N \setminus \emptyset \)) and for any \( z \in [x, \beta] \) (\( z \in [\alpha, \beta] \)), \( \mathcal{W}(z) \neq \emptyset \) and \( \emptyset \notin \mathcal{W}(z) \).

2. Coalition monotonicity: if \( W \subseteq \mathcal{W}(z) \) and \( W' \subseteq W \), then \( W' \in \mathcal{W}(z) \).

3. Outcome monotonicity: if \( z' < z \) (\( z' > z \)) and \( W \in \mathcal{W}(z) \), then \( W \in \mathcal{W}(z') \).

4. Upper semicontinuity: for any \( W \subseteq N \), \( z \in Z \), and sequence \( \{z'_t\}_{t \in N} \subseteq Z \) such that \( \lim_{t \rightarrow \infty} z'_t = z \), if for any \( t, W \in \mathcal{W}(z_t) \), then \( W \in \mathcal{W}(z) \).

Given a right (left) coalition system \( \mathcal{R} \ (\mathcal{L}) \), let us define its associated generalized median voter scheme.

Definition 11. Let \( \mathcal{R} \ (\mathcal{L}) \) be a right (left) coalition system. The generalized median voter scheme induced by \( \mathcal{R} \ (\mathcal{L}) \) is the social choice function \( f : (\mathcal{P}_N)^n \rightarrow Z \) such that for any \( u \in (\mathcal{P}_N)^n \),

\[
\begin{align*}
   f(u) &= \max \{ z \in Z : \{ i \in N : \tau(u_i) \geq z \} \in \mathcal{R}(z) \}, \\
   (f(u)) &= \min \{ z \in Z : \{ i \in N : \tau(u_i) \leq z \} \in \mathcal{L}(z) \}.
\end{align*}
\]

Observe that since these social choice functions are tops-only, they can be also defined on any domain of preferences with unique maximal elements.
Any coalition in \( R(z) \) (\( L(z) \)) is said to be a right (left) winning coalition for \( z \).

Remark 1. There is a relationship between the right (\( R \)) and the left (\( L \)) coalition system associated with a generalized median voter scheme. Given \( R \), define \( L^* \) as follows:

\[
L^*(z) = \{ W \in 2^N \setminus \emptyset : \text{for any } z' > z \text{ and } W' \in R(z'), W \cap W' \neq \emptyset \}.
\]

According to the above definition, we can see that given \( R \) and \( L \) they select the same outcome for any \( n \)-tuple of peaks \( (\tau(u^1), \ldots, \tau(u^n)) \) if and only if \( L = L^* \).

Definition 12. Let \( R (L) \) be a right (left) coalition system and \( z \in Z \). We say that \( W \) is a minimal right (left) winning coalition for \( z \) if

(i) \( W \) is a right (left) winning coalition for \( z \), and

(ii) for any \( W' \not\subseteq W, W' \) is not a right (left) winning coalition for \( z \).

Given \( R (L) \), denote by \( R^m (L^m) \) the corresponding sets of minimal right (left) winning coalitions.

We show how to rewrite two specific rules as generalized median voter schemes.

Example 1. Suppose that \( n \) is odd and consider majority voting. Consider a right coalition system \( R \) such that for any \( z \in Z \setminus \{ x \} \), \( R(z) = \{ W \subseteq N : \# W \geq (n+1)/2 \} \) where \( \# W \) denotes the cardinality of \( W \), and \( R(x) = 2^N \setminus \emptyset \). The generalized median voter scheme induced by \( R \) is the median voter scheme.

Example 2. Consider the minimal voters scheme, that is, the social choice function which chooses the minimum of the agents’ peaks. Let \( R \) be a right coalition system such that for any \( z \in Z \setminus \{ x \} \), \( R(z) = \{ N \} \) and \( R(x) = 2^N \setminus \emptyset \). Observe that the minimal voter scheme is the generalized median voter scheme induced by \( R \).

From now on to the end of the section, let \( f \) be a generalized median voter scheme whose associated right and left coalition systems are \( R \) and \( L \), respectively. We define different agents, veto voters, and decisive voters, depending on the power over the social outcome that \( f \) gives to them. It is worth mentioning that, in general, agents’ power depends on the alternative one considers.

Definition 13. Given \( z \in Z \), we say that agent \( i \) is a left (right) veto voter at \( z \) if \( i \in \bigcap_{S \in L^m(z)} S \setminus \bigcap_{S \in R^m(z)} S \).
We say that agent $i$ is a left (right) decisive voter at $z$ if $\{i\} \in \mathcal{L}(z)$ ($\{i\} \in \mathcal{R}(z)$).\footnote{Observe that no agent is either a left veto voter at $\beta$ or a right veto voter at $\alpha$. However, every agent is a left decisive voter at $\beta$ and a right decisive voter at $\alpha$.}

The following remark states the relationship between left and right veto and decisive voters.

**Remark 2.** Given $z \in Z$, an agent is a left (right) veto voter at $z$ if and only if he is a right (left) decisive voter at $z$.\footnote{This relationship comes from Remark 1.}

Note also that the following result holds:

**Remark 3.** If $f$ satisfies the no vetoer condition, then for any $z \in Z$, no agent is either a left or a right veto voter at $z$, except that every agent is both a left veto voter at $\alpha$ and a right veto voter at $\beta$.

Considering the exceptions at points $\alpha$ and $\beta$, we abuse definition and we say that such a generalized median voter scheme has neither veto voters nor decisive voters.

### 3. MAIN RESULTS

As mentioned in the Introduction, strategy-proofness on the domain of quadratic preferences is the weakest condition in the literature that characterizes generalized median voter schemes. One possible further question is how much can we weaken the assumption of “strategy-proofness on a domain of preferences” while preserving generalized median voter schemes as the unique class of strategy-proof rules? Note that by definition of strategy-proofness, the smaller the domain, the weaker the assumption. Therefore, we propose the following notion of a “minimally rich” domain:

**Definition 14.** A domain $\mathcal{P}$ is minimally rich if

(i) $\mathcal{P} \subset \mathcal{P}_S$,

(ii) for any $z \in Z$, there is a unique preference $u^o \in \mathcal{P}$ such that $\tau(u^o) = z$, and

(iii) for any $x, y \in Z$, $x \neq y$, there exists $u^o \in \mathcal{P}$ such that $u^o(x) > u^o(y)$ and $\tau(u^o) \in \{\min\{x, y\}, \max\{x, y\}\}$.

Let $\mathcal{P}_m$ be our generic notation for minimally rich domains. The following are examples:
the domain of quadratic preference $\mathcal{P}_Q$.

(2) $\mathcal{P} = \{ u_z : z \in Z \}$ where $u_z(x) = x - z$ if $x \leq z$ and $u_z(x) = (z - x)/2$ otherwise.

(3) $\mathcal{P} = \{ u_z : z \in Z \}$ where for any $x \in Z$, $u_z(x) = u_o(x + b - z)$, and $u_o$ is a continuous utility function whose peak is $b$. $u_o$ is strictly increasing up to $b$ and strictly decreasing after it.

Observe that Conditions (i) and (ii) together say that for any alternative $z$, there exists only one single-peaked preference with maximal element $z$. In this sense, a minimally rich domain is as “small” as the quadratic one. Also observe that Condition (iii) is satisfied by a diversity of domains. The domain of quadratic preferences is one example, and domains which include preferences asymmetric around their peaks are also admissible (see Examples (2) and (3) above). Thus, Condition (iii) is weaker than requiring symmetry around the peak. In addition to the examples above, there are many cases in which the three conditions hold.

Now we establish a characterization of generalized median voter schemes on minimally rich domains which turns out to be the most powerful result for our object of studying the domain maximality.

**Theorem 1.** A social choice function on a minimally rich domain is strategy-proof if and only if it is a generalized median voter scheme.

Since a minimally rich domain is strictly smaller than the single-peaked domain and may be different from the quadratic domain, the previous results in the literature cannot exclude the possibility that there exist more strategy-proof rules on a minimally rich domain besides the generalized median voters schemes. Theorem 1 says that this is not possible, and thus restricting preference domains in such a way cannot enlarge the class of social choice functions. See the proof of Theorem 1 in the Appendix.

Since $\mathcal{P}_Q$ itself is a minimally rich domain, the characterization below on $\mathcal{P}_Q$ follows from Theorem 1.

**Corollary 1 (Border and Jordan [9]).** A social choice function on the domain of quadratic preferences is strategy-proof if and only if it is a generalized median voter scheme.

By Theorem 1, the existence of a variety of nondictatorial strategy-proof rules on minimally rich domains is guaranteed. From now until the end of the section, we study another question: how much larger can preference domains...
domains be while still allowing for strategy-proof rules? This question is qualified in several ways. Note that dictatorships are strategy-proof on the universal domain. To eliminate such trivial rules, we impose the no veto condition. Therefore, here we search for maximal domains compatible with strategy-proofness and the no vetoer condition. To state this maximality question formally, we need to given the following definition.

**Definition 15.** We say that \(\mathcal{P} \subseteq \mathcal{U}\) is a maximal domain for a list of properties if

(i) there exists a social choice function on \(\mathcal{P}\) satisfying the properties, and

(ii) for any \(\mathcal{P}', \mathcal{P} \not\subseteq \mathcal{P} \subseteq \mathcal{U}\), there does not exist any social choice function on \(\mathcal{P}'\) satisfying the same properties.

Note that a maximal domain for desirable properties such as strategy-proofness, and the no vetoer condition may not be unique. Also note that it is possible to construct trivial domains for these properties if the variation of the preferences is artificially restricted in a nonnatural way. However, such artificial domain maximality results cannot be applied to interesting cases. Therefore, we only pay attention to domains that include at least some natural preferences. Note that the smaller the set of preferences required to be included in the domain, the stronger the domain maximality result, and the more the cases it can be applied to. Thus, we qualify the previous question as follows: how much can we enlarge preference domains for strategy-proofness and the no vetoer condition from a minimally rich domain? We establish that there is a unique maximal domain including a minimally rich domain for strategy-proofness and the no vetoer condition and it is the domain of convex preferences.

Before establishing the maximality result, we emphasize the role of our characterization theorem in the maximality problem. Let \(f\) be a strategy-proof social choice function defined on the domain \(\mathcal{P} \supseteq \mathcal{P}_0\). Then, by Theorem 1, the restriction of \(f\) to \(\mathcal{P}_0\), say \(f|_{\mathcal{P}_0}\), is a generalized median voter scheme. By Remark 3, if \(f\) satisfies the no vetoer condition, then the restriction of \(f\) to \(\mathcal{P}_0\) has neither veto voters nor decisive voters. Therefore, for any \(z \in Z \setminus \{x\}\) and any \(W \in \mathcal{P}(z)\), \(2 \leq \# W \leq n - 1\) if \(z \neq \beta\) and \(2 \leq \# W \leq n\) if \(z = \beta\). Note that if there was an alternative \(z \in Z \setminus \{x\}\) with a minimal right winning coalition of cardinality one, then this agent would be a right decisive voter at \(z\), which would contradict Remark 3. Moreover, if \(N\) was a minimal right winning coalition for \(z \in Z \setminus \{x, \beta\}\), any agent would be a right veto voter at \(z\), which would also contradict Remark 3. By a symmetric argument, we have that for any \(z \in Z \setminus \{\beta\}\) and any \(W \in \mathcal{P}(z)\), \(2 \leq \# W \leq n - 1\) if \(z \neq x\) and \(2 \leq \# W \leq n\) if \(z = x\). Thus, we must consider societies with at least three agents.
Now we are able to state our domain maximality result for strategy-proofness.

**Theorem 2.** Consider societies with at least three agents. The domain of convex preferences is the unique maximal domain including a minimally rich domain for strategy-proofness and the no vetoer condition.

The proof is found in the Appendix. Let us mention that its crucial step is the use of the characterization result stated in Theorem 1.

Observing that $\mathcal{P}_Q$ is a minimally rich domain and $\mathcal{P}_S$ contains it, Theorem 2 leads to the following results.

**Corollary 2.** Consider societies with at least three agents. The domain of convex preferences is the unique maximal domain including the quadratic preferences for strategy-proofness and the no vetoer condition.

**Corollary 3.** Consider societies with at least three agents. The domain of convex preferences is the unique maximal domain including the single-peaked preferences for strategy-proofness and the no vetoer condition.

We close this section by both stating a question related to the no vetoer condition and comparing our result with other kinds of maximality results. Note that we require the no vetoer condition to exclude trivial rules such as dictatorships. However, as Example 3 below shows, this condition is stronger than required for this purpose since some interesting nondictatorial rules are also excluded. Moreover, observe that the no vetoer condition turns out to be crucial to obtain convex preferences as a maximal domain: if we rule out this condition, our result does not hold and there exist nondictatorial strategy-proof rules with vetoers defined on domains with nonconvex preferences. In order to see that, consider the following example.

**Example 3.** We say that a preference $\pi$ is left-side single-peaked if there is a unique maximal element $\tau(\pi)$ such that $\pi$ is strictly increasing in $[\pi, \tau(\pi)]$. Denote $\mathcal{P}_L$ the domain of left-side single-peaked preferences. Let $f: (\mathcal{P}_L)^n \to Z$ be the minimal voter scheme defined in Example 2; that is, for any $u \in (\mathcal{P}_L)^n$, $f(u) = \min\{ \tau(u'i); i \in N\}$. Observe that $\mathcal{P}_L$ includes a minimally rich domain and nonconvex preferences and that $f$ satisfies strategy-proofness on $\mathcal{P}_L$. However, each agent is a veto voter everywhere except for $\pi$, that is, the condition of no vetoer fails although $f$ is not a dictatorship.

It is easy to construct the similar example for the framework of finite alternatives and strict preferences.
Example 3 also shows the difference between our result and a result due to Kalai and Müller [12]. They study social choice functions that are rationalized by an underlying social welfare function and that are defined both on preference profiles and feasible sets. They establish that a domain of preferences allows the existence of strategy-proof, nondictatorial, and rationalizable rules if and only if it satisfies the condition that they call “decomposability.” The requirement that the social choice function be both strategy-proof and rationalizable is stronger than our requirement that the social choice function be strategy-proof for a fixed feasible set. This permits us to show for finite alternatives that there is no strategy-proof and nondictatorial rule in the sense of Kalai and Müller [12] on $\mathcal{P}_L$. Thus, it follows that $\mathcal{P}_L$ does not satisfy decomposability. Therefore, an adequate modification of Example 3 for the finite framework illustrates that their result does not hold when the feasible set is fixed, as is often the case in economics. Our result can be applied to these cases outside the purview of Kalai and Müller [12].

Other papers in the literature such as Barberá, Sonnenschein, and Zhou [5] and Serizawa [17] study a maximality problem parallel to ours but restricting to the class of generalized median voter schemes. Consider the following example.

**Example 4.** Let $N = \{1, 2, 3\}$, and $Z = [0, 2]$. Let $\mathcal{R}$ be the right coalition system such that $\mathcal{R}(z) = \{W \subseteq N : \# W \geq 1\}$ if $z \leq 1$ and $\mathcal{R}(z) = \{W \subseteq N : \# W \geq 2\}$ if $z > 1$, and let $g$ be the generalized median voter scheme associated with $\mathcal{R}$. Let $u_o$ be a preference on $Z$ with the utility representation such that $u_o(z) = 1 - z$ if $z \in [0, 1]$ and $u_o(z) = 2(z - 1)$ if $z \in (1, 2]$. Note that $\tau(u_o) = 2$. Let $\mathcal{P} = \mathcal{P}_L \cup \{u_o\}$ and $f$ be the social choice function on $\mathcal{P}$ specified as follows: if one agent, say agent 1, announces $u_o$, and the other two agents, say agents 2 and 3, announce single-peaked preferences $u_2$ and $u_3$ such that $\tau(u_2) < 1$ and $\tau(u_3) < 1$, then $f(u) = \max\{\tau(u_2), \tau(u_3)\}$; otherwise $f(u) = g(u)$. It is obvious that $\mathcal{P}$ includes a minimally rich domain and a nonconvex preference $u_o$. $f$ is not a generalized median voter scheme, but $f$ is strategy-proof on $\mathcal{P}$.

This example illustrates two important points. First, $f$ fails the no vetoer condition, but is “anonymous” in the sense that the votes of all agents are treated equally. Therefore, the maximal domain including single-peaked preferences for strategy-proofness and the condition of anonymity is not the domain of convex preferences but rather some yet to be determined domain. Thus altering the side condition that rules out trivial rules such as dictatorship changes the maximal domain. Second, since $f$ is not a

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11 The proof of this statement is available on request.
12 The associate editor pointed out this point.
generalized median voter scheme, the example demonstrates that restricting consideration to that class of rules, as has been the case in earlier papers, has the cost of excluding some strategy-proof rules that may be of interest.

Furthermore it is worth mentioning that the result in Serizawa [17] is mathematically independent from ours. Serizawa [17] studies maximality of domains with a property weaker than the no vetoer condition; he analyzes parallel problems for generalized median voter schemes allowing for agents to have veto power for some alternatives. His domain maximality result is modified as follows: any preference in the domain may not be multidimensional single-peaked over all the set of alternatives, but it must be multidimensional single-peaked on some restricted region, which reflects the power structure of the rule. Thus, our result does not imply Serizawa’s [17] results. And, of course, Serizawa’s [17] result does not imply ours because of the discussion in the previous paragraph.

4. CONCLUDING REMARKS

In this paper, we address the question: how much can we enlarge a domain of preferences allowing the existence of strategy-proof rules satisfying the no vetoer condition? We have given an answer by establishing a domain maximality result: there is a unique maximal domain including a minimally rich domain for strategy-proofness and the no vetoer condition and it is the domain of convex preferences. As Example 3 in Section 3 shows, our maximality result does not hold without assuming the no vetoer condition. Therefore, an interesting open problem is how our domain maximality result would be modified if we did not use the no vetoer condition. Serizawa [17] studies the problem of maximal domains for generalized median voter schemes to be strategy-proof. He obtains that preferences in the maximal domain must be multidimensional single-peaked on some restricted region which depends on the power of agents. We believe that a similar result would be obtained without assuming the prespecified structure of rules. However, separate arguments are necessary to prove this conjecture.

5. APPENDIX A

The Proofs

In order to establish our characterization result, we need some previous results. Some of them are similar to the ones used by Barberá and Jackson [2].

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13 Multidimensional single-peaked preferences are called “cross-shaped” preferences in Serizawa [17].
In the following lemmas, let \( f : (\mathcal{P}_R)^n \to Z \) be a strategy-proof social choice function.

**Lemma 1.** Let \((u^C, u^-)\) be such that for some \( a \in Z \), \( \max_{x \in \sigma_f(u^-)} u'(x) = \{ a \} \) for any \( i \notin C \). Then, \( f(u^C, u^-) = a \).

**Proof of Lemma 1.** Since \( a \in \sigma_f(u^-) \), there exists \( \hat{u}^- \in (\mathcal{P}_R)^{-C} \) such that \( f(u^C, \hat{u}^-) = a \). Let \( j \notin C \). Consider the preference profile \( v = (u^C, u'_j, \hat{u}^-) \). Since \( f(v) \in \sigma_f(u^-) \) and \( \max_{x \in \sigma_f(u^-)} u'(x) = \{ a \} \), if \( f(v) \neq a \), agent \( j \) would manipulate \( f \) at \( v \) via \( u'_j \), which contradicts strategy-proofness. Thus, \( f(v) = a \). Repeating the same argument for any \( j \notin C \), we get that \( f(u^C, u^-) = a \).

**Remark.** Note that from Lemma 1 applied to \( C = \emptyset \), if \( f \) satisfies strategy-proofness on a minimally rich domain, then \( f \) respects unanimity. Thus, if the domain of a strategy-proof rule contains a minimally rich domain \( \mathcal{P}_R \), then the rule restricted to \( \mathcal{P}_R \) also respects unanimity.

**Lemma 2.** For any \( C \subseteq N \) and \( u^C \in (\mathcal{P}_R)^C \), \( \sigma_f(u^C) \) is closed and convex.

**Proof of Lemma 2.** Let \( C \subseteq N \) and \( u^C \in (\mathcal{P}_R)^C \). Without loss of generality, let \( C = \{1, \ldots, c\} \) and \( -C = \{c+1, \ldots, n\} \) where \( c = \# C \).

First, let us prove closedness of \( \sigma_f(u^C) \). Suppose the contrary, that is, there exists a sequence \( \{x_k\}_{k \in \mathbb{N}} \) in \( \sigma_f(u^C) \) such that \( x_k \) converges to \( x \in \mathbb{R}^c \) as \( k \) goes to infinity and \( x \notin \sigma_f(u^C) \). Since \( Z \) is closed, \( x \in Z \). By minimal richness of the domain, there is \( u_a \in \mathcal{P}_R \) such that \( \tau(u_a) = x \). Consider the preference profile \( u = (u^C, u^-) \) such that for any \( j \in -C \), \( u'_j = u_a \), and let \( a = f(u) \). Then, \( a \neq x \) and \( u_a(x) > u_a(a) \). Moreover, since \( x = \lim_{k \to +\infty} x_k \) and \( u_a \) is continuous, there exists \( K \in \mathbb{N} \) such that \( u_a(x_K) > u_a(a) \). Since \( x_K \in \sigma_f(u^C) \), there exists \( \bar{u}^- \in (\mathcal{P}_R)^{-C} \) such that \( f(u^C, \bar{u}^-) = x_K \). By strategy-proofness, \( u_a(f(u^C, \bar{u}^-)) \leq u_a(f(u^C, x_K)) \). Moreover, \( u_a(f(u^C, \bar{u}^-)) \leq u_a(f(u^C, x_K)) \). Repeating this reasoning for any agent in \(-C\), we get \( u_a(x_K) = u_a(f(u^C, \bar{u}^-)) \leq u_a(f(u^C, x_K)) \), which is a contradiction. Thus, \( \sigma_f(u^C) \) is closed.

To prove convexity, consider \( C = \{1\} \). Suppose that \( \sigma_f(u^1) \) is not convex. That is, there exist \( x, y \in \sigma_f(u^1) \) and \( z \in (x, y) \) by convexity of \( Z \). Since by Lemma 1, \( \tau(u^1) \subseteq \sigma_f(u^1) \), \( z \notin \tau(u^1) \). Let \( \tau(u^1) = z \) otherwise, a similar reasoning works. Let \( x' = \sup \left\{ x' : x' \in \sigma_f(u^1), x' < z \right\} \) and \( y' = \inf \left\{ y' : y' \in \sigma_f(u^1), y' > z \right\} \). By closedness of \( \sigma_f(u^1) \), \( x', y' \in \sigma_f(u^1) \). By definition of a minimally rich domain, given \( x' \) and \( y' \), there exists \( u_a \in \mathcal{P}_R \) such that \( \tau(u_a) = (x', y') \) and \( u_a(x') > u_a(y') \). Since for any \( z' \neq x' \) with \( z' \in \sigma_f(u^1) \), \( u_a(x') > u_a(z') \), from Lemma 1, \( f(u^1, u_a^{-1} = x' \) where \( u_a^{-1} = (u_a, \ldots, u_a) \). Moreover, since \( f \) respects unanimity, \( f(u_a^{-1} = (u_a, \ldots, u_a) \). Then, agent 1 would manipulate
f at \((u', u^{-1})\) via \(u_{ors}\), which is the desired contradiction. Thus, \(\sigma_i(u')\) is convex.

To show that \(\sigma_i(u', u^2)\) is convex, note that \(f(u', \cdot)\) is a \((n-1)\)-agent strategy-proof rule onto \(\sigma_i(u')\). Therefore, by the same reasoning, taking into account its closedness, we obtain the convexity of \(\sigma_i(u', u^2)\). To prove convexity of \(\sigma_i(u^c)\), we iterate the procedure for any agent \(f \in C\).

The next property assures us that if an agent has his peak strictly above or below the social outcome, say \(y\), and his preferences change in such a way that his peak remains on the same side of \(y\), then the social outcome does not change.

**Definition 16.** A social choice function \(f: (\mathcal{P}_R)^n \rightarrow Z\) is uncompromising if for any \(i \in N\), \(u', v' \in \mathcal{P}_R\) and \(u^{-1} \in (\mathcal{P}_R)^{n-1}\) such that \(f(u) \neq \tau(u')\) and either (i) \(f(u) > \tau(u')\) and \(f(u) \geq \tau(v')\) or (ii) \(f(u) < \tau(u')\) and \(f(u) \leq \tau(v')\) holds, then \(f(u) = f(v', u^{-1})\).

**Proposition 1.** If \(f: (\mathcal{P}_R)^n \rightarrow Z\) is a strategy-proof social choice function, then \(f\) is uncompromising.

**Proof of Proposition 1.** Suppose the contrary; that is, there exist \(i \in N\), \(u', v' \in \mathcal{P}_R\) and \(u^{-1} \in (\mathcal{P}_R)^{n-1}\) such that \(\tau(u') > f(u)\), \(\tau(v') \geq f(u)\) (a similar argument applies if \(f(u) > \tau(u')\) and \(f(u) \geq \tau(v')\)) and \(f(v') \neq f(u)\) where \(v = (v', u^{-1})\). Let \(r \in Z\) be such that \(u'(r) = u'(f(u)) \neq f(u)\). Note that agent \(i\) would manipulate \(f\) either at \(v\) via \(u'\) if \(f(v) < f(u)\), or at \(u\) via \(v'\) if \(f(v) \in f(u, r)\). Thus, by strategy-proofness, \(f(v) \geq r\). By Lemma 2, \(\sigma_i(u^{-1})\) is convex. Then, for any \(z \in f(u, r)\), \(z \in \sigma_i(u^{-1})\), there exists \(w' \in \mathcal{P}_R\) such that \(f(w', u^{-1}) = z\) and then \(i\) manipulates \(f\) at \(u\) via \(w'\) which is the desired contradiction.

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** The proof of sufficiency is omitted since it is straightforward from definition. To prove necessity, given \(f\), a strategy-proof social choice function on a minimally rich domain \(\mathcal{P}_R\), we construct a right coalition system, say \(\mathcal{W}\), such that for any \(u \in (\mathcal{P}_R)^n\), \(f(u) = g(u)\) where \(g\) is the generalized median voter scheme induced by \(\mathcal{W}\). To build \(\mathcal{W}\), we proceed as follows:

given a nonempty coalition \(W \subseteq N\), let \(\vec{u}_W \in (\mathcal{P}_R)^n\) be such that \(\tau(\vec{u}_W) = \beta\) if \(i \in W\) and \(\tau(\vec{u}_W) = \alpha\) otherwise. Now let \(\mathcal{W}: Z \rightarrow 2^N\) be the correspondence such that for any \(z \in Z\), \(\mathcal{W}(z) = \{W \in 2^N \setminus \emptyset: f(\vec{u}_W) \geq z\}\).

To ensure that \(\mathcal{W}\) is a right coalition system, we have to verify Conditions (1) to (4) in Definition 10.

Note that to obtain Condition (1) two facts remain to be shown: (a) for any \(z \in Z\), \(\mathcal{W}(z) \neq \emptyset\), and (b) \(\mathcal{W}(z) = 2^N \setminus \emptyset\). Note that since \(f\) is onto and strategy-proof, \(f\) respects unanimity and therefore \(f(\vec{u}_W) = \beta\). Moreover,
for any $W \in 2^N \setminus \emptyset$, $f(\bar{u}_W) \geq x$. By the definition of $\mathcal{W}$, for any $z \in Z$, $N \in \mathcal{W}(z)$ and any nonempty coalition $W \in \mathcal{W}(x)$. Thus, facts (a) and (b) hold.

Clearly Conditions (3) and (4) follow straightforwardly from the definition of $\mathcal{W}$: let $z' < z$ and $W \in \mathcal{W}(z)$. Then, $f(\bar{u}_W) \geq z'$ which implies $W \in \mathcal{W}(z')$. This shows (3). Let $W \subseteq N$, $z \in Z$, $\{z_i\}_{i \in N} \subseteq Z$, $z_i$ converges to $z$ as $t$ goes to infinity such that for any $t \in \mathbb{N}$, $W \in \mathcal{W}(z_t)$. Since $f(\bar{u}_W) \geq z_i$ for any $i$, $f(\bar{u}_W) \geq z$. Therefore, $W \in \mathcal{W}(z)$ which shows (4).

It still remains to prove Condition (2) which comes from strategy-proofness. Let $z \in Z$, $W \in \mathcal{W}(z)$, and $W \not\subseteq \emptyset$. Then, $f(\bar{u}_W) \geq z$ and we want $f(\bar{u}_W') \geq z$, that is, $W' \in \mathcal{W}(z)$. Let $j \in W \setminus W$. Consider $u_{W \setminus \{j\}}$. If $f(\bar{u}_{W \setminus \{j\}}) < f(\bar{u}_W)$, then agent $j$ would manipulate $f$ at $u_{W \setminus \{j\}}$ via $\bar{u}_W$. Thus, by strategy-proofness, $f(\bar{u}_W \cup \{j\}) \geq f(\bar{u}_W)$. We can repeat this argument for any agent in $W \setminus W$. Thus, we obtain $f(\bar{u}_W) \geq f(\bar{u}_W') \geq z$, which establishes that $\mathcal{W}$ satisfies Condition (2). Therefore, $\mathcal{W}$ is a right coalition system.

Let $g$ be the generalized median voter scheme induced by $\mathcal{W}$. Observe that from the “if part” of this Theorem, $g$ is strategy-proof. Moreover, since $g$ is onto $Z$, $g$ respects unanimity (see Condition (1) in Definition 10).

Now, it only remains to show that for any $u \in (\mathcal{P}_\mathcal{W})^n$, $f(u) = g(u)$. Note that any preference profile $u$ satisfies one of the following cases:

**Case (1).** $u = \bar{u}_W$ for some $W \subseteq \emptyset \neq W \subseteq N$, or

**Case (2).** $u \in (\mathcal{P}_\mathcal{W})^n$ such that for some $C \subseteq C \subseteq N$, $\tau(u') = f(u)$ for any $i \in C$.

Let $u \in (\mathcal{P}_\mathcal{W})^n$. In Case 1, by definition of $g$, $f(u) = g(u)$. Observe that by definition of $\mathcal{W}$, for any $W \subseteq N$, $g(\bar{u}_W)$ is the maximum alternative where $W$ is a right winning coalition. For Case 2, if $C = N$, by respecting for unanimity of both $f$ and $g$, $f(u) = g(u) = \tau(u')$. If $C \neq N$, let $T = \{i \in N : \tau(u') > f(u)\}$. Then, $N \setminus (C \cup T)$ is the set of agents with peak less than $f(u)$. Define $\tilde{u}$, $\tilde{u} \in (\mathcal{P}_\mathcal{W})^n$ such that $\tau(\tilde{u}') = \beta$ if $i \in C \cup T$ and $\tau(\tilde{u}') = x$ otherwise; $\tau(\tilde{u}') = \beta$ if $i \in T$ and $\tau(\tilde{u}') = x$ otherwise; and $\tau(\tilde{u}') = \beta$ if $i \in T$, $\tau(\tilde{u}') = f(u)$ if $i \in C$ and $\tau(\tilde{u}') = x$ otherwise. By uncompromisingness of $f$ (see Proposition 1), $f(\bar{u}) = f(u)$. Moreover, by definition of $g$, $g(\bar{u}) \leq g(\tilde{u})$, and from Case 1, $\{f(u), f(\bar{u})\} = \{g(\bar{u}), g(\tilde{u})\}$. Thus, from uncompromisingness of $f$, we get the Condition (**): $f(\bar{u}) \in \{g(\bar{u}), g(\tilde{u})\}$.

Let us prove that $g(\bar{u}) = f(\bar{u})$. Without loss of generality, let $C = \{1, \ldots, c\}$. Note that if $g(\bar{u}, \tilde{u}^{-1}) > g(\bar{u})$, then agent 1 would manipulate $g$ at $\bar{u}$ via $\tilde{u}$. If $g(\bar{u}^{1}, \tilde{u}^{-1}) < f(\bar{u})$, then by uncompromisingness of $g$, $g(\bar{u}) = (\bar{u}^{1}, \tilde{u}^{-1})$ which would contradict Condition (**). Thus, by strategy-proofness and uncompromisingness of $g$, $g(\bar{u}^{1}, \tilde{u}^{-1}) = \{f(\bar{u}), g(\tilde{u})\}$. Using the same reasoning, $g(\bar{u}^{(1,2)}, \tilde{u}^{-1}) \in \{f(\bar{u}), g(\tilde{u})\}$. Thus,
replacing one by one, \( \tilde{u}^i \) by \( \hat{u}^i \), for any \( i \in C, i \neq 1, 2 \), we get \( g(\tilde{u}) \in [f(\hat{u}), g(\tilde{u})] \).

By the same argument, we also obtain that \( g(\tilde{u}) \in \{g(y), f(\hat{u})\} \). Therefore, \( g(\tilde{u}) = f(\hat{u}) \).

Then, it is straightforward by uncompromisingness of \( g \) that \( g(u) = g(\tilde{u}) \). Therefore, \( f(u) = g(u) \) which ends the proof of Theorem 1.

To prove Theorem 2, we use some interesting results presented in some Lemmas and the following definition.

**Definition 17.** Let \( f \) be a real-valued function on \([a, b] \). A subdivision of \([a, b] \), say \( X \), is a finite set \( \{x_1, ..., x_k\} \) of points in \([a, b] \) such that \( a = x_1 < x_2 < \cdots < x_k = b \). Define three real values, \( p_X, n_X, \) and \( t_X \), which depend on \( X \), as follows:

\[
p_X = \sum_{i=1}^{k} \max\{f(x_i) - f(x_{i-1}), 0\},
\]

\[
n_X = \sum_{i=1}^{k} \max\{-f(x_i) + f(x_{i-1}), 0\}
\]

and

\[
t_C = n_X + p_X = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|.
\]

Set \( P = \sup p_X, N = \sup n_X, \) and \( T = \sup t_X \) where we take the suprema over all subdivisions of \([a, b] \). Note that they depend on \( f \) and \([a, b] \).

Clearly, \( P \leq T \leq P + N \). We call \( P, N, \) and \( T \) the positive, negative, and total variation of \( f \) on \([a, b] \), respectively. If \( T < \infty \), we say that \( f \) is of bounded variation on \([a, b] \).

**Fact** (See p. 104 in Royden [15]). If \( f \) is of bounded variation on \([a, b] \), then for almost all \( x \in [a, b] \), the derivative of \( f \) at \( x \), say \( f'(x) \), exists.

**Remark.** According to Definition 17, if \( f \) is increasing then \( P = T = f(b) - f(a) \) is finite. Thus, any increasing real-valued function \( f \) on \([a, b] \) is of bounded variation on \([a, b] \).

**Lemma 3.** Let \( g \) be a continuous real-valued function defined on \( Z \) such that for some \( x, y, z \in Z, x < y < z, g(x) > g(y) \) and \( g(z) > g(y) \). Then,

(a) there exists a triple \( a, b, r \in Z \), and for any \( \varepsilon > 0 \), there exists \( c \) such that \( c \in (r, r + \varepsilon) \) which satisfy the following four conditions: (i) \( x \leq a < b < r < c \leq z \); (ii) \( g(a) = g(r) \); (iii) \( g(a) > g(x') \) if \( x' \in (a, r) \); and (iv) \( g(c) > g(x') \) if \( x' \in (a, e) \), and
there exists a triple \( r', b', c' \in Z \), and for any \( \varepsilon > 0 \), there exists \( a' \) such that \( a' \in (r' - \varepsilon, r') \) which satisfy the following four conditions: (i) \( x \leq a' < r' < b' < c' \leq z \), (ii) \( g(r') = g(c') \), (iii) \( g(c') > g(x') \) if \( x' \in (r', c') \), and (iv) \( g(a') > g(x') \) if \( x' \in (a', c'] \).

Proof of Lemma 3. To prove part (a), let \( m = z \) if \( g(z) < g(x) \) and \( m = x \), otherwise. Define \( d \equiv \min \{ z' \in [y, z] : g(z') = g(m) \} \). Let \( B = \text{Arg min}_{x' \in [x, d]} g(x') \). Consider \( h : [g(b), g(d)] \to [b, d] \) such that for any \( t \in [g(b), g(d)] \), \( h(t) = \min \{ y' \in [b, d] : t = g(y') \} \). Note that since the inverse image by \( g \) of any such \( t \) is not empty, \( h \) is well defined. Also observe that \( h \) is strictly increasing (although it may not be continuous). Then, \( h \) is of bounded variation on \([g(b), g(d)]\). Since from the above fact, \( h \) is differentiable at some \( t_o \in (g(b), g(d)) \), \( h \) is also continuous at \( t_o \). Then, jointly with increasingness of \( h \), we have that for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( t \in (t_o, t_o + \delta) \) then \( 0 < h(t) - h(t_o) < \varepsilon \).

Thus, by letting \( r = h(t_o) \), we have that for any \( \varepsilon > 0 \) there exists \( c \in (r, r + \varepsilon) \) such that \( c = h(t) \) for some \( t \in (t_o, t_o + \delta) \); that is, \( g(c) = g(r) \) (note that by definition of \( h \), \( g(c) = g(r) = h(t_o) \)).

We now define \( a = \max \{ x' \in (x, b) : g(x') = g(r) \} \) and consider the triple \( a, b, r \) defined as above and for any \( \varepsilon > 0 \), choose \( c' \) in \((r, r + \varepsilon)\) such that \( g(c') > g(r) \). Clearly \( x \leq a < b < r < c \leq z \) and \( g(a) = g(r) \). Also for any \( x' \in (a, r) \), \( g(a) > g(x') \); observe that by definition of \( a \), for any \( x' \in (a, b) \), \( g(a) > g(x') \) and on the other hand since \( r = h(t_o) \), for any \( y' \in (b, r) \), \( g(y') < g(r) = g(a) \). Thus, Conditions (i) to (iii) hold. Condition (iv) is straightforward from Condition (iii), the fact that \( g(c) > g(r) \), and the definition of \( c \) according to \( h \).

A symmetric argument applies to prove part (b) where \( h \) is the maximum of the set of the inverse images.

From now on, consider \( \mathcal{P} \) as set of preferences containing a minimally rich domain, say \( \mathcal{P} \), and \( f : \mathcal{P} \to Z \) a strategy-proof social choice function.

The next result states that if \( f \) is strategy-proof and if for some profile and coalition \( W \), for any agent \( i \in N \setminus W \), his best alternative is one of two points in the range, then \( W \) is a winning coalition for the outcome whenever it is not in the closed subinterval determined by both points.

Lemma 4. Let \( z, z' \in Z \) be such that \( z' > z \). Let \( W \subseteq N \) and \( u \in \mathcal{P}^n \) be such that for any \( j \in N \setminus W \), \( u' \in \mathcal{P}_j \) and \( \tau(u') \) is either \( z \) or \( z' \). If \( f(u) > z' \) (respectively, \( f(u) < z \) then \( W \) is a right (respectively, left) winning coalition for \( f(u) \).

Proof of Lemma 4. Let \( z, z' \in Z \), \( W \subseteq N \) and \( u \in \mathcal{P}^n \) be as above. For any \( i \in W \), define \( v_i' \in \mathcal{P}_i \) such that \( \tau(v_i') = f(u) \). By strategy-proofness, \( f(v_i', u^{N \setminus W}) = f(u) \). Let \( \mathcal{A} \) and \( \mathcal{L} \) be the right and left coalition systems
associated to \( f \mid_{\mathcal{P}_W} \), respectively. If \( f(u) > z' \), by Definition 10, then \( W \in R(f(u)) \). Similarly, if \( f(u) < z \), then \( W \in \mathcal{D}(f(u)) \).

Now we state a Corollary of Lemma 4 which applies to strategy-proof social choice functions without vetoers and where \( W \) is a right domain, \( z \) and \( z' \) are points independently of the preference of the agent in \( W \).

**Lemma 5.** Let \( f \) satisfy the no vetoer condition and let \( z, z' \in Z, z < z' \), \( i \in N, u \in (\mathcal{P}_W)^{n-1} \) be such that for any \( j \in N \setminus \{i\} \), \( \tau(u') = z \) or \( \tau(u') = z' \). Then, for any \( u' \in \mathcal{P}, f(u') \in [z, z'] \).

The proof of Lemma 5 is straightforward from Lemma 4. We only need to apply Lemma 4 to the case where \( W = \{i\} \). Since by Remark 3, \( f \mid_{\mathcal{P}_W} \) has neither veto nor decisive voters in \( Z \), \( W \) can be neither a right winning coalition for points strictly above \( z' \) nor a left winning coalition for points strictly below \( z \). Then, we obtain that \( z \leq f(u) \leq z' \).

**Proof of Theorem 2.** To prove that \( \mathcal{P}_c \) satisfies Condition (ii) in Definition 15 of maximal domain, note that the rule \( f \) on \( \mathcal{P}_c \) defined below satisfies the no vetoer condition and it is strategy-proof on \( \mathcal{P}_c \). Let \( u \in (\mathcal{P}_W)^n \). For any agent \( i \in N \), let \( h_i(u) = \min \{ x : x \in \mathcal{F}(u') \} \). Consider the generalized median voter scheme \( g \) by right quota 2, that is, whose associated right coalition system \( R \) is such that for any \( z \in Z \setminus \{x\} \), \( R(z) = \{ W \subseteq N : \# W \geq 2 \} \), and \( R(x) = 2^N \setminus \emptyset \). Then, define \( f(u) = g((h_i(u))_{i \in N}) \).

To establish both Condition (ii) for \( \mathcal{P}_c \) and uniqueness of maximal domains, we proceed as follows: let \( \mathcal{P} \) be a maximal domain including a minimally rich one for the properties of strategy-proofness and the no vetoer condition. That is, there is a rule \( f \) on \( \mathcal{P} \) satisfying these two properties. We want to show that \( \mathcal{P} \subseteq \mathcal{P}_c \). By contradiction, suppose that there exists a nonconvex preference \( u_a \in \mathcal{P}\setminus \mathcal{P}_c \). By nonconvexity of \( u_a \), there exist \( x, y, z \in Z \), \( x < y < z \) such that \( u_a(x) > u_a(y) \) and \( u_a(z) > u_a(y) \). By Lemma 3(a) applied to \( u_a \), there exist \( a, b, r \) and for any \( e > 0 \), there exists \( c \in (r, r+e) \) which satisfy Conditions (i) to (iv). Furthermore, by Condition (ii) in Definition 14 of minimally rich domain, given \( a, r \in Z \), there exists \( u_d \in \mathcal{P}_R \) such that \( \tau(u_d) = d \in (a, r) \) and \( u_d(r) > u_d(a) \). Choose \( e > 0 \), and \( c \in (r, r+e) \) such that \( u_d(c) > u_d(a) \). Note that by continuity of \( u_a \), such \( c \) exists. Consider \( u_e, u_i \in \mathcal{P}_R \) such that \( \tau(u_e) = a \) and \( \tau(u_i) = c \). Given \( C \subseteq N \), let \( u_C^e = (u_i^e)_{i \in C} \) be such that for any \( i \in C, u_C^e = u_i^e \). Similarly define the preference profiles \( u_C^{z'} \) and \( u_C^{z'} \) of coalition \( C \). Observe that we can similarly define \( u_d, u_c, \) and \( u_a \) for any such a quadruplet \( a, b, r, c \).

By Theorem 1, the restriction \( f \mid_{\mathcal{P}_R} \) of \( f \) to \( (\mathcal{P}_R)^n \) is a generalized median voter scheme. Let \( \mathcal{R} \) and \( \mathcal{L} \) be the corresponding right and left condition systems, respectively. We want to find a contradiction to strategy-proofness
by induction over the cardinality of a minimal right winning coalition for \( c \), say \( W \), for any quadruplet \( a, b, r, c \) defined above. Thus, the two steps of our induction argument are the following.

**Step 1.** For any quadruplet \( a, b, r, c \) satisfying Conditions (i) to (iv) in Lemma 3(a) such that \( \#W = 2 \), \( f \) fails to be strategy-proof.

**Step 2.** Suppose that for any quadruplet \( a, b, r, c \) satisfying Conditions (i) to (iv) in Lemma 3(a) such that \( \#W = k - 1 \), \( f \) fails to be strategy-proof. Then, for any quadruplet \( a, b, r, c \) satisfying Conditions (i) to (iv) in Lemma 3(a), \( f \) also fails to be strategy-proof when \( \#W = k \).

Proof of **Step 1.** Let \( a, b, r, c \) be any such a quadruplet with \( \#W = 2 \). Without loss of generality, assume that \( W = \{1, 2\} \). Clearly, \( f(u_1^a, u_2^r, u_a^{W_a}) = c \). Note also that since for any \( z \not\in [a, c] \), agent 2 is not a right veto voter at \( z \), then \( f(u_1^a, u_2^b, u_a^{W_a}) = a \). By Lemma 5 and strategy-proofness, we have

\[
f(u_1^a, u_2^b, u_a^{W_a}) = a, \tag{1}
\]
and

\[
f(u_1^a, u_2^r, u_a^{W_a}) = c. \tag{2}
\]

Then, from expressions (1) and (2) and since \( u_d(c) > u_d(a) \), agent 2 manipulates \( f \) at \( (u_1^a, u_2^b, u_a^{W_a}) \) via \( u_c \).

Observe that by using a symmetric argument, we can also prove that for any quadruplet \( a', b', r', c' \) satisfying Conditions (i) to (iv) in Lemma 3(b), if \( \# V = 2 \) then \( f \) is not strategy-proof, where \( V \) is the minimal left winning coalition for \( a' \).

Proof of **Step 2.** Let \( a, b, r, c \) be any such a quadruplet and \( W \) be a minimal right winning coalition for \( c \) with cardinality \( k \), \( k \geq 3 \). Without loss of generality, let \( W = \{1, ..., k\} \). Consider the following cases:

(2.1) \( W \) is not a minimal right winning coalition for some \( x' \in [r, c] \), and

(2.2) \( W \) is a minimal right winning coalition for any \( x' \in [r, c] \).

Case (2.1): Suppose that \( W \) is not a minimal right winning coalition for some \( x' \in [r, c] \). Without loss of generality, and by outcome monotonicity, there exists a minimal right winning coalition for \( r \), \( W_f = \{1, ..., l\} \), such that \( W_f \subseteq W \). Let us apply Lemma 3(a) to \( a, b, r \) as \( x, y, z \), respectively. Then, there exist \( a, b, r, c \) satisfying Conditions (i) to (iv). Note that \( c \not\leq r \), and \( W_f \) is a right winning coalition for \( c \). Let \( W_m \subseteq W_f, m \leq 1 \) be a minimal right winning coalition for \( c \). Then, since \( m \leq k - 1 \), by the induction
hypothesis, we get the desired contradiction to strategy-proofness which ends the proof of Case (2.1).

Case (2.2): Suppose that \( W \) is a minimal right winning coalition for any \( x' \in [r, c] \). Let \( W' = \{1, \ldots, k-1\} \). Note that \( f(u_W^W, u_d^k, u_a^-W) = a \) and \( f(u_W^W, u_a^-W) = c \). We change the preferences of agents in \( W' \) into \( u_j^1 \) one by one from the preference profiles \((u_W^W, u_d^k, u_a^-W)\) and \((u_W^W, u_a^-W)\) to establish that

\[
\begin{align*}
    f(u_o^W, u_d^k, u_a^-W) &= a \\
    f(u_o^W, u_d^k, u_a^-W) &= c.
\end{align*}
\]

Let us define for any \( j \in \{0, \ldots, k-2\} \), \( W_{j+1} = W_j \cup \{j+1\} \), where \( W_0 = \emptyset \), and let us proceed stating the following result:

**Claim.** For any \( j \in \{0, \ldots, k-2\} \),

1. if \( f(u_o^W, u_d^W, u_a^-W) = a \) then \( f(u_o^{W_{j+1}}, u_d^{W_{j+1}}, u_a^-W) = a \), and
2. if \( f(u_o^W, u_d^W, u_a^-W) = c \) then \( f(u_o^{W_{j+1}}, u_d^{W_{j+1}}, u_a^-W) = c \).

Let \( j \in \{0, \ldots, k-2\} \). To prove part (i), suppose that \( f(u_o^W, u_d^W, u_a^-W) = a \) and let \( z' \equiv f(u_o^{W_{j+1}}, u_d^{W_{j+1}}, u_a^-W) \). By strategy-proofness, \( u_o(z') \supseteq u_o(a) \); that is, either \( z' \geq r \) or \( z' \leq a \).

If \( z' \geq r \), by Lemma 4, \( W_{j+1} \) is a right winning coalition for \( z' \). Since \( j+1 \leq k-1 \), this contradicts that \( W \) is a minimal right winning coalition for \( r \).

If \( z' < a \), by Lemma 4, \( W_{j+1} \) is a left winning coalition for \( z' \). When \( j = 0 \), this contradicts that \( f \) satisfies the no vetoer condition. Otherwise, applying Lemma 3(b) to \( u_o \) and \( a \), \( b \), \( r \) corresponding to \( x \), \( y \), \( z \), there exist \( r' \), \( b' \), \( c' \in Z \) and for any \( \varepsilon > 0 \), there exists \( a' \in (r' - \varepsilon, r') \) which satisfy Conditions (i) to (iv). Observe that \( W_{j+1} \) is also a left winning coalition for \( a' \). By definition of minimally rich domain, given \( r' \), \( c' \in Z \) there exists \( u_o \in \mathcal{P}_R \) such that \( \tau(u_o) = d' \in (r', c') \) and \( u_o(r') > u_o(c') \). Choose \( \varepsilon > 0 \), and thus \( a' \) such that \( u_o(a') > u_o(c') \). Note that by continuity of \( u_o \), such \( \varepsilon \) exists. Consider \( V \) a minimal left winning coalition for \( a' \) where \( v = \# V \), \( 2 \leq v \leq j+1 \). If \( v = 2 \), by step 1, we get a contradiction to strategy-proofness. If \( v > 2 \), consider \( u_v \in \mathcal{P}_R \) such that \( \tau(u_v) = c' \), \( \tau(u_v) = a' \) and distinguish the following two subcases:

(2.2.1) \( V \) is a minimal left winning coalition for any \( x'' \in [a', r') \), and

(2.2.2) \( V \) is not a minimal left winning coalition for some \( x'' \in [a', r') \).

Subcase (2.2.1): Suppose that \( V \) is a minimal left winning coalition for any \( x'' \in [a', r') \). We can repeat the same argument from the beginning of
Case (2.2) replacing \( W \) by \( V \), \( W \) by \( V' = \{1, \ldots, v - 1\} \), \( u_a \) by \( u_{a'} \), \( u_\alpha \) by \( u_{a'} \), \( u_d \) by \( u_{d'} \) and \( k \) by \( \gamma \). Thus, we obtain that either

\[
f(u_{a'}, u_d', u_{a'}' - V) = c' \quad \text{and} \quad f(u_{a'}, u_d', u_{a'}' - V) = a',
\]

or we have a quadruplet \( \alpha, \beta, \rho, \gamma \in \mathbb{Z} \) satisfying Conditions (i) to (iv) in Lemma 3(a) such that a minimal right winning coalition for \( \gamma \), say \( W_\gamma \), has cardinality less than \( k - 1 \). If the former holds, agent \( v \) manipulates \( f \) at \( (u_{a'}, u_d', u_{a'}' - V) \) via \( u_{a'} \). If the latter holds, by the induction hypothesis we get the desired contradiction.

Subcase (2.2.2): Suppose that \( V \) is not a minimal left winning coalition for some \( x' \in (a', r') \). By a symmetric argument to Case (2.1), applying Lemma 3(b) to \( r', b', c' \), there exists a quadruplet \( a', r', b', c' \) satisfying Conditions (i) to (iv) such that \( V_m \subseteq V \) is a minimal left winning coalition for \( a' \). If \( m = 2 \), by step 1 we get a contradiction. If \( 2 < m < v \), we apply again Subcases (2.2.1) and (2.2.2) to the new minimal left winning coalition till by strategy-proofness, by induction hypothesis, or by step 1, we get the desired contradiction. Therefore, \( z' = a \) which shows part (i) of the Claim.

A similar argument proves part (ii). Then, applying this Claim from \( j = 1 \) to \( j = k - 2 \) we obtain

\[
f(u_a^W, u_k^W, u_a^W - W) = a \quad \text{and} \quad f(u_a^W, u_k^W, u_a^W - W) = c.
\]

That is, agent \( k \) manipulates \( f \) at \( (u_a^W, u_k^W, u_a^W - W) \) via \( u_a \) which ends the proof of Case (2.2). Thus, Theorem 2 holds.

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