Inefficiency of Strategy-Proof Rules for Pure Exchange Economies

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In his pioneering article, (in “Decision and Organization” (C. B. McGuire and R. Radner, Eds.), pp. 297–336, North-Holland, Amsterdam, 1972) Hurwicz establishes that there is no strategy-proof, Pareto-efficient, and individually rational rule for pure exchange economies with two agents and two goods, provided that the domain includes a sufficiently wide class of classical preferences. In this article, we extend his result to pure exchange economies with any finite number of agents and goods. We establish that (i) there is no strategy-proof, Pareto-efficient, and individually rational rule on the class of classical, homothetic, and smooth preferences; and (ii) there is no strategy-proof, Pareto-efficient, and symmetric rule on the same class of preferences. Journal of Economic Literature Classification Numbers: D78, D71, C72. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

For pure exchange economies with only two traders, Hurwicz [2] establishes an impossibility result concerning the existence of incentive compatible allocation mechanisms. Although his result is proved only for two-trader case, it makes clear that strategic behavior is a pervasive economic problem. Hurwicz’s result has led to substantial further research on incentive compatibility, much of it generalizing his result in one direction or another. This article extends Hurwicz’s result, as well as many of subsequent results, to many traders case, with the traders allowed to have any preference drawn from a broad class of preferences.

When a collection of individuals has to choose an allocation, the procedure they use for making their choice should take into account their preferences. Procedures are formally represented as functions from the class of
possible preference profiles into the feasible set, and they are called social choice functions, or social choice rules. In this article, we call procedures rules and the classes of possible preference profiles domains for short. Since preferences are usually privately known, agents may strategically misrepresent their preferences so as to manipulate the final outcome in their favor. As a result of such behavior, the actual outcome may be far from satisfactory for the true preference profile. Thus it is important for a rule to be immune to strategic behavior. If a rule is immune to strategic behavior by any agent, that is, if it is a dominant strategy for each agent to announce his true preferences, then the rule is said to be strategy-proof.

Also, individual rationality is an indispensable condition; it requires that a rule should never assign an allocation which makes an agent worse off than he would be by choosing not to participate (i.e., by simply consuming his endowment). Another condition, symmetry is a distributional one; it requires that if two agents have the same preference, they should receive indifferent consumption bundles.

In this article, we establish that for pure exchange economies with any finite number of agents and any finite number of goods, there is no strategy-proof, Pareto-efficient, and individually rational rule on the class of classical, homothetic, and smooth preferences. We also establish that for pure exchange economies with any finite number of agents and any finite number of goods, there is no strategy-proof, Pareto-efficient, and symmetric rule on the same domain. We remark that these results on the class of classical, homothetic, and smooth preferences imply the same conclusions on any superdomain of the class of those preferences.

Hurwicz [2] establishes that there is no strategy-proof, Pareto-efficient, and individually rational rule for pure exchange economies with two agents and two goods, provided that the domain includes a sufficiently wide class of classical preferences. However, he left open the many-agent case. By citing the following example from Satterthwaite and Sonnenschein [8], Zhou [13] later explained that the results of the two-agent case may not extend to the many-agent case: Suppose that there are two goods and three agents, say, agents 1, 2, and 3. Agent 1 gets the whole endowment if agent 3’s marginal rate of substitution at (1, 1) is greater than 1. Otherwise, agent 2 gets the whole endowment. This rule is strategy-proof, Pareto-efficient, and nondictatorial, although Zhou [13] proves the non-existence of such a rule in the two-agent case. He establishes that in pure exchange economies with two agents and any finite number of goods, there is no strategy-proof, Pareto-efficient, and individually rational rule on the class of classical, homothetic, and smooth preferences.

1 A preference is classical if it is continuous on the consumption set, and monotonic and strictly convex on the interior of the consumption set.

2 A rule is dictatorial if there is some agent such that the outcome of the rule is always his most preferred element in the feasible set; it is nondictatorial otherwise.
Pareto-efficient, and nondictatorial rule on the class of classical preferences. Schummer [10] recently has established the same conclusion as Zhou [13] on the class of classical and homothetic preferences.

Pure exchange economies with more than two agents have also been investigated under different assumptions. Hurwicz and Walker [3] analyze strategy-proof rules on the class of quasi-linear preferences for pure exchange economies with two or more agents. They establish that \textit{if a rule is strategy-proof, then the set of preference profiles at which the rule yields non-Pareto-efficient outcomes, is dense in the set of the preference profiles at which the consumption bundle of each agent is in the interior of his consumption set.} They show further that \textit{if a rule is continuous as well, then the former set of preference profiles is open and dense in the latter set of preference profiles.} If it is assumed that the consumption bundle of each agent is in the interior of the consumption set for many preference profiles, the implications of these results will be remarkably stronger than those described in the previous paragraph. However, it is also desirable to analyze strategy-proof rules without such a technical assumption. This is because when preferences are quasi-linear, there are many Pareto-efficient allocations at which some agents receive bundles on the boundary of their consumption sets. For instance, Satterthwaite and Sonnenschein's [8] rule described in the previous paragraph does not satisfy either the interiority of consumption bundles or the continuity of rules, notwithstanding it is strategy-proof, Pareto-efficient, and nondictatorial.

Compared with Hurwicz and Walker's [3] result, ours dispenses with the interiority of bundles and the continuity of rules. Furthermore, we replace the class of quasi-linear preferences by the class of classical, homothetic, and smooth preferences. They write in their concluding remark, “There remain a number of open questions that ought to be resolved. Perhaps the most obvious are the following two: First, there is no reason to believe that Theorem 1\textsuperscript{3} depends upon the quasi-linearity on the individuals' preferences; however, it is not clear how to obtain the result without the quasi-linearity assumption....” Our result does not include their result or does not answer this question completely. However, we believe that our result is a promising first step for solving the parallel question on the class of classical, homothetic, and smooth preferences, an important class of preferences studied in economics.

Some authors give up or weaken the requirements imposed by Hurwicz [2] to avoid negative results. Robert and Postlewaite [6] introduce “limiting incentive compatibility,” which states that the incentive to misrepresent,

\textsuperscript{3} Theorem 1 in Hurwicz and Walker [3] corresponds to their results mentioned above, but it is actually more general in that it can be applied to other kind of economies than pure exchange economies.
as measured by the gain from misrepresentation, become arbitrarily small as the number of agents increases. Under the topological assumptions on the limiting distribution of agents’ types, they establish that the Walrasian rule is limiting incentive compatible. Recently, Satterthwaite and Williams [9] apply the equilibrium concept of Bayesian Nash, and show that the double auction market mechanism possesses certain desirable asymptotic properties as the number of agents increases. Barberà and Jackson [1] give up Pareto-efficiency, and identify strategy-proof, nonbossy, and anonymous rules for pure exchange economies with any finite number of agents.

Many economists anticipate that the result of Hurwicz [2] can extend to the many-agent case without additional assumptions such as those imposed by Hurwicz and Walker [3]. While this expectation has inspired a substantial number of papers, all have fallen short of the general result that most investigators have thought to be true. In this article, we succeed in taking Hurwicz’s two-agent result [2] to any finite number of agents without imposing any of the additional technical assumptions that other authors have grudgingly employed.

We organize the article as follows: In Section 2, we set up the model, formally define the concepts introduced above, and state the main result. In Section 3, we give the proofs of the main result after discussing the techniques of the proofs we develop in the article. In Section 4, we formally introduce symmetry and establish an impossibility result that is parallel to the main one where individual rationality is replaced by symmetry. In Section 5, we mention an open question that remains.

2. THE MODEL AND THE MAIN RESULT

There are \( n \geq 2 \) agents. We denote the set of agents by \( N = \{1, 2, \ldots, n\} \).

Given a subset \( N’ \) of \( N \) and an agent \( i \in N \), we denote the set \( N \setminus N’ \) by \( -N’ \) and the set \( N \setminus \{i\} \) by \( -i \). There are \( m \geq 2 \) goods. We denote the set of goods by \( M = \{1, 2, \ldots, m\} \). Given \( y = (y_1, \ldots, y_m) \in \mathbb{R}^m \), we write \( x \succeq y \) if \( x_l \geq y_l \) for any \( l \in M \); we write \( x > y \) if \( x \succeq y \) and \( x \not= y \); we write \( x \gg y \) if \( x_l > y_l \) for any \( l \in M \). Each agent \( i \) is faced with his consumption set \( Z^i = \mathbb{R}^m \). His (consumption) bundle \( z^i = (z^i_1, \ldots, z^i_m) \) is an element of \( Z^i \) and

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4 A rule is nonbossy if by changing his announced preferences, no agent can change other agents’ consumption bundles without changing his own bundle.

5 A rule is anonymous if when the preferences of two agents are switched, their assigned consumption bundles are also switched.

6 Many economists anticipate that the result of Hurwicz [2] can extend to the many-agent case without additional assumptions such as those imposed by Hurwicz and Walker [3]. While this expectation has inspired a substantial number of papers, all have fallen short of the general result that most investigators have thought to be true. In this article, we succeed in taking Hurwicz’s two-agent result [2] to any finite number of agents without imposing any of the additional technical assumptions that other authors have grudgingly employed.

7 For example, after Satterthwaite [7] mentions the literature from Hurwicz [2] to Schummer [10], he writes “... even though a completely general theorem has not yet been formulated and proved, it seems clear that no attractive social choice functions exist for market settings that are both strategy-proof and efficient.” (p. 44)
his endowment is $e'i \in Z^i \setminus \{0\}$. We assume that $\sum_{i \in N} e'i \in \mathbb{R}_+^m$. The feasible set is the set $Z = \{ z = (z^1, ..., z^m) \in Z^1 \times \cdots \times Z^m \mid \sum_{i \in N} e'i \geq \sum_{i \in N} z^i \}$. An allocation $z = (z^1, ..., z^m)$ is an element of $Z$. Let $e = (e^1, ..., e^m) \in Z$ be the endowment point. Let $U^0$ be the class of preferences on $\mathbb{R}_+^m$ that are represented by continuous utility functions. We shall often abuse language and identify preferences with the continuous utility functions that represent them. Given $z' \in Z^i$ and $u' \in U^0$, let the upper contour set $UC(u', z') = \{ z' \in Z \mid u'(z') \geq u'(z') \}$.

**Definition 1.** A preference $u' \in U^0$ is **classical** if its utility function is continuous on $\mathbb{R}_+^m$, and strictly quasi-concave and strictly monotonic on the interior of $\mathbb{R}_+^m$.

We denote the class of classical preferences by $U^C$.

**Definition 2.** A preference $u' \in U^0$ is **homothetic** if for any $z' \in Z^i$, any $z'' \in Z^i$, and any $\lambda \in \mathbb{R}_+$, whenever $u'(z') \geq u'(z'')$, $u'(\lambda \cdot z) \geq u'(\lambda \cdot z')$.

We denote the class of classical and homothetic preferences by $U^H$.

**Definition 3.** A preference $u' \in U^0$ is **smooth** if for any $z' \in \mathbb{R}_+^m$, there is a unique vector in the unit simplex $\{ q \in \mathbb{R}_+^m : \| q \| = 1 \}$ that generates the hyperplane supporting $UC(u', z')$ at $z'$.

We denote the class of classical and smooth preferences by $U^S$.

Let $U' \subset U^0$ for each $i \in N$, and $U = U^1 \times \cdots \times U^n$. A rule on $U$ is a function $f$ from $U$ to $Z$. The set $U$ is called the **domain** of $f$. Throughout the paper, we assume that $U^1 = \cdots = U^n$. A preference profile is an element of $U$. Given $N' \subseteq N$, let $U^{N'} = \prod_{i \in N'} U^i$. We denote generic elements of $U$, $U^{N'}$ and $U^{-i}$ by $u$, $u^{N'}$ and $u^{-i}$ respectively. If $u = (u^1, ..., u^n) \in U$, $N' \subseteq N$, and $i \in N$ are given in advance, $u^{N'}$ denotes $(u^i)_{i \in N'}$ and $u^{-i}$ denotes $(u^i)_{i \in N \setminus \{i\}}$. Given a rule $f : U \rightarrow Z$ and a preference profile $u \in U$, we write the outcome $f(u) = (f^1(u), ..., f^n(u))$, and $f^{-i}(u)$ denotes $(f^j(u))_{j \neq i}$. Although $u'(z^1, ..., z^m)$ actually depends only on $z^i$, we treat $u' \in U^i$ as a function on $\mathbb{R}_+^m$ when it simplifies notation without creating confusion.

**Definition 4.** An allocation $z \in Z$ is **Pareto-efficient** for a preference profile $u \in U$ if for any $z' \in Z$,

$$[\exists i \in N \text{ such that } u'(z') > u'(z) ] \Rightarrow [\exists j \in N \text{ such that } u'(z) > u'(z')].$$

A rule $f$ is **Pareto-efficient** if for any $u \in U$, $f(u)$ is Pareto efficient for $u$.

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7 A vector $p \in \mathbb{R}_+^m$ supports $UC(u', z')$ at $z'$ if for any $z' \in UC(u', z')$, $p \cdot z' \geq p \cdot z'$. 
Definition 5. A rule $f$ is individually rational if for any $u \in U$ and any $i \in N$, $u'(f(u)) \geq u'(e)$.

Definition 6. A rule $f$ is strategy-proof if for any $u \in U$, any $i \in N$, and any $\hat{u} \in U^i$, $u'(f(u)) \geq u'(f(\hat{u}, u^{-i}))$.

Theorem 1. There is no strategy-proof, Pareto-efficient, and individually rational rule on the class of classical, homothetic, and smooth preferences.

We remark that the conclusion of Theorem 1 holds on a more restricted class of preferences, the class of what are called CES type preferences, i.e., preferences whose elasticities of substitution are constant.

Corollary 1. Let the domain $U$ be a superset of the class of classical, homothetic, and smooth preferences. Then there is no strategy-proof, Pareto-efficient, and individually rational rule on $U$.

Proof of Corollary 1. By contradiction, suppose that there is a strategy-proof, Pareto-efficient, and individually rational rule $f$ on $U$. Let $f'$ be the restriction of $f$ onto the class of classical, homothetic, and smooth preferences. Then, $f'$ is a strategy-proof, Pareto-efficient, and individually rational on the class of classical, homothetic, and smooth preferences. This is a contradiction to Theorem 1.

3. PROOFS

We devote this Section to establish Theorem 1 stated in Section 2. The proof of the theorem applies many techniques invented in previous literature. In addition, it develops several new techniques. As a result, the whole process of the proof becomes relatively long and complex. Thus, it is helpful to discuss the difficulties in extending Hurwicz’s [2] result to the many-agent case, and to explain how we overcome them before we start proving the theorem formally.

One difficulty in extending Hurwicz’s [2] result is on the visualizability of the locus of the Pareto-efficient allocations. Zhou [13] explains the difficulty in strengthening the results of the two agent-case to the many-agent case as follows; “The technical difficulty we encounter when we try to prove [his result mentioned in the introduction of our article] for many-agent economies is that the geometry of the set of efficient allocations becomes too complicated. Although we know that it is generally an $(m-1)$ dimensional manifold (here $m$ is the number of agents), it is difficult to visualize in the commodity space with only exception $m = 2$ where it can be represented by a contract curve.” The proofs of Hurwicz [2], Zhou [13], and Schummer [10] all depend upon visualizability of the locus of Pareto-
efficient allocations in the Edgeworth box diagram. In this article, we find
that if one agent, say agent 1, has a classical preference, and if the remain-
ing agents all have identical preferences which are also classical and
homothetic, then the locus of the Pareto-efficient allocations can be
visualized from agent 1’s viewpoint (Lemma 2). This result is simple and
limited, but it contributes satisfactorily to proving the theorem.

Another difficulty in extending Hurwicz’s [2] result is on the trackability
of the outcomes of rules. Examining the condition of strategy-proofness is
nothing but scrutinizing how strategy-proof rules assign outcomes to dif-
ferent preference profiles. To extend his result, we also need to track how
the outcomes will change as the preferences are replaced. “Nonbossiness”
introduced by Satterthwaite and Sonnenschein [8] is a useful property to
track the outcomes of strategy-proof rules. A rule is nonbossy if by
changing his announced preferences, no agent can change other agents’
consumption bundles without changing his own bundle. Nonbossiness
implies that when the preferences of an agent change but the rule assigns to
him the same bundle, the entire outcome remains the same. If nonbossi-
ness is satisfied together with strategy-proofness, this implication is quite
useful to track outcomes. Barberà and Jackson [1] intensively employed it
to identify strategy-proof, nonbossy, and anonymous rules for pure
exchange economies with many agents (Fact 3).

Note that Pareto-efficiency requires that the sum of the agents’ bundles
should be equal to the total endowment. Thus, in the case that there are
only two agents, an agent’s consumption bundle assigned by a Pareto-effi-
cient rule moves if and only if that of the other agent moves. Therefore,
Pareto-efficiency implies nonbossiness in the two-agent case. This special
feature of the two-agent case provides Hurwicz [2], Zhou [13], and
Schummer [10] the way to track the outcomes in order to prove their
results. However, in the case that there are at least three agents, Pareto-
efficient rules do not necessarily satisfy nonbossiness. When the preference
of an agent is changed, even if his consumption bundle remains the same,
the other agents’ bundles may change. For instance, Satterthwaite and
Sonnenschein’s [8] example mentioned in Section 1 is a strategy-proof,
Pareto-efficient, but bossy rule. Therefore, it is difficult to trace how the
assigned bundles of agents will change without nonbossiness provided that
the preferences of several agents are changed one by one, or their pref-
erences are changed simultaneously. In this article, we find that if rules are
individually rational as well as strategy-proof and Pareto-efficient, and if
preferences are smooth as well as classical and homothetic, then this
problem can be avoided without nonbossiness (Lemma 5).

\[ \text{In fact, nonbossiness is formally defined by the condition: For any } u \in U, \text{ any } i \in N, \text{ and any } \bar{a} \in U^i, \text{ if } f'((\bar{a}, u^-)) = f'((a), f'((a', u^-)) = f'((a)). \]
3.1. Preliminary Results

We say that \( \hat{u} \in U^0 \) is a Maskin Monotonic Transformation\(^9\) of \( u' \) at \( z' \) if (i) \( UC(\hat{u}, z') \subseteq UC(u', z') \) holds, and (ii) \( \hat{z}' \in UC(\hat{u}', z') \) and \( \hat{z}' \neq z' \) together imply that \( u'(\hat{z}') > u'(z') \). Let \( M(u', z') \) be the set of Maskin Monotonic Transformations of \( u' \) at \( z' \). Given \( u' \in U^0 \) and \( z \in Z \), we also denote \( M(u', z) = M(u', z') \).

**Fact 1.**\(^10\) Let \( f \) be a strategy-proof rule. For any \( u \in U \), any \( i \in N \), and any \( \hat{u} \in M(u', f(u)) \), \( f'((\hat{u}', u'')) = f'(u) \).

**Proof.** Let \( u \in U \), \( i \in N \), and \( \hat{u} \in M(u', f(u)) \). We show that \( f'((\hat{u}', u'')) = f'(u) \). By contradiction, suppose that \( f'((\hat{u}', u'')) \neq f'(u) \). Strategy-proofness for agent \( i \) implies \( \hat{u}'(f(\hat{u}', u'')) > \hat{u}'(f(u)) \). Then the condition (ii) of Maskin Monotonic Transformation implies \( u'(f(\hat{u}', u'')) > u'(f(u)) \). This contradicts strategy-proofness. Thus \( f'((\hat{u}', u'')) = f'(u) \).

Fact 2 below says that for any classical, homothetic, and smooth preference \( u^0 \) and any consumption bundle \( z^0 \) in the interior of the consumption set, there exists a CES type preference that has the same rate of substitution as \( u^0 \) at \( z^0 \) and is a Maskin Monotonic Transformation of \( u^0 \) at \( z^0 \).

**Fact 2.** Let \( u^0 \) be a classical, homothetic, and smooth preference, and let \( \hat{u}^0 \in \hat{U}^0 \). Given \( \rho \in (-\infty, 1) \), let \( \hat{u}^0(\cdot; \rho) = A^{-1/\rho} \cdot \left[ \sum_{i \in M} a_i \cdot (z^0)^{1-\rho} \cdot (x_i)^{\rho} \right]^{1/\rho} \) for any \( x \in R^m_+ \), where \( A = \sum_{i \in M} a_i \cdot (z^0)^{1-\rho} \). Then the function \( \hat{u}^0(\cdot; \rho) \) on \( R^m_+ \) is a classical, homothetic, and smooth preference. If \( a_i = [\partial u^0(x)/\partial x_i] \cdot A \) for each \( i \in M \), then the following holds:

(i) \( \partial \hat{u}^0(x^0; \rho)/\partial x_i = \partial u^0(x^0)/\partial x_i \) for any \( i \in M \).

(ii) There is \( \rho_0 \in (-\infty, 1) \) such that for any \( \rho \in (-\infty, \rho_0) \), \( u^0(\cdot; \rho) \in M(u^0, z^0) \).

Fact 2(i) can be shown by direct computation. Fact 2(ii) follows from the well-known fact that as \( \rho \) goes to negative infinity, the preference \( u^0 \) converges to the Leontief type preference whose indifferent surface is kinked at \( z^0 \) and the curvature of the indifferent surface at \( z^0 \) goes to infinity.

Lemma 1 below says that if a rule is Pareto-efficient and if two agents have identical preferences that are classical and homothetic, then the rule...

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\(^9\)To be precise, this is the definition of what is called “Strictly Maskin Monotonic Transformation.” Here, we call this notion “Maskin Monotonic Transformation” for short.

\(^10\)This result has already appeared in literature. See, for example, Zhou [13], Barberà and Jackson [1], etc.
assigns them the proportional bundles. The proof of Lemma 1 is similar to that of Lemma 2 of Schummer [10], which formally states that the outcomes of Pareto-efficient rules of two-agent pure exchange economies must be on the diagonal of the Edgeworth box diagram when the two agents have identical preferences that are classical and homothetic. This lemma is based on the property of classical and homothetic preference that the marginal rate of substitutions at two consumption bundles coincide if and only if the two bundles are proportional.

**Lemma 1.** Let $f$ be a Pareto-efficient rule, and let $u^0 \in U^H$. Let $i \in N$, $j \in N$, and $u \in U$ be such that $u_i = u^0 = u_j$. If $f'(u) \neq 0$, there is $\lambda \in \mathbb{R}_+$ such that $f'(u) = \lambda \cdot f'(u)$.

Lemma 2 below says that if one agent, say agent 1, has classical preferences, and the other agents all have the common preference that is also classical and homothetic, then the locus of the Pareto-efficient allocations looks like a contract curve of a two-agent economy from the viewpoint of agent 1. To treat the feasible sets and the sets of Pareto-efficient allocations of economies with different number of agents, we introduce new notations. Given $W \in \mathbb{R}^{m+}$ and $n' \geq n$, let $Z(n', W) = \{ z \in \mathbb{R}^{m \cdot n'} | \sum_{i=1}^{n'} z_i = W \}$; that is, $Z(n', W)$ is the feasible set of the pure exchange economy with the resource bundle $W$ and $n'$ agents. Given $W \in \mathbb{R}^{m+}$, $n' \leq n$ and a preference profile $u \in (U^0)^n$ for $n'$ of agents, denote the set of Pareto-efficient allocations for $u$ on $Z(n', W)$ by $P(u, Z(n', W))$, and the projection of $P(u, Z(n', W))$ on $Z'$ by $P'(u, Z(n', W))$. Note that the notation $Z$ defined in Section 2 coincides with $Z(n, \sum_{i \in N} e_i)$.

**Lemma 2.** Let $\Omega \in \mathbb{R}^{m+}$, $u^0 \in U^H$, $i \in N$, and $u' \in U^C$. Then $P'(u', u^0, Z(2, \Omega)) = P'(u', u^0, \ldots, u^0, Z(n, \Omega))$.

Lemma 2 is related to a well known fact on demand aggregation. It states that when agents have identical, classical, and homothetic preferences, their demands can be aggregated, that is, the sum of the individual demands depends only on the price and the sum of the individual endowments. The intuition behind the proof is that the consumption bundles of Pareto-efficient allocations for agents $j \neq i$ can be also aggregated owing to the First and Second Welfare Theorems, which state that an allocation $z$ is Pareto-efficient for $u \in U$ if and only if there is a price vector $p \in \mathbb{R}_{++}^{m}$ that generates a hyperplane supporting the upper contour set $UC(u, z')$ of any agent $j \neq i$ at $z'$.

**Proof.** Without loss of generality, let $i = 1$.

We first show $P^1(u', u^0, Z(2, \Omega)) \subseteq P^1(u', u^0, \ldots, u^0, Z(n, \Omega))$. Let $z^1 \in P^1(u', u^0, Z(2, \Omega))$, $z^0 = (\Omega - z^1)/(n-1)$, and $z = (z^1, z^0, \ldots, z^0)$. Note
Thus $z \in Z(n, \Omega)$. We claim that $z \in P(u^1, u^0, \ldots, u^0, Z(n, \Omega))$, which implies $z^1 \in P^i(u^1, u^0, \ldots, u^0, Z(n, \Omega))$. Since the case that $z^1 = 0$ or $z^1 = \Omega$ is trivial, we consider only the case that $z^1 \neq 0$ and $z^1 \neq \Omega$. Since $z^1 \in P(u^1, u^0, Z(2, \Omega))$, there is $p \in \mathbb{R}^n_+$ such that for any $z^1 \in UC(u^1, z^1)$, $p \cdot (z^1 - z^1) \geq 0$, and such that for any $z^2 \in UC(u^0, \Omega - z^1)$, $p \cdot [z^2 - (\Omega - z^1)] \geq 0$. Since $u^0$ is classical and homothetic, and $z^0 = (\Omega - z^1)/(n - 1)$, it follows that for any $z^0 \in UC(u^0, z^0)$, $p \cdot (z^0 - z^0) \geq 0$. Thus $z \in P(u^1, u^0, \ldots, u^0, Z(n, \Omega))$.

We next show $P^1(u^1, u^0, \ldots, u^0, Z(n, \Omega)) \subseteq P^i(u^1, u^0, Z(2, \Omega))$. Let $z^1 \in P^i(u^1, u^0, \ldots, u^0, Z(n, \Omega))$. Then there is $(z^2, \ldots, z^n) \in Z^2 \times \cdots \times Z^n$ such that $z = (z^1, z^2, \ldots, z^n) \in P(u^1, u^0, \ldots, u^0, Z(n, \Omega))$. We claim that $(z^1, \sum_{i=2}^n z^i) \in P(u^1, u^0, Z(2, \Omega))$, which implies $z^1 \in P^i(u^1, u^0, Z(2, \Omega))$. Since the case that $z^1 = 0$ or $z^1 = \Omega$ is trivial, we consider only the case that $z^1 \neq 0$ and $z^1 \neq \Omega$. Since $z \in P(u^1, u^0, \ldots, u^0, Z(2, \Omega))$, there is $p \in \mathbb{R}^n_+$ such that for any $z^1 \in UC(u^1, z^1)$, $p \cdot (z^1 - z^1) \geq 0$, and such that for any $i \neq 1$ and any $z^i \in UC(u^0, z^i)$, $p \cdot (z^i - z^i) \geq 0$. Since $u^0$ is classical and homothetic, by Lemma 1, there is $l = (l^2, \ldots, l^n) \in \mathbb{R}_+^{n-1}$ such that $z^i = l^i \cdot \sum_{j=2}^n z^j$, and it follows that for any $z^0 \in UC(u^0, \sum_{i=2}^n z^i)$, $p \cdot (z^0 - \sum_{i=2}^n z^i) \geq 0$. Thus $(z^1, \sum_{i=2}^n z^i) \in P^i(u^1, u^0, Z(2, \Omega))$.

**Lemma 3.** Let $f$ be a strategy-proof rule. Let $u \in U$ and $z = f(u)$. Let $i \in N$ and $z^i \in Z$ be such that $z^i < z^i$ or $z^i > z^i$. Then for any $u_i^i \in U^i$, $z \neq f(u^i)$.

**Proof.** By contradiction, suppose that there is $u_i^i \in U^i$ such that $z = f(u^i)$. If $z^i < z^i$, then by monotonicity, $u^i(z^i) < u^i(z^i)$. If $z^i > z^i$, then by monotonicity, $u^i(z^i) > u^i(z^i)$. Both cases contradict strategy-proofness.

**Lemma 4.** Let $f$ be a Pareto-efficient and individually rational rule. Let $i \in N$ be such that $e^i$ is proportional to $\Omega = \sum_{j \in N} e^j$. Then for any $u_i^i \in U^i$, $f_i^i(u_i^i, \ldots, u^0) \geq e^i$.

**Proof.** Let $u_i^0 \in U_i^0$. Since $e^i$ is proportional to $\Omega$, there is $\mu \in \mathbb{R}_+$ such that $e^i = \mu \cdot \Omega$. By Lemma 1, there is $l \in \mathbb{R}^n_+$ such that $f_i^i(u_i^i, \ldots, u_i^0) = l \cdot \sum_{j \in N} f(u_i^i, \ldots, u_i^0) = \lambda \cdot \Omega$. If $l < \mu$, $f_i^i(u_i^i, \ldots, u_i^0) \ll e^i$. Then by monotonicity, $u_i^i(f_i^i(u_i^i, \ldots, u_i^0)) < u_i^0(e^i)$, contradicting individual rationality. Thus $l \geq \mu$, so that $f_i^i(u_i^i, \ldots, u_i^0) \geq e^i$.

To explain Lemma 5 below, we introduce Fact 3, which is employed in Barberà and Jackson [1].

---

This result is called “diagonality” in Barberà and Jackson [1].
Fact 3. Let \( f \) be a strategy-proof and nonbossy rule. Let \( f(u^1, u^2, \ldots, u^n) = z \). For each \( j \neq 1 \), let \( \hat{u}^j \) be a Maskin monotonic transformation of \( u^j \) at \( z^j \). Then \( f(u^1, \hat{u}^2, \ldots, \hat{u}^n) = z \).

Proof of Fact 3. Since \( \hat{u}^2 \) be a Maskin monotonic transformation of \( u^2 \) at \( z^2 \), it follows from strategy-proofness, Fact 1, and \( f(u^1, u^2, \ldots, u^n) = z \) that \( f^2(u^1, \hat{u}^2, u^3, \ldots, u^n) = z^2 \). Then nonbossiness implies \( f(u^1, \hat{u}^2, u^3, \ldots, u^n) = z \). Repeating this argument for agents \( j = 3, \ldots, n \), we get \( f(u^1, \hat{u}^2, \hat{u}^3, \ldots, \hat{u}^n) = z \).

As we mentioned before, when there are only two agents, Pareto-efficiency implies nonbossiness. Thus, Fact 4 below follows from Fact 3, which is employed in Zhou [13] and Schummer [10].

Fact 4. Assume that there are only two agents. Let \( f \) be a strategy-proof and Pareto-efficient rule. Let \( f(u^1, u^2) = z \), and let \( \hat{u}^2 \) be a Maskin monotonic transformation of \( u^2 \) at \( z^2 \). Then \( f(u^1, \hat{u}^2) = z \).

In Lemma 5, we do not assume that rules are nonbossy or that there are only two agents. We instead assume that (i) a rule is individually rational as well as strategy-proof and Pareto-efficient; (ii) every agent has positive endowments of all goods; (iii) one agent, say agent 1, has a classical, homothetic, and smooth preference; (iv) the remaining agents, \( j = 2, \ldots, n \), all have the common preference \( u^0 \) that is also classical, homothetic, and smooth; (v) the consumption bundle assigned to agent 1 is not proportional to the total endowment; and (vi) the preferences of agents, \( j = 2, \ldots, n \), are all changed to a common CES type preference \( \hat{u}^0 \) that is a Maskin Monotonic Transformation of \( u^0 \). Then, we get the conclusion that the rule assigns to agent 1 the same bundle as before. This conclusion is weaker than those of Facts 3 and 4. Nevertheless, when used together with Lemma 2, Lemma 5 powerfully helps us to trace how the outcomes will change as preferences are replaced.

Lemma 5. Let \( f \) be a strategy-proof, Pareto-efficient, and individual rational rule. Let \( e^i \gg 0 \) for each \( i \in N \). Let \( u^0 \) and \( u^1 \) be classical, homothetic, and smooth preferences. Let \( z = f(u^1, u^0, \ldots, u^0) \), \( z^1 \in \mathbb{R}^m_+ \), \( z^1 \) be not proportional to \( \Omega \), and \( z^0 = \sum_{i \in M} z^i \in \mathbb{R}^m_+ \). For every \( x \in \mathbb{R}^n_+ \), let \( \hat{u}^0(x) = A^{-1/\rho} \cdot \left[ \sum_{i \in M} a_i \cdot (z^0_i)^{1-\rho} \cdot (x^i)^{\rho} \right]^{1/\rho} \), where \( a_i = \left[ \partial u^0(z^0)/\partial x_i \right] \cdot A \) for each \( i \in M \), \( A = \sum_{i \in M} a_i \cdot (z^0_i) \), and \( \rho \in (-\infty, 1) \) is chosen so that the function \( \hat{u}^0 \) is a Maskin Monotonic Transformation of \( u^0 \) at \( z^0 \). (Note that such \( \rho \) exists, by Fact 2.) Then \( f^1(u^1, \hat{u}^0, \ldots, \hat{u}^0) = z^1 \).

Proof. For each \( k \in \{1, \ldots, n\} \), let \( \hat{u}^{[2, \ldots, k]} \) denote the preference profile of the set \( \{2, \ldots, k\} \) of agents in which \( \hat{u}^i = \hat{u}^0 \) for each \( i \in \{2, \ldots, k\} \). When
$k = 1$, let $\bar{u}^{[2,\ldots,k]}$ denote the empty profile (no agents). We establish by induction that for every $k \in \{1, \ldots, n\}$,

$$\forall i \in \{2, \ldots, n\}, \exists \lambda_i \in \mathbb{R}_+ \text{ s.t. } f^i(u^1, \bar{u}^{[2,\ldots,k]}, u^0, \ldots, u^0) = \lambda_i \sum_{i=2}^{i=n} z^i, \quad (1)$$

and

$$\sum_{i=2}^{i=n} f^i(u^1, \bar{u}^{[2,\ldots,k]}, u^0, \ldots, u^0) = \sum_{i=2}^{i=n} z^i. \quad (2)$$

Note that when $k=1$, Eq. (2) implies that $\sum_{i=2}^{i=n} f^i(u^1, \bar{u}^{0}, \ldots, \bar{u}^0) = z^1$, which is the result we wish to prove.

Note that Eqs. (1) and (2) both hold when $k=1$, according to our assumption and Lemma 1. As the induction hypothesis, assume that

$$\forall i \in \{2, \ldots, n\}, \exists \lambda_i \in \mathbb{R}_+ \text{ s.t. } f^i(u^1, \bar{u}^{[2,\ldots,k-1]}, u^0, \ldots, u^0) = \lambda_i \cdot C_{i=n} z^i, \quad (1')$$

and

$$\sum_{i=2}^{i=n} f^i(u^1, \bar{u}^{[2,\ldots,k-1]}, u^0, \ldots, u^0) = \sum_{i=2}^{i=n} z^i. \quad (2')$$

We show that

$$\forall i \in \{2, \ldots, n\}, \exists \lambda_i \in \mathbb{R}_+ \text{ s.t. } f^i(u^1, \bar{u}^{[2,\ldots,k]}, u^0, \ldots, u^0) = \lambda_i \cdot \sum_{i=2}^{i=n} z^i, \quad (1'')$$

and

$$\sum_{i=2}^{i=n} f^i(u^1, \bar{u}^{[2,\ldots,k]}, u^0, \ldots, u^0) = \sum_{i=2}^{i=n} z^i. \quad (2'')$$

Denote $f(u^1, \bar{u}^{[2,\ldots,k]}, u^0, \ldots, u^0) = \bar{z}$, and $f(u^1, \bar{u}^{[2,\ldots,k-1]}, u^0, \ldots, u^0) = \bar{z}$. Since $u^0$, $u^1$ and $\bar{u}^0$ are classical preferences, and since $e' \gg 0$ for any $i \in N$, it follows from individual rationality that $\bar{z}^i \neq 0$ for each $i \in N$. Then, since $z^i \in \mathbb{R}_{++}^n$ and $z^0 \in \mathbb{R}_{++}^n$, Eqs. (1') and (2') imply that $\bar{z}^i \in \mathbb{R}_{++}^n$ for each $i \in N$. Since $u^0$ and $\bar{u}^0$ are homothetic and since $\bar{u}^0(\cdot, \rho) \in M(u^0, \sum_{i=2}^{i=n} z^i)$, it follows from Eqs. (1') and (2') that for any $i \in N \setminus \{1\}$, $\bar{u}^0 \in M(u^0, \bar{z})$. Since $\bar{u}^0 \in M(u^0, \bar{z})$, by Fact 1, it holds that $\bar{z}^k = \bar{z}$. In this paragraph, we show Eq. (1''). Let $i \in \{2, \ldots, n\}$. First consider the case that $i \in \{2, \ldots, k\}$. Since agents $k$ and $i$ have the same classical and homothetic preference $\bar{u}^0$, it follows from Lemma 1 that $\bar{z}^i$ is proportional to $\bar{z}^k = \bar{z}^k$. Next consider the case that $i \in \{k+1, \ldots, n\}$. Since $\bar{z}$ is Pareto-
inefficiency of strategy-proof rules

efficient, there is \( p \in \mathbb{R}^n_+ \) that generates a hyperplane supporting \( UC(u^0, z^0) \) at \( z^k \) and a hyperplane supporting \( UC(u^j, z^j) \) at \( z^j \). By Fact 2(i) and \( u^0 \in M(u^0, z^0), p \) also generates a hyperplane supporting \( UC(u^0, z^j) \) at \( z^j \).

Since \( u^0 \) is classical and homothetic, and since \( p \) generates a hyperplane supporting \( UC(u^0, z^j) \) at \( z^j \), it follows that \( z^j \) is proportional to \( z^k \) too.\(^2\) Therefore, by \( z^k = z^k \neq 0 \), for any \( i \in N \setminus \{1\} \), there is \( \lambda_i \in \mathbb{R} \) such that \( \lambda_i = \lambda_i \cdot z^k \). Since \( z^k = \lambda_k \cdot \sum \lambda_j z^j \) by (1'), we have \( z^i = \lambda_i \cdot (\lambda_k \cdot \sum \lambda_j z^j) = (\lambda_i, \lambda_k) \cdot \sum \lambda_j z^j \). We have shown (1').

In this paragraph, we show Eq. (2'). By Eq. (1'), we have:

\[
\sum_{i=2}^{i=n} z_i = \sum_{i=2}^{i=n} \left( \lambda_i \cdot \sum_{j=2}^{j=n} z_j \right) = \left( \sum_{i=2}^{i=n} \lambda_i \right) \cdot \sum_{j=2}^{j=n} z_j.
\]

Denote \( \mu = \sum_{i=2}^{i=n} \lambda_i \). Then \( \sum_{i=2}^{i=n} z_i = \mu \cdot \sum_{i=2}^{i=n} z_i \). Now, we have only to show \( \mu = 1 \). By contradiction, suppose \( \mu \neq 1 \). Since \( z^i \) is not proportional to \( \Omega \) by (2'), \( z^1 \) is not proportional to \( z^1 \) either.\(^3\) Since \( z \in P(u^1, u^{[k-1]}, u^0, \ldots, u^0) \) and \( z^i \in \mathbb{R}^n_+ \) for each \( i \in N \), there is \( p \in \{ q \in \mathbb{R}^n_+: ||q|| = 1 \} \) that generates the hyperplane supporting \( UC(u^1, z^i) \) at \( z^1 \) and the hyperplane supporting \( UC(u^0, z^i) \) at \( z^k \). Since \( z^k = z^k \in \mathbb{R}^n_+ \),

\(^2\) To verify this statement, suppose by contradiction that \( z^i \) is not proportional to \( z^1 \). Then, \( z^i \neq z^1 \). Since \( u^0 \) is classical, we have: (a) any \( z^0 \in \mathbb{R}^n_+ \) is a unique \( u^0 \) maximizer on a hyperplane supporting \( UC(u^0, z^0) \) at \( z^0 \). Since \( u^0 \) is homothetic, we also have: (b) for any \( z^0 \in \mathbb{R}^n_+ \), if \( p' \in \mathbb{R}^n_+ \) generates a hyperplane supporting \( UC(u^0, z^0) \) at \( z^0 \), then for any \( \lambda \in \mathbb{R}^n_+ \), \( p' \) also generates a hyperplane supporting \( UC(u^0, \lambda \cdot z^0) \) at \( \lambda \cdot z^0 \). If \( p \cdot z^i = p \cdot z^1 \), then since \( p \) generates a hyperplane supporting \( UC(u^0, z^i) \) at \( z^1 \) and a hyperplane supporting \( UC(u^0, z^i) \) at \( z^1 \), it follows from (a) that \( z^i \) and \( z^i \) are both unique \( u^0 \) maximizers on the same hyperplane \( \{ z^0 \in \mathbb{R}^n_+: p \cdot z^0 = p \cdot z^i \} \). This contradicts to \( z^i \neq z^i \). Thus, \( p \cdot z^i \neq p \cdot z^i \), so that \( p \cdot z^i < p \cdot z^i \) or \( p \cdot z^i > p \cdot z^i \). Consider the case that \( p \cdot z^i < p \cdot z^i \). Then there is \( z^i \in \{ 0, z^i \} \) such that \( p \cdot z^i = p \cdot z^i \). Since \( z^i \) is not proportional to \( z^i \), we have \( z^i \notin \{ 0, z^i \} \), so that \( z^i \neq z^i \). By (b), \( z^i \notin \{ 0, z^i \} \) implies that \( p \) also generates a hyperplane supporting \( UC(u^0, z^i) \) at \( z^i \). Since \( p \cdot z^i = p \cdot z^i \), it follows from (a) that \( z^i \) and \( z^i \) are both unique \( u^0 \) maximizers on the same hyperplane \( \{ z^0 \in \mathbb{R}^n_+: p \cdot z^0 = p \cdot z^i \} \). This contradicts to \( z^i \neq z^i \). We can similarly derive a contradiction in the case that \( p \cdot z^i > p \cdot z^i \).

\(^3\) To verify this statement, suppose by contradiction that \( z^i \) is proportional to \( z^1 \). Since \( z^i \neq 0 \), there is \( v \in \mathbb{R} \) such that \( z^i = v \cdot z^i \). Note

\[
z^i = \Omega - \sum_{i=2}^{i=n} z_i = \Omega - \mu \sum_{i=2}^{i=n} z_i = \Omega - \mu (\Omega - z^i) = (1 - \mu) \cdot \Omega + \mu \cdot z^i.
\]

Thus

\[
v \cdot z^i = (1 - \mu) \cdot \Omega + \mu \cdot z^i
\]

\[
(v - \mu) \cdot z^i = (1 - \mu) \cdot \Omega.
\]

Since \( \mu \neq 1 \), and since \( z^i \neq 0 \) is not proportional to \( \Omega \neq 0 \), this is a contradict.
and since \( u^0 \) is classical, homothetic, and smooth, Fact 2(i) implies that \( p \) is the unique vector in the unit simplex that generates the hyperplane supporting \( UC(\hat{u}^0, \hat{z}^k) \) at \( \hat{z}^k \). Since \( p \) generates a hyperplane supporting \( UC(u^1, \hat{z}^3) \) at \( \hat{z}^3 = z^1 \), since \( u^1 \) is classical, homothetic, and smooth, and since \( z^1 \) is not proportional to \( \hat{z}^1 \), \( p \) cannot generate the hyperplane supporting \( UC(u^1, \hat{z}^1) \) at \( \hat{z}^1 \). This contradicts Pareto-efficiency of \( \hat{z} \). Therefore, \( \mu = 1 \), so that

\[
\sum_{i=2}^{i=n} f'(u^1, \hat{u}^{[2, \ldots, k]}, u^0, \ldots, u^0) = \sum_{i=2}^{i=n} z^i. 
\]

Figure 1 illustrates the proof of Lemma 5.

---

To verify this statement, suppose by contradiction that \( p \) supports a hyperplane supporting \( UC(u^1, \hat{z}^3) \) at \( \hat{z}^3 \). Note that \( u^1 \) satisfies the properties (a) and (b) of \( u^0 \) in footnote 12. Since \( \hat{z}^3 \) is not proportional to \( \hat{z}^1 \), and since \( p \) generates a hyperplane supporting \( UC(u^1, \hat{z}^1) \) at \( \hat{z}^1 \) and a hyperplane supporting \( UC(u^1, \hat{z}^3) \) at \( \hat{z}^3 \), we can derive a contradiction similarly to Footnote 12.
3.2. Proof of Theorem 1

We now prove Theorem 1. By contradiction, suppose that there is a strategy-proof, Pareto-efficient, and individually rational rule \( f \) on the class of classical, homothetic, and smooth preferences. We will derive a contradiction. Let \( \Omega = \sum_{i \in N} e^i \). We distinguish between the following two cases.

Case 1: For some \( i \in N \), \( e^i \) is not proportional to \( \Omega \).

Case 2: For any \( i \in N \), \( e^i \) is proportional to \( \Omega \).

We apply the Lemmas above to derive a contradiction in each case.

Case 1. Given \( \rho \in (-\infty, 1) \) and \( x \in \mathbb{R}^m_+ \), let \( u^0(x; \rho) = \left[ \sum_{i \in M} \left( \Omega_i \right)^{1-\rho} (x_i)^\rho \right]^{1/\rho} \), and let \( u^*(\cdot; \rho) = (u^0(\cdot; \rho), \ldots, u^0(\cdot; \rho)) \in (U^c \cap U^d \cap U^e)^N \). By Lemma 1, for every \( i \in N \) and any \( \rho \in (-\infty, 1) \), \( f^*(u^*(\cdot; \rho)) \) is proportional to \( \Omega \). For each \( i \in N \), let \( w^i \in \mathbb{R}^m_+ \) be such that \( w^i \) is proportional to \( \Omega \), and such that \( \sum_{i \in M} w^i = \sum_{i \in M} e^i \). Then \( \sum_{i \in N} w^i = \sum_{i \in N} e^i \), and \( w = (w^1, \ldots, w^n) \) is a Walrasian equilibrium allocation for \( u^*(\cdot; \rho) \) with arbitrary \( \rho \in (-\infty, 1) \).\(^15\) For each \( i \in N \), let \( M(i) = \{ l \in M : w^l < e^l \} \), and \( p^i = (p^i_1, \ldots, p^i_m) \in \mathbb{R}^m_+ \) be such that \( p^i_l = 2 \) for \( l \in M(i) \) and \( p^i_l = 1 \) otherwise. For each \( i \in N \), given \( \sigma \in (-\infty, 1) \) and \( x \in \mathbb{R}^m_+ \), let \( \tilde{u}^i(x; \sigma) = \left[ \sum_{i \in M} P_i \cdot (\Omega_i)^{1-\sigma} (x_i)^\sigma \right]^{1/\sigma} \). For each \( i \in N \), let \( \tilde{w}^i \in \mathbb{R}^m_+ \) be such that \( \tilde{w}^i \) is proportional to \( \Omega \), and such that \( p^i \cdot \tilde{w}^i = p^i \cdot e^i \).

In this paragraph, we show that for any \( i \in N \), as \( \rho \to -\infty \) and \( \sigma \to 1 \), \( f^*(\tilde{u}^i(\cdot; \sigma), u^{-i}(\cdot; \rho)) \) converges to the segment \( [\tilde{w}^i, \Omega] \). Note that as \( \rho \to -\infty \) and \( \sigma \to 1 \), for each \( i \in N \), \( P^i(\tilde{u}^i(\cdot; \sigma), u^{-i}(\cdot; \rho), Z(2, \Omega)) \)

\(^{15}\) To verify this statement, note that for each \( i \in N \), since \( w^i \) is proportional to \( \Omega \), there is \( \lambda^i \in \mathbb{R} \) such that \( w^i = \lambda^i \cdot \Omega \). By contradiction, suppose that \( \sum_{i \in N} w^i \neq \sum_{i \in N} e^i \) for some good \( l \). First consider the case that \( \sum_{i \in N} w^i_l < \sum_{i \in N} e^i_l = \Omega_l \). Since \( w^i_l = \lambda^i \cdot \Omega_l \) for each \( i \in N \), \( \sum_{i \in N} \lambda^i < 1 \). Then it follows that \( \sum_{i \in N} w^i_l < \sum_{i \in N} e^i_l = \Omega_l \) for any good \( l \in M \). But this implies that \( \sum_{i \in N} \sum_{i \in M} w^i_l < \sum_{i \in N} \sum_{i \in M} e^i_l \), which is a contradiction to our assumption that \( \sum_{i \in N} \sum_{i \in M} w^i_l = \sum_{i \in N} \sum_{i \in M} e^i_l \). For each \( i \in N \). In the case that \( \sum_{i \in N} \sum_{i \in M} w^i_l > \sum_{i \in N} \sum_{i \in M} e^i_l \), we can similarly get \( \sum_{i \in N} \sum_{i \in M} w^i_l > \sum_{i \in N} \sum_{i \in M} e^i_l \), which is also a contradiction.

\(^{16}\) To verify this statement, note that for each \( l \in M \),

\[
\frac{\partial u^l(x; \rho)}{\partial x_l} = (\Omega_l)^{1-\rho} (x_l)^\rho \left[ \sum_{i \in M} (\Omega_i)^{1-\rho} (x_i)^\rho \right]^{(1-\rho)/\rho - 1}.
\]

Thus for each \( i \in N \) and each \( l \in M \), since \( w^i \) is proportional to \( \Omega \), \( \frac{\partial u^l(w^i; \rho)}{\partial x_l} = \frac{[\sum_{i \in M} (\Omega_i)]^{(1-\rho)/\rho - 1}}{[\sum_{i \in M} (\Omega_i)]^{(1-\rho)/\rho - 1}}. \) Hence, the allocation \( w \) together with the price vector \((1, \ldots, 1) \in \mathbb{R}^n_+\) constitute a Walrasian equilibrium.
converges to the segment \([0, \Omega]\).\(^{17}\) and so does \(f'(\tilde{u}(\cdot; \sigma), u^{-}(\cdot; \rho))\) by Pareto-efficiency and Lemma 2. For any \(i \in N\), since as \(\sigma \to 1\), the preference \(\tilde{u}(\cdot; \sigma)\) converges to the flat preference for which the indifference surfaces all have normal \(p'\), it follows that as \(\sigma \to 1\), the upper contour set \(U_C(\tilde{u}(\cdot; \sigma), e')\) converges to the half space \(\{x \in \mathbb{R}^n_+: p'i \geq p'i e'\}\), and so does \(f'(\tilde{u}(\cdot; \sigma), u^{-}(\cdot; \rho))\) by individual rationality. Since \([\tilde{w}^i, \Omega] = [0, \Omega] \cap \{x \in \mathbb{R}^n_+: p'i x \geq p'i e'\}\), Pareto-efficiency and individual rationality imply that for any \(i \in N\), as \(\rho \to -\infty\) and \(\sigma \to 1\), \(f'(\tilde{u}(\cdot; \sigma), u^{-}(\cdot; \rho))\) converges to the segment \([\tilde{w}^i, \Omega]\).

In this paragraph, we show that there are \(\rho_0\) and \(\sigma_0\) such that for any \(i \in N\) with \(M(i) \neq \emptyset\), \(u^l(f'(\tilde{u}(\cdot; \sigma_0), u^{-}(\cdot; \rho_0)); \rho_0) > u^0(w^i; \rho_0)\). Note that for any \(i \in N\), if \(M(i) \neq \emptyset\),

\[
p' \cdot (e' - w') = \sum_{l \in M(i)} 2 \cdot (e'_l - w'_l) + \sum_{l \notin M(i)} (e'_l - w'_l) > \sum_{l \in M} (e'_l - w'_l) = 0,
\]

so that \(p' \cdot e' > p' \cdot w'). For any \(i \in N\), since \(w^i\) and \(\tilde{w}^i\) are both proportional to \(\Omega\), \(\Omega \in \mathbb{R}^n_+\), and \(w^i \in \mathbb{R}^n_+\), it follows that there is \(\lambda' \in \mathbb{R}^n_+\) such that \(\tilde{w}^i = \lambda' \cdot w^i\). Thus for any \(i \in N\), if \(M(i) \neq \emptyset\), then

\[
\lambda' \cdot p' \cdot w^i = p' \cdot (\lambda' \cdot w^i) = p' \cdot \tilde{w}^i = p' \cdot e' > p' \cdot w',
\]

so that \(\lambda' > 1\). That is, for any \(i \in N\), if \(M(i) \neq \emptyset\), then \(\tilde{w}_i > w'_i\) for any \(l \in M\). Therefore, by the result of the previous paragraph, there are \(\rho\) sufficiently small and \(\sigma\) sufficiently close to 1 such that for any \(i \in N\) with \(M(i) \neq \emptyset\), \(f'[\tilde{u}(\cdot; \sigma), u^{-}(\cdot; \rho)] > w'_i\) for any \(l \in M\), so that \(u^0(f'(\tilde{u}(\cdot; \sigma_0), u^{-}(\cdot; \rho_0)); \rho_0) > u^0(w^i; \rho_0)\). Hence there are \(\rho_0\) and \(\sigma_0\) such that for any \(i \in N\) with \(M(i) \neq \emptyset\), \(u^l(f'(\tilde{u}(\cdot; \sigma_0), u^{-}(\cdot; \rho_0)); \rho_0) > u^0(w^i; \rho_0)\).

Note that for any \(i \in N\) such that \(e'\) is not proportional to \(\Omega\), \(M(i) \neq \emptyset\). Also note that \(\sum_{i \in N} f'(u(\cdot; \rho_0)) = \sum_{i \in N} w^i\), and that by Lemma 4, for any \(i \in N\) such that \(e'\) is proportional to \(\Omega\), \(f'(u(\cdot; \rho_0)) \equiv w^i\). Thus, there is \(i \in N\) whose \(e'\) is not proportional to \(\Omega\) such that \(f'(u(\cdot; \rho_0)) \leq w^i\), and so \(u^0(w^i; \rho_0) \geq u^0(f'(u(\cdot; \rho_0)); \rho_0)\). Now by the result of the previous paragraph, we have:

\[
u^l(f'(\tilde{u}(\cdot; \sigma_0), u^{-}(\cdot; \rho_0)); \rho_0) > u^0(w^i; \rho_0) \geq u^0(f'(u(\cdot; \rho_0)); \rho_0)\).

This contradicts strategy-proofness.

\(^{17}\) To verify this statement, remember that as \(\rho \to -\infty\), the preference \(u^l(\cdot; \rho)\) converges to the Leontief type preference whose indifference surfaces are kinked on the segment \([0, \Omega]\), and that as \(\sigma \to 1\), the preference \(\tilde{u}(\cdot; \sigma)\) converges to the preference with flat indifference surfaces.
Case 2. For any $i \in N$, since $e_i$ is proportional to $\Omega \in \mathbb{R}_+^m$ and $e_i' \in \mathbb{R}_+^m \setminus \{0\}$, we have $e_i' \in \mathbb{R}_+^m$. Given $x \in \mathbb{R}_+^m$, let $u^0(x) = x_1 \times \cdots \times x_m$, $u^1(x) = (x_1)^2 \times x_2 \times \cdots \times x_m$, and $z = f(u^1, u^0, \ldots, u^0)$.

Note that by Pareto-efficiency, $z^1 = 0$, or $z^1 = \Omega$, or $z^1 \notin [0, \Omega]$. Since $e_i' \in \mathbb{R}_+^m$, individual rationality implies that $z^1 \neq 0$. Since $e_i' \in \mathbb{R}_+^m$ for any $i \neq 1$, individual rationality also implies that $z^1 \neq \Omega$. Thus $z^1 \notin [0, \Omega]$, and

Figure 2 illustrates the proof of Case 1. In Fig. 2, $f(u(\cdot; \rho_0))$, $P^i(\tilde{u}^i(\cdot; \sigma), u^i(\cdot; \rho), Z(2, \Omega))$, and $f^i(\tilde{u}^i(\cdot; \sigma), u^i(\cdot; \rho))$ are respectively denoted by $z^i(\rho)$, $P^i(\sigma, \rho)$, and $z^i(\sigma, \rho_0)$ for short.

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so \( z^1 \neq e^1 \). Also note that since \( e^i \in \mathbb{R}^n_+ \) for any \( i \in N \), individual rationality implies that \( z^i \in \mathbb{R}^n_+ \) for any \( i \in N \).

By Lemma 4, \( f(u^0, u^{i_1}, ..., u^{i_m}) = e \). Thus by Lemma 3, neither \( z^1 < e^1 \) nor \( z^1 > e^1 \). Thus by \( z^1 \neq e^1 \), there is \( p \in \mathbb{R}^m_+ \) such that \( p \cdot (z^1 - e^1) < 0 \). Given \( \sigma \in (0, 1) \) and \( x \in \mathbb{R}^m_+ \), let \( \hat{u}^1(x; \sigma) = \left[ \sum_{i \in M} p_i \cdot (z^i)^{1-\sigma} \cdot (x_i)^{\sigma} \right]^{1/\sigma} \). Note that for any \( x \in [z^1, \Omega] \), the vector \( q = (p_i \cdot (z^i)^{1-\sigma} \cdot (x_i)^{\sigma} - 1)_{i \in M} \) is normal to the support of \( UC(\hat{u}^1(\cdot; \sigma), x) \) at \( x \). As a special case, \( p \) is normal to the support of \( UC(\hat{u}^1(\cdot; \sigma), z^1) \) at \( z^1 \). Thus since \( p \cdot (z^1 - e^1) < 0 \), there is \( \sigma \) close to 1 such that \( \hat{u}^1(e; \sigma) > \hat{u}^1(z^1; \sigma) \).

Let \( \sigma \) be so chosen. Since \( \hat{u}^1(\Omega; \sigma) > \hat{u}^1(e; \sigma) > \hat{u}^1(z^1; \sigma) \), by the continuity of \( \hat{u}^1(\cdot; \sigma) \), there is \( z^1 \in [z^1, \Omega] \) such that \( \hat{u}^1(z^1; \sigma) = \hat{u}^1(e; \sigma) \). For any \( x \in [z^1, \Omega] \), since

20 To verify this statement, note that if \( z^1 \neq 0 \) for some \( i \in M \), then \( u^i(z^1) = 0 < u^i(e^1) \) contradicting individual rationality, and that if there are \( i \neq 1 \) and \( i \in M \) such that \( z^1 = 0 \), then \( u^i(z^1) = 0 < u^i(e^1) \), contradicting individual rationality.

21 To verify this statement, remember that as \( \sigma \to 1 \), the upper contour set \( UC(\hat{u}^1(\cdot; \sigma), z^1) \) converges to the half space \( \{ x \in \mathbb{R}^n_+ : p \cdot x \geq p \cdot z^1 \} \).
$x \geq z^1$, it holds that $\hat{u}^i(x; \sigma) \geq \hat{u}^i(z^1; \sigma) = \hat{u}^i(e; \sigma) > \hat{u}^i(z^1; \sigma)$. For any $x \in [z^1, \Omega]$, since $x \geq z^1 > z^1$, it also holds that $u^i(x) > u^i(z^1)$.

Let $\hat{u}^i(\cdot; \rho)$ be the preference specified in Fact 2 for the case that $u^0$ is specified above and $z^0 = \sum_{j=2}^{\infty} z^j \gg 0$. By Lemma 5, $f^i(u^1, \hat{u}^0(\cdot; \rho), \ldots, \hat{u}^0(\cdot; \rho)) = z^1$ for any $\rho < \rho_0$, where $\rho_0$ is specified in Fact 2. Note that as $\rho \to -\infty$, $P(\hat{u}^i(\cdot; \sigma), \hat{u}^i(\cdot; \rho), Z(2, \Omega)) \cap UC(\hat{u}^i(\cdot; \sigma), z^1)$ converges to $[z^1, \Omega]$. Thus by Lemma 2, it follows from Pareto-efficiency and individual rationality that as $\rho \to -\infty$, $f^i(\hat{u}^i(\cdot; \sigma), \hat{u}^i(\cdot; \rho), \ldots, \hat{u}^i(\cdot; \rho))$ converges to $[z^1, \Omega]$. Since $u^i(x) > u^i(z^1)$ for any $x \in [z^1, \Omega]$, there is $\rho$, such that for any $\rho < \rho_1$, $u^i(f^i(\hat{u}^i(\cdot; \sigma), \hat{u}^i(\cdot; \rho), \ldots, \hat{u}^i(\cdot; \rho))) > u^i(z^1)$.

Therefore, for $\rho < \min\{\rho_0, \rho_1\}$, we have:

$$u^i(f^i(\hat{u}^i(\cdot; \sigma), \hat{u}^i(\cdot; \rho), \ldots, \hat{u}^i(\cdot; \rho))) > u^i(f^i(u^1, \hat{u}^i(\cdot; \rho), \ldots, \hat{u}^i(\cdot; \rho))).$$

This contradicts strategy-proofness.

Figure 3 illustrates the proof of Case 2. In the figure, $z^{-1}_j$ denotes $\sum_{j=1}^{\infty} z^j$ for every $l \in M$.

## 4. Symmetric Rules

In this section, we establish an impossibility result that is parallel to Theorem 1 where individual rationality is replaced by a distributional condition of symmetry. This condition is a requirement that equals should be treated equally.

**Definition 7.** A rule $f$ is symmetric if for any $u \in U$, any $i \in N$, and any $j \in N$,

$$[u^i = u^j] \Rightarrow u^i(f^i(u)) = u^j(f^j(u)).$$

**Definition 8.** A rule $f$ is strongly symmetric if for any $u \in U$, any $i \in N$, and any $j \in N$,

$$[u^i = u^j] \Rightarrow f^i(u) = f^j(u).$$

Strong symmetry implies symmetry, but not vice versa. However, if preferences are classical, then Pareto-efficiency and symmetry imply strong symmetry.

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22 To verify this statement, remember that as $\rho \to -\infty$, the preference $\hat{u}^i(\cdot; \rho)$ converges to the Leontief type preference whose indifference surfaces are kinked on the segment $[0, \Omega - z^1]$. 
Theorem 2. There is no strategy-proof, Pareto-efficient, and symmetric rule on the class of classical, homothetic, and smooth preferences.

Similarly to Theorem 1, we remark that the conclusion of Theorem 2 holds on the class of CES type preferences. Similarly to Corollary 1, we can also prove Corollary 2.

Corollary 2. Let the domain $U$ be a superset of the class of classical, homothetic, and smooth preferences. There is no strategy-proof, Pareto-efficient, and symmetric rule on $U$.

In the rest of this section, we discuss how to prove Theorem 2. The basic structure of Theorem 2’s proof is similar to that of Case 2 of Theorem 1’s proof. In Theorem 1’s proof, Lemma 5 plays a crucial role, and it assumes individual rationality. In Theorem 2’s proof, we employ Lemma 6 below instead of Lemma 5. Lemma 6 assumes symmetry instead of individual rationality. Another difference between the two lemmas is that the assumption on endowments of Lemma 5 is removed in Lemma 6.

Lemma 6. Let $f$ be a strategy-proof, Pareto-efficient, and symmetric rule. Let $u^0$ and $u^1$ be classical, homothetic, and smooth preferences. Let $z = f(u^1, u^0, \ldots, u^0)$, $z^1$ be not proportional to $\Omega$, and $z^0 = \sum_{i=1}^{n} z^i \in \mathbb{R}^n_+$. Given $x \in \mathbb{R}^n_+$, let $\hat{u}^0(x) = A^{-1/p} \cdot \left( \sum_{i \in M} a_i \cdot (z^0_i)^{1-p} \cdot (x^i) \right)^{1/p}$, where $a_i = \left[ \frac{\partial u^0(z^0)}{\partial x_i} \cdot A \right]$ for each $i \in M$, $A = \sum_{i \in M} a_i \cdot (z^0_i)$, and $p \in (-\infty, 1)$ is chosen so that the function $\hat{u}^0$ is a Maskin Monotonic Transformation of $u^0$ at $z^0$. Then $f^i(u^1, \hat{u}^0, \ldots, \hat{u}^0) = z^i$.

Proof. Note that since preferences are classical, Pareto-efficiency and symmetry imply strong symmetry. For each $k \in \{1, \ldots, n\}$, let $\hat{u}^{[2, \ldots, k]}$ denote the preference profile of the set $\{2, \ldots, k\}$ of agents in which $\hat{u}^i = \hat{u}^0$ for each $i \in \{2, \ldots, k\}$. When $k = 1$, let $\hat{u}^{[2, \ldots, k]}$ denote the empty profile (no agents). We establish by induction that for every $k \in \{1, \ldots, n\}$ and any $i \in N$,

$$f^i(u^1, \hat{u}^{[2, \ldots, k]}, u^0, \ldots, u^0) = z^i.$$  

Note that when $k = n$, this claim implies that $f^i(u^1, \hat{u}^0, \ldots, \hat{u}^0) = z^i$.

Note that when $k = 1$, by our assumption, our claim holds. As the induction hypothesis, assume that

$$\forall i \in N, f^i(u^1, \hat{u}^{[2, \ldots, k-1]}, u^0, \ldots, u^0) = z^i.$$  

We show that

$$\forall i \in N, f^i(u^1, \hat{u}^{[2, \ldots, k]}, u^0, \ldots, u^0) = z^i.$$  

Denote \( f(u^1, \tilde{a}^{[2 \ldots k]}, u^2, \ldots, u^k) = \tilde{z} \), and \( f(u^1, \tilde{a}^{[2 \ldots k-1]}, u^2, \ldots, u^k) = \tilde{z} \).

Note that by symmetry, \( f(u^0, \ldots, u^k) = d \), where \( d = (\Omega/n, \ldots, \Omega/n) \).

Thus by Lemma 3, \( z^i \neq 0 \) and \( z^i \neq \Omega \); and so by symmetry, \( z^i = \cdots = z^k = (\Omega - z^i)/(n-1) \neq 0 \).

Since \( u^0 \) and \( \tilde{a}^0 \) are homothetic, for any \( i \in N \setminus \{1\} \), \( \tilde{a}^0 \in M(u^0, z^i) \). Since \( \tilde{a}^0 \in M(u^0, z^i) \) and \( z^k = z^i \) by our induction hypothesis, it follows from Fact 1 that \( \hat{z}^k = \hat{z}^i = z^k = (\Omega - z^i)/(n-1) \). For any \( i \in \{2, \ldots, k\} \), by symmetry, \( \hat{z}^i = \hat{z}^i = (\Omega - z^i)/(n-1) = z^i \).

In this paragraph, we show that for any \( i \in N \setminus \{1\} \), \( \hat{z}^i \) is proportional to \( \hat{z}^k \). For any \( i \in \{2, \ldots, k\} \), since \( \hat{z}^i = \hat{z}^k \), this holds. Thus consider the case that \( i \in \{k+1, \ldots, n\} \). Since \( \hat{z} \) is Pareto-efficient, there is \( p \in \mathbb{R}^n_+ \) that generates a hyperplane supporting \( UC(u^0, \hat{z}^i) \) at \( \hat{z}^k \) and a hyperplane supporting \( UC(u^0, \hat{z}^i) \) at \( \hat{z}^i \). By Fact 2(i) and \( \tilde{a}^0 \in M(u^0, \hat{z}^i) \), \( p \) also generates the hyperplane supporting \( UC(u^0, \hat{z}^i) \) at \( \hat{z}^k \). Since \( u^0 \) is classical and homothetic, and since \( p \) generates a hyperplane supporting \( UC(u^0, \hat{z}^i) \) at \( \hat{z}^k \) and a hyperplane supporting \( UC(u^0, \hat{z}^i) \) at \( \hat{z}^i \), it follows that \( \hat{z}^i \) is proportional to \( \hat{z}^k \).

Since for any \( i \in N \setminus \{1\} \), \( \hat{z}^i \) is proportional to \( \hat{z}^k \) and since \( \hat{z}^k = \hat{z}^k \neq 0 \), it follows that for any \( i \in N \setminus \{1\} \), there is \( \hat{\lambda}^i \in \mathbb{R}_+ \) such that \( \hat{z}^i = \hat{\lambda}^i \cdot \hat{z}^k \).

Since \( \hat{z}^k = (\Omega - z^i)/(n-1) \), we have \( \hat{z}^i = \hat{\lambda}^i \cdot (\Omega - z^i)/(n-1) = [\hat{\lambda}^i/(n-1)] \cdot (\Omega - z^i) \) for any \( i \in N \setminus \{1\} \). Denote \( \mu = [1/(n-1)] \cdot \sum_{i=2}^n \hat{\lambda}^i \).

Then, we get \( \sum_{i=2}^n \hat{z}^i = \mu \cdot (\Omega - z^i) \). Similarly to the proof of Lemma 5, we can also show that \( \mu = 1 \), so that \( \sum_{i=2}^n \hat{z}^i = (\Omega - z^i) = \sum_{i=2}^n z^i \). Thus \( \hat{z}^i = \Omega - \sum_{i=2}^n z^i = \Omega - \sum_{i=2}^n z^i = z^i \). Since \( \hat{z}^i = z^i \) and \( \hat{z}^i = \hat{z}^i = (\Omega - z^i)/(n-1) \), for any \( i \in \{2, \ldots, k\} \), it follows from symmetry that for any \( i \in \{k+1, \ldots, n\} \),

\[
\hat{z}^i = \Omega - z^i - (k-1) \cdot ((\Omega - z^i)/(n-1)) = \Omega - z^i = z^i. 
\]

Now, we complete the proof of Theorem 2. By contradiction, suppose that there is a strategy-proof, Pareto-efficient, and symmetric rule \( f \) on the class of classical, homothetic, and smooth preferences. We will derive a contradiction similarly to Case 2 in the proof of Theorem 1.

Note that since preferences are classical, Pareto-efficiency and symmetry imply strong symmetry. Let \( \Omega = \sum_{i=1}^n e^i \). Let \( \tilde{a}^0(x_1, \ldots, x_m) = x_1 \times \cdots \times x_m \), \( u^i(x_1, \ldots, x_m) = (x_1)^{\mu} \times \cdots \times (x_m) \), and \( z = f(u^1, u^0, \ldots, u^k) \). By symmetry, \( f(u^0, u^i, \ldots, u^k) = d \), where \( d = (\Omega/n, \ldots, \Omega/n) \) in \( \mathbb{R}^n_+ \). By Pareto-efficiency, it must hold that \( z^i = 0 \), or \( z^i = \Omega \), or \( z^i \notin [0, \Omega] \). By \( f(u^0, u^i, \ldots, u^k) = d \) and Lemma 3, neither \( z^i < d^i \) nor \( z^i > d^i \). Thus \( z^i \neq 0 \), and \( z^i \neq \Omega \), so that \( z^i \notin [0, \Omega] \) and \( z^i \neq d^i \). By \( f(u^0, u^i, \ldots, u^k) = d \) and strategy-proofness, \( z^i \in \mathbb{R}^n_+ \). By \( z^i \notin \Omega \), Pareto-efficiency and symmetry, \( z^i = (\Omega - z^i)/(n-1) \in \mathbb{R}^n_+ \) for any \( i \in N \setminus \{1\} \).
Since $z^1 \neq d^1$, and neither $z^1 < d^1$ nor $z^1 > d^1$, there is $p \in \mathbb{R}^m_{++}$ such that \( p \cdot (z^1 - d^1) < 0 \). Given $\sigma \in (0, 1)$, let $\hat{u}^i(x; \sigma) = \left[ \sum_{j \in M} p_i \cdot (z^i_j) \right]^{1 - \sigma} \cdot (x_i)^{\sigma}$.

Note that for any $x \in [z^1, \Omega)$, the vector $q = (p_i \cdot (z^i_j))^{1 - \sigma} \cdot (x_i)^{\sigma}$ generates the hyperplane supporting $UC(\hat{u}^i(\cdot; \sigma), x)$ at $x$. As a special case, $p$ generates the hyperplane supporting $UC(\hat{u}^i(\cdot; \sigma), z^i)$ at $z^i$. Thus there is $\sigma$ close to $1$ such that $\hat{u}^i(d^i; \sigma) > \hat{u}^i(z^i; \sigma)$. Let $\sigma$ be so chosen. Since $\hat{u}^i(\Omega; \sigma) > \hat{u}^i(d^i; \sigma) > \hat{u}^i(z^i; \sigma)$, by the continuity of $\hat{u}^i(\cdot; \sigma)$, there is $\hat{z}^i \in [z^i, \Omega]$ such that $\hat{u}^i(\hat{z}^i; \sigma) = \hat{u}^i(d^i; \sigma)$. For any $x \in [\hat{z}^i, \Omega]$, since $x \geq z^i$, it holds that $\hat{u}^i(x; \sigma) \geq \hat{u}^i(z^i; \sigma) = \hat{u}^i(d^i; \sigma) > \hat{u}^i(z^i; \sigma)$. For any $x \in [\hat{z}^i, \Omega]$, since $x \geq z^i > z^1$, it also holds that $u^i(x) > u^i(z^i)$.

Let $\hat{u}^i(\cdot; \rho)$ be the preference specified in Fact 2 for the case that $u^\theta$ is specified above and $z^i = \sum_{j \in M} z^j$. By Lemma 6, \( f(\hat{u}^i, \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho)) = z \) for any $\rho < \rho_0$, where $\rho_0$ is specified in Fact 2. Note that as $\rho \rightarrow -\infty$, \( P^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), Z(2, \Omega)) \cap UC(\hat{u}^i(\cdot; \rho), z^i) \) converges to $[z^i, \Omega]$. Also note that by symmetry, \( f(\hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho)) = d \) for any $\rho \in (-\infty, 1)$, so that by strategy-proofness, for any $\rho \in (-\infty, 1)$, we have:

\[
\hat{u}^i(\hat{f}^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho))) \geq \hat{u}^i(d^i) = \hat{u}^i(z^i).
\]

Thus for any $\rho \in (-\infty, 1)$,

\[
f^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho)) \in P^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), Z(2, \Omega)) \cap UC(\hat{u}^i, z^i).
\]

Therefore, as $\rho \rightarrow -\infty$, \( f^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho)) \) converges to $[z^i, \Omega]$. Since $u^i(x) > u^i(z^i)$ for any $x \in [z^i, \Omega]$, there is $\rho_1$ such that for any $\rho < \rho_1$, \( u^i(\hat{f}^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho))) > u^i(z^i) \). Hence, for $\rho < \min(\rho_0, \rho_1)$, we have:

\[
u^i(\hat{f}^i(\hat{u}^i(\cdot; \sigma), \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho))) > u^i(f^i(u^i, \hat{u}^0(\cdot; \rho), ..., \hat{u}^0(\cdot; \rho))).
\]

This contradicts strategy-proofness.

5. CONCLUDING REMARK

In this article, extending Hurwicz’s [2] result, we have established that for pure exchange economies with any finite number of agents and any finite number of goods, (i) there is no strategy-proof, Pareto-efficient, and individually rational rule on the class of classical, homothetic, and smooth preferences; and (ii) there is no strategy-proof, Pareto-efficient, and symmetric rule on the same class of preferences. Here we mention questions related to the conjecture raised in Zhou [13]. The conjecture states that for pure exchange economies with any finite number of agents and any finite number
of goods, there is no strategy-proof, Pareto-efficient, and non-inverse-dictatorial rule. Non-inverse-dictatoriality is the requirement that there should exist no agent who is always assigned no consumption bundle. Since non-inverse-dictatoriality is much weaker than individual rationality and symmetry, Zhou’s conjecture is stronger than our results. Recently, Kato and Ohseto [5] have found a counter-example to the conjecture. However, we believe that the results similar to Zhou’s conjecture still hold, and that the techniques we have developed in this article are also useful to prove them.

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