A multi-criteria optimal replacement policy for a system subject to shocks

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\begin{abstract}
A system is subject to shocks that arrive according to a non-homogeneous Poisson process. As these shocks occur, the system experiences one of two types of failures: a type-I failure (minor), rectified by a minimal repair; or a type-II failure (catastrophic) that calls for a replacement. In this study, we consider a multi-criteria replacement policy based on system age, nature of failure, and entire repair-cost history. Under such a policy, the system is replaced at planned life time \(T\), or at the nth type-I failure, or at the nth type-II failure \((k \leq n)\) at which the accumulated repair cost exceeds the pre-determined limit, or at the first type-II failure, whichever occurs first. An optimal policy over the control parameters is studied analytically by showing its existence, uniqueness, and structural properties. This model is a generalization of several existing models in the literature. Some numerical examples are presented to show several useful insights.
\end{abstract}

\section{1. Introduction}
Consider a system subject to shocks that weaken the system and make it more expensive to run. It is worth finding the optimal replacement policy for such a system to reduce the operating costs and catastrophic breakdown risk. In the classical age-based replacement policy, an operating system is replaced at age \(T\) or at a failure, whichever occurs first (Barlow & Hunter, 1960). In another classical block replacement policy, an operating system is replaced at times \(kT\) \((k = 1, 2, \ldots)\) and at failures (Barlow & Proschan, 1965). Later, Boland and Proschan (1982) considered the case of periodic replacement at times \(kT\) \((k = 1, 2, \ldots)\) and minimal repairs upon failures which is a variation of block replacement model. Besides, Makabe and Morimura (1963a, 1963b, 1965) proposed another periodic replacement model in which a system is replaced at the nth failure, and they also discussed the determination of the optimal policy. These classical replacement policies have been further studied and extended by Cléroux, Dubuc, and Tilquin (1979), Nakagawa (1982), Boland and Proshan (1983), Nguyen and Murthy (1984), Berg, Bievrem, and Cléroux (1986), Park (1987), Ait Kadi and Cléroux (1988), Sheu (1992), Sheu, Griffith, and Nakagawa (1995), and Sheu and Chang (2009), among others. In this article, a bivariate replacement policy \((n, T)\) is presented under which the system is replaced at the life age \(T\) or at the time of the nth minor failure, whichever occurs first. In addition, we introduce more criteria for the replacement action.

It is a common assumption that a minor failure can be rectified by a minimal repair (Barlow & Hunter, 1960). A minimal repair restores the system to its functioning condition just prior to the failure, and the failure rate is not affected by minimal repairs. Drinkwater and Hastings (1967) presented a classical repair-cost limit policy: the system is replaced if the repair cost exceeds a certain threshold, otherwise it is repaired. A disadvantage of this type of policy is that a system with frequent “not-very-costly” failures and consequently high accumulated repair costs will continue to be repaired rather than replaced. Some improved policies based on the entire repair history have been proposed. Beichelt (2001) proposed a cumulative repair-cost limit replacement policy where a system is replaced as soon as the cumulative repair cost reaches or exceeds a given limit, but the cost function was determined exogenously and was not based on the lifetime repair history. Chien, Sheu, and Chang (2009) presented an improved self-cumulative repair-cost limit replacement policy which has some similar aspects to the cumulative damage model (Nakagawa & Kijima, 1989; Qian, Ito, & Nakagawa, 2005). In Chien et al. (2009), the accumulated repair costs are endogenous and can be evaluated from the shock process. Here, we adopt the self-cumulative repair-costs in our proposed bivariate replacement policy \((n, T)\). Nakagawa and Kijima (1989) discussed several individual replacement policies for a cumulative damage model with only minimal repair at failure. Qian et al. (2005) applied a preventive maintenance policy to a cumulative damage model. However, the
Nomenclature

\({N(t): t \geq 0}\) NHPP with intensity \(r(t)\)

\(A(t) = \int_0^t r(u)\,du\)

\(S_k\) arrival instant of the \(k\)th shock for \(k = 1, 2, 3, \ldots\)

\(W_i\) minimal repair cost due to the \(i\)th type-I failure for \(i = 1, 2, 3, \ldots\)

\(G_i(w), g_i(w)\) cumulative distribution function (CDF), probability density function (PDF) of the random variable \(W_i\)

\(c_w\) mean cost of the random variable \(W_i\)

\(Z_i\) accumulated repair cost until the \(j\)th type-I failure for \(j = 1, 2, 3, \ldots; Z_0 = \sum_{k=1}^\infty W_k\)

\(G^{(j)}(z), g^{(j)}(z)\) CDF, PDF of the random variable \(Z_j\); the \(j\)-fold Stieltjes convolution of the distributions \(G_1, G_2, \ldots, G_j\)

\(M\) number of shocks preceding type-II failure

\(P_k, P_\infty\) survival function (SF), probability mass function of \(M\):

\[P_k = P(M = k) = Pr[\text{first } k \text{ shocks are type-I failures}] \]

where the domain of \(P_k\) is \((0, 1, 2, \ldots)\) and

\[1 = P_0 \geq P_1 \geq P_2 \geq \cdots; \quad P_k = P(M = k) = P_{k-1} - P_k = P_{k-1} - (1 - P_{k-1}/P_{k-1})\]

\(h_k\) a sequence of \(P_k\)

\(q_k\) \(Pr(\text{a type-I failure when shock } k \text{ arrives})\); \(q_k = P_k/P_{k-1}\)

\(t_k\) \(Pr(\text{a type-II failure when shock } k \text{ arrives})\); \(t_k = 1 - q_k\)

\(T\) planned replacement age of an operating system

\(n\) number of minimal repairs (or type-I failures) before replacement

\((n, T)\) replacement policy based on age \(T\) and number \(n\) of minimal repairs (or type-I failures) before replacement

Cumulative repair-cost limit

\(B(n, T; L; \{P_k\})\) \(s\)-expected cost rate under policy \((n, T)\) for an infinite time horizon

\(T^*\) optimal \(T\) that minimizes \(B(n, T; L; \{P_k\})\)

\(n^*\) optimal \(n\) that minimizes \(B(n, T; L; \{P_k\})\)

\(c_0\) cost of a planned replacement

\(c_i\) cost of an unplanned replacement

\(Y\) waiting time until first unplanned replacement

\(H(t), h(t)\) CDF, PDF of the random variable \(Y\)

\(\Pi(t)\) SF of \(Y\) which is \(1 - h(t)\)

\(r_Y(t)\) failure (hazard) rate of \(Y\); \(r_Y(t) = h(t)/\Pi(t)\)

\(U_i\) length of successive replacement cycle for \(i = 1, 2, \ldots\)

\(V_i\) operational cost over \(U_i\)

\(D(t)\) \(s\)-expected cost of the operating system over \([0, t]\)

Probability of a failure type is constant. Chien et al. (2009) considered an age replacement policy with the constant minimal repair probability. By the inspiration of the above models, we propose a cumulative repair-cost limit model and consider further a bivariate replacement policy \((n, T)\) while the probability of failure types depends on shock number. Compared with the model of Chien et al. (2009), we also obtain several merits as follows. The bivariate optimal policy \((n, T)\) would be better than the univariate optimal policy \(T\) due to the extra control variable. Furthermore, the history-dependent failure type probability is more general than the constant failure type probability. In addition, our optimization analysis is also different from the previous models.

In this paper, a multi-criteria replacement policy for a system subject to random shocks is considered. The decision to repair or replace depends on multiple factors including the system life age \((T)\), nature of failure, number \((n)\) of random shocks, and accumulated repair costs. The rest of the paper is organized as follows: Section 2 presents the model formulation and policy optimization. Section 3 shows that several classical maintenance models are special cases of our model. Section 4 develops some algorithms for determining the optimal policy parameters, and some computational examples are provided to demonstrate the use of the algorithms. Section 5 concludes.

2. System description and model formulation

We consider a system that is subject to shocks of a non-homogeneous Poisson process (NHPP). As a shock occurs, the system experiences one of two types of failures: a type-I failure (minor) rectified by a minimal repair; or a type-II failure (catastrophic) fixed by a replacement. We focus on the following multi-criteria replacement policy: the planned replacement occurs whenever an operating system reaches age \(T\); the unplanned replacement occurs at the \(n\)th minor failure, or at the \(k\)th type-I failure at which the accumulated repair cost exceeds the pre-determined limit \(L\), or at the first catastrophic failure. A replacement cycle is defined as the time interval between two consecutive replacements. In this framework, replacement cycles constitute a regenerative process.

Assumptions:

1. The intensity function \(r(t)\) is positive, continuous, and increasing in \(t \geq 0\).
2. All failures are instantly detected and corrected.
3. The minor failure is rectified by a minimal repair. A minimal repair restores the system to its functioning condition just prior to failure.
4. After a replacement, the system is restored to as good as new state and the procedure is repeated.
5. The times for minimal repair and replacement are negligible.
6. The mean values of minimal repair cost are increasing with the number of minor failure, i.e., \(c_{m1} \leq c_{m2} \leq c_{m3} \leq \ldots\)
7. The probability for the type of shock (type-I or type-II) is independent of the shock arrival process \({N(t): t \geq 0}\), and is also \(s\)-independent of random variables \(W_i\).

2.1. General model

We formulate a generalized replacement model for a system subject to shocks with minimal repairs or replacements according to the following schemes:

1. Shocks arrive according to a NHPP \({N(t): t \geq 0}\) with a intensity function \(r(t)\). A shock can cause either a minor failure fixed by a minimal repair or a catastrophic (type-II) failure corrected by a replacement.
2. The probability of a type-II failure may depend on the number of minor shocks since the last replacement. Let \(M\) be the number of minor shocks until type-II failure from the last replacement, and \(\overline{P}_k = P(M > k)\) \(i.e., the probability that first \(k\) shocks of the system are type-I failures). When the \(k\)th shock occurs, the system requires either an unplanned replacement (type-II failure) with probability \(h = 1 - \overline{P}_k/\overline{P}_{k-1}\) or a minimal repair (type-I failure) with probability \(q_k = \overline{P}_k/\overline{P}_{k-1}\).
3. A planned replacement is carried out at age \(T\). An unplanned replacement due to \(n\) type-I failures is carried out at the nth type-I failure where \(n\) is a pre-determined threshold value. An unplanned replacement due to \(k < n\) type-I failures is executed at the failure time when the accumulated repair
cost up to this kth minor failure exceeds a pre-determined limit L. However, if the accumulated repair at the kth failure is less than L, a minimal repair is executed immediately to correct this failure. An unplanned replacement occurs at any type-II failure.

The following structure is imposed on the system.

1. The costs of unplanned (due to either type-I or type-II failure) and planned (due to age) replacement are \( c_1 \) and \( c_0 \), respectively.

2. Suppose that a random minimal repair cost \( W_i \) due to the ith type-I failure has a non-negative independent CDF \( G_i(w) = P(W_i \leq w) \) for \( i = 1, 2, 3, \ldots \) with a finite mean \( c_i \). Then, the accumulated repair cost denoted by \( Z_j = \sum_{i=1}^{j} W_i \) up to the jth type-I failure has a CDF

\[
P(Z_j \leq z) = G^j(z) = \int G_1 G_2 \cdots G_j(z), \quad j = 1, 2, 3, \ldots, \quad f = 0.
\]

which is the j-fold convolution of distributions \( G_1, G_2, \ldots, G_j \).

**Remark 1.** In a system with only minimal repairs, the failure rate \( r(t) \) remains undisturbed by any repair. Furthermore, if the repair time is negligible, the failures occur according to a NHPP \( N(t): t \geq 0 \) with intensity function \( r(t) \) (see Block, Borges, & Savits, 1985; Céroux et al., 1979; Nakagawa & Kowada, 1983; Sheu & Griffin, 1991).

### 2.2. Long-term average cost

If no planned replacements are allowed (i.e., \( T \rightarrow \infty \)), then the SF of the time between the two consecutive unplanned replacements is given by

\[
\Pi(t) = \sum_{k=0}^{\infty} P(N(t) = k, Z_k < L, M > k)
\]

\[
= \sum_{k=0}^{\infty} \frac{e^{-4\lambda t} A(t)^k}{k!} G^k(L) \Pi_k.
\]

The density \( h(t) = -d(\Pi(t))/dt \) is given by

\[
h(t) = \sum_{k=0}^{\infty} \frac{e^{-4\lambda t} A(t)^k}{k!} r(t) G^k(L) \Pi_k - \frac{e^{-4\lambda t} A(t)^{k+1}}{(k+1)!} \Pi_{k+1}.
\]

Let \( Y \) represent the time until either the nth type-I failure, or the kth type-I failure with the accumulated repair cost above the limit \( L \), or the first type-II failure for our model (the case of \( T \rightarrow \infty \)). That is, \( Y \) is the random variable with SF \( \Pi(t) \) and hazard function \( r_Y(t) \) given by

\[
r_Y(t) = \frac{h(t)}{\Pi(t)} = [1 - \tau(t)] r(t),
\]

where

\[
\tau(t) = \sum_{k=0}^{\infty} \frac{A(t)^k}{k!} G^k(L) \Pi_{k+1}.
\]

Let \( Y_1, Y_2, \ldots \) be independent and identically distributed random variables with SF \( \Pi(t) \). For the model with finite \( T \), we denote \( U_t \) the length of the \( i \)th successive replacement cycle for \( i = 1, 2, \ldots \), and \( V_i \), the operational cost over the time interval \( U_t \). Thus \( \{U_t, V_i\} \) constitutes a renewal reward process. Letting \( D(t) \) be the s-expected cost of operating the system over the time interval \([0, t] \), we have the long-term average cost, denoted by \( B(n, T; L, \{P_k\}) \), as

\[
B(n, T; L, \{P_k\}) = \lim_{t \to \infty} \frac{D(t)}{t} = \frac{E[V_1]}{E[U_t]},
\]

(see e.g., Ross, 1970, p.52).

For a finite replacement age \( T \), we have

\[
U_t = \begin{cases} Y_1, & \text{if } Y_1 \leq T, \\ T, & \text{if } Y_1 > T. \end{cases}
\]

Thus, the \( s \)-expected length of a successive replacement cycle is given by

\[
E[U_t] = \int_0^T t dH(t) + T \cdot \Pi(t) = \int_0^T \Pi(t) dt.
\]

To find \( E[V_1] \), we introduce the indicator function \( I_k \) which equals 1 if event \( A \) occurs, otherwise 0. It is easy to write

\[
V_1 = c_1 \cdot I_{\{Y_1 \leq T\}} + c_0 \cdot I_{\{Y_1 > T\}} + \sum_{k=1}^{n-1} W_k \cdot I_{\{Z_k \leq L\}} I_{\{S_k \leq T\}}.
\]

Therefore, the \( s \)-expected total cost in a replacement cycle is

\[
E[V_1] = c_1 \cdot H(T) + c_0 \cdot \Pi(T) + \sum_{k=1}^{n-1} c_k \int_0^T P_k G^k(L) P(S_k \in (t, t+dt)),
\]

where \( P(S_k \in (t, t+dt)) \) is the probability that the kth shock occurs during \((t, t+dt), 0 \leq t < T \).

\[
P(S_k \leq t) = P(N(t) \geq k) = \sum_{j=k}^{\infty} \frac{e^{-4\lambda t} A(t)^j}{j!},
\]

then

\[
P(S_k \in (t, t+dt)) = \left( \frac{dP(S_k \leq t)}{dt} \right) = \frac{\sum_{j=k}^{\infty} \frac{e^{-4\lambda t} A(t)^j}{j!}}{(k-1)!} r(t) dt.
\]

Thus,

\[
E[V_1] = c_0 + (c_1 - c_0) H(T) + \sum_{k=1}^{n-1} c_k \int_0^T \frac{e^{-4\lambda t} A(t)^{k-1}}{(k-1)!} r(t) G^k(L) P(S_k \leq t) dt.
\]

Therefore, the long-run \( s \)-expected operating cost per unit time under policy \( n, T, L, \{P_k\} \)

\[
B(n, T; L, \{P_k\}) \quad \frac{E[V_1] - c_0 \cdot H(T) - \sum_{k=1}^{n-2} c_k \int_0^T e^{-4\lambda t} A(t)^{k-1} r(t) G^k(L) P(S_k \leq t) dt}{E[U_t]}.
\]

### 2.3. Optimization

For an infinite time horizon, we want to obtain the optimal policy \( n, T^* \) to minimize \( B(n, T; L, \{P_k\}) \) for a given \( L \). We first minimize \( B(n, T; L, \{P_k\}) \) with respect to \( n \) for a given \( T \). We see that the inequalities

\[
B(n+1, T; L, \{P_k\}) \geq B(n, T; L, \{P_k\}) \quad \text{and} \quad B(n, T+1; L, \{P_k\}) \geq B(n, T; L, \{P_k\})
\]

hold if and only if

\[
A(n+1, T; L, \{P_k\}) \geq c_0 \quad \text{and} \quad A(n+1, T; L, \{P_k\}) < c_0,
\]

(13) where
\( A(n,T,L,\{P_k\}) = \begin{cases} 
\int_0^T \mathcal{H}(t) \, dt - \left( (c_1 - c_0) H(T) + \sum_{k=0}^{n-2} c_{w_{n-1}} \int_0^T \frac{e^{-\lambda t} \rho(t)}{k!} \, r(t) \, G^{(k+1)}(L) \, P_{k+1} \, dt \right), & n = 1, 2, 3, \ldots, \\
0, & n = 0.
\end{cases} \)

The following lemma is needed and its detailed proof is presented in Appendix A.

**Lemma 1.** If \( r(t) \) is continuous and increasing in \( t \), and \( r(t) \to \infty \) as \( t \to \infty \), then the following results are true.

(i) \( \int_0^T e^{-\lambda t} \rho(t) r(t) \, dt / \int_0^T e^{-\lambda t} \rho(t) \, dt \) is increasing in \( n \) and converges to \( r(T) \) as \( n \to \infty \) for any \( T > 0 \).

(ii) \( e^{-\lambda t} \rho(t) / \int_0^T e^{-\lambda t} \rho(t) \, dt \) is increasing in \( n \) and diverges to \( \infty \) as \( n \to \infty \) for any \( T > 0 \).

The structural properties of the optimal \((n,T)\) that minimize \( B(n,T,L,\{P_k\}) \) are summarized below.

**Main Theorem.** Suppose that \( r(t) \) is continuously increasing in \( t \) and \( r(t) \to \infty \) as \( t \to \infty \), we have:

(i) If \( c_1 \geq c_0 + c_0 \) for all \( k = 1, 2, \ldots \) then there exists a finite \( T \) which satisfies \( Q(T,\infty; L,\{P_k\}) = c_0 \).

(ii) If \( c_0 < c_1 \leq c_0 + c_0 \) for all \( k = 1, 2, \ldots \) then there exists a finite \( n^* \) and \( T^* \) which satisfy (13) and (16), respectively.

(iii) If \( c_1 < c_0 \), then there exists a finite and unique \( n^* \) which satisfies (13) for all \( T > 0 \).

The detailed proof of Main Theorem is presented in Appendix B.

**Remark 3.**

(1) In the main theorem, Cases (i) and (ii) are more common in practice. Case (iii) is less common. One possible case that the planned replacement cost is more than the unplanned replacement cost is when the major cost of unplanned replacement is covered by the insurance or the system supplier and the cost of planned replacement (at age \( T \)) must be paid by the system manager.

(2) We have derived the optimal \((n,T)\) to minimize \( B(n,T,L,\{P_k\}) \) for a given repair-cost limit \( L \). If we need to search for the optimal cumulative repair-cost limit \( L^* \) that minimizes the total expected cost per unit time in (12) for any \( n = 1, 2, 3, \ldots \) and \( T > 0 \), then the necessary condition \( dB(n,T,L,\{P_k\})/dt = 0 \) holds if and only if

\[
B(n,T,L,\{P_k\}) = \sum_{k=0}^{n-1} \frac{e^{-\lambda k t} \rho^k(t)}{k!} G^{(k+1)}(L) P_{k+1} dt - (c_1 - c_0) \sum_{k=0}^{n-1} \frac{e^{-\lambda k t} \rho^k(t)}{k!} G^{(k+1)}(L) P_{k+1} dt
\]

(19)

**3. Special cases**

The replacement policy \((n,T)\) for a given \( L \) can be considered as a generalization of several past studies. Below are a few examples.

(1) If \( n \to \infty \) (the unplanned replacement policy due to \( n \) type-I failures is removed), the long-term average cost becomes

\[
B(\infty; L,\{P_k\}) = c_0 + c_0 + c_0 \sum_{k=0}^{n-1} \frac{e^{-\lambda k t} \rho^k(t)}{k!} G^{(k+1)}(L) P_{k+1} dt
\]

(20)

(2) If \( T \to \infty \) (the planned age-replacement policy is removed), the long-term average cost becomes

\[
B(n,\infty; L,\{P_k\}) = c_0 + \sum_{k=0}^{n-1} \frac{e^{-\lambda k t} \rho^k(t)}{k!} G^{(k+1)}(L) P_{k+1} dt
\]

(21)
(3) If the repair-cost limit $L \to \infty$ (i.e., all type-I failures can be rectified through minimal repairs), the long-term average cost becomes

$$
B(n, T; \infty; (P_k)) = c_q + (c_1 - c_2) \int_0^\infty \left( \sum_{i=0}^{n-1} e^{-(x-a)/T} P_i \right) \left( \sum_{i=0}^{n-2} c_{i+1} e^{-(x-a)/T} G_i \right) dr dt
$$

(22)

(4) When the $k$th shock occurs, the system experiences a type-I failure with probability $q_k$. If we assume that $P_i = q^i$ where $q$ is the initial probability of type-I failure, then $q_k = P_k/P_{k-1} = q$, which represents the case where the probability of type-I failure remains constant.

To show that our model provides a general framework for analyzing the systems subject to shocks, we list some classical models as special cases:

**Case 1.** $n = \infty$, $P_k = q^k$ for $k = 1, 2, 3, \ldots$: This is the model considered by Lai (2007), and the mean cost rate becomes $B(\infty, T; L; (q^k))$.

**Case 2.** $n = \infty$, $L = \infty$, $P_k = q^k$, and $W_k = C$ for $k = 1, 2, 3, \ldots$: This is the case considered by Cléroux et al. (1979). They considered that $C$ is the random cost of repair with PDF $f(x)$ for $0 < x < \infty$ and a minimal repair at failure is carried out if $C \leq L$ (i.e., $q = \int_0^L f(x)dx$). They defined further that $C$ is a truncated random variable of $C$ with PDF $f(x)/q$ and mean value $E[C] = (1/q) \int_0^L x f(x)dx$ in the range $0 < x \leq L$. Here, if we let the minimal repair cost $W_k = C$, then the mean cost rate $B(\infty, T; \infty; (q^k))$ with $c_n = E[C]$ is considered by Cléroux et al. (1979).

**Case 3.** $n = \infty$, $L = \infty$, $P_k = 1$, and $W_k = c_k$ for $k = 1, 2, 3, \ldots$: This is the case considered by Boland and Proschan (1982). In particular, they considered the cost structure $c_k = a + kc$ where $a$ and $c$ are constant. The mean cost rate becomes $B(\infty, T; \infty; (1))$ with $c_n = c_1 = \sum_{i=1}^{n-1} c_i$.

**Case 4.** $T = \infty$, $L = \infty$, $P_k = q^k$, and $W_k = C$ for $k = 1, 2, 3, \ldots$: This is the case considered by Park (1987). Again, $C$ is defined as in Case 2. Then the mean cost rate becomes $B(n, \infty; (q^k))$ with $c_n = E[C] = (1/q) \int_0^L x f(x)dx$. If $W_k = C$ (constant) for $k = 1, 2, 3, \ldots$, this is the case considered by Nakagawa (1981).

4. Numerical example

In this section, a system with a Weibull failure rate distribution is used as a numerical example. That is

$$
r(t) = \alpha t^{\beta-1}, \quad \alpha > 0, \quad \beta > 1,
$$

and the cumulative hazard function is $A(t) = \int_0^t r(u)du = (\alpha/\beta)t^\beta$.

We consider the shape parameter $\beta = 2$. Thus, we have $r(t) = \alpha t$. Assume that the minimal repair cost $W_k$ due to the $k$th type-I failure has an exponential distribution $G(\lambda)$ with parameter $\lambda$ and finite mean $E[W_k] = 1/\lambda$ for $i = 1, 2, 3, \ldots$.

Based on the discussions of Main Theorem, we propose Algorithm A for computing the optimal replacement policy $(n, T)$ under a pre-determined repair-cost limit $L$ as follows.

**Algorithm A.** Find the optimal replacement schedule $(n, T)$

```
Input: $c_0, c_1, L, r(t), A(t), G(\lambda); (P_k)$.
For case (i) of Main Theorem:
Step 1. Set $n = \infty$.
Step 2. Compute $\tilde{H}(t)$ and $\tilde{z}(t)$ as defined by (2) and (5), respectively.
Step 3. Find the solution $T'$ of (16) (i.e., $Q(T', \infty; L; (P_k)) = c_0$).
Step 4. Compute $B(\infty, T'; L; (P_k))$ as defined by (12).
Output: $T', B(\infty, T'; L; (P_k))$.
Stop.
For case (ii) of Main Theorem:
Step 1. Set $n = 1$ and $B(0, T_0; L; (P_k)) = \infty$.
Step 2. Compute $\tilde{H}(t)$ and $\tilde{z}(t)$ as defined by (2) and (5), respectively.
Step 3. Find the solution $T_{n-1}, T_n, T_{n+1}$ which satisfy (16) (i.e., $Q(T_{n-1}, n-1; L; (P_k)) = c_0$).
$Q(T_{n+1}, n; L; (P_k)) = c_0$. $Q(T_{n+1}, n-1; L; (P_k)) = c_0$.
Step 4. Compute $B(n-1, T_{n-1}; L; (P_k))$, $B(n, T_n; L; (P_k))$, $B(n+1, T_{n+1}; L; (P_k))$ as defined by (12).
Step 5. If $B(n+1, T_{n+1}; L; (P_k)) > B(n, T_n; L; (P_k))$ and $B(n, T_n; L; (P_k)) < B(n-1, T_{n-1}; L; (P_k))$,
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(continued on next page)
then \( n' = n \), \( T' = T_n \), \( B(n', T', L; (P_1)) = B(n, T_n, L; (P_1)) \)
and go to Output; Otherwise, make \( n = n + 1 \) and go to Step 2.

Output: \( n', T', B(n', T', L; (P_1)) \).

Stop.

For case (iii) of Main Theorem:

Step 1. Input \( T \).

Step 2. Set \( n = 1 \) and \( B(0, T, L; (P_1)) = \infty \).

Step 3. Compute \( \pi(T) \) and \( \tau(T) \) as defined by (2) and (5), respectively.

Step 4. Compute \( B(n - 1, T, L; (P_1)) \), \( B(n, T, L; (P_1)) \), \( B(n + 1, T; (P_1)) \) as defined by (12).

Step 5. If \( B(n + 1, T, L; (P_1)) \geq B(n, T, L; (P_1)) \)
and \( B(n, T; (P_1)) < B(n - 1, T, L; (P_1)) \),
then \( n' = n \), \( B(n', T, L; (P_1)) = B(n, T, L; (P_1)) \)
and go to Output;

Otherwise, make \( n = n + 1 \) and go to Step 3.

Output: \( n', T', B(n', T', L; (P_1)) \).

Stop.

In this paper, the optimal replacement policy \((n, T)^*\) based on the cumulative repair-cost limit \( L \) is studied analytically. \( L \) be regarded as a tolerable threshold of repair action. A proper \( L \) can be set by considering several factors such as replacement, repair costs and maintenance experiences or budget. In this numerical example, \( L \) is set approximately as the multiples of replacement cost for the ease of computation. We can also develop a procedure for obtaining the optimal \( L' \) along with the optimal schedule \((n, T')\) to minimize the total \( z \)-expected cost per unit time as follows.

**Algorithm B.** Find the optimal cumulative repair-cost limit \( L' \) along with the optimal replacement schedule \((n, T')^*\).

Input: \( c_0, c_1, r(\cdot), A(\cdot), G(\cdot); (P_1), T_0 \)

Step 1. Set \( j = 1, n = 1 \) and \( T_j = T_0 \).

Step 2. Set \( n = n_1, T = T_j \), and find the solution \( L_j \) which satisfies (19).

Step 3. If \( j = 1 \), then go to Step 5.

Step 4. If \( L_j = L_j - 1 \), then \( n' = n_j - 1 \).

Step 5. Set \( L = L_j \) perform Algorithm A, and find \( n_j, T_j \).

Step 6. Make \( n_{j+1} = n_j \), \( T_{j+1} = T_j \), \( j = j + 1 \), and go to Step 2.

Output: \( n', T', B(n', T', L'; (P_1)) \).

Stop.

In addition, a flow sheet representing the use of these two algorithms is also given in Fig. 1.

For the ease of computation, a sequential \( \{P_1\} \) is defined as \( P_k = 2^q \). We consider the following three \( \{P_1\} \) cases in the numerical analysis:

**Case 1.** \( P_k = q_k; k = 0, 1, 2, \ldots, 0 \leq q \leq 1 \) (i.e., \( q_k = q \)).

**Case 2.** \( P_k = q_k^{1/\delta}; k = 0, 1, 2, \ldots, 0 \leq q \leq 1 \) (i.e., \( q_k = q^{1/\delta} \)).

**Case 3.** \( P_k = q_k^{1/\delta}; k = 0, 1, 2, \ldots, 0 \leq q \leq 1 \) (i.e., \( q_k = q^{1/\delta} \)).

By changing \( q \) we can determine the impact of type-I failure probability on the optimal policy and its average cost. Note that if \( P_k = 2^q \) and \( \delta = 1 \) (i.e., \( \delta = 1 \)), the probability of the 4th minimal repair

**Table 1**

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<tr>
<th>( q )</th>
<th>Case 1. ( P_k = q )</th>
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<th>( T )</th>
<th>( B(n, T; L; (P_1)) )</th>
<th>Case 2. ( P_k = q^{1/\delta} )</th>
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<th>( T )</th>
<th>( B(n, T; L; (P_1)) )</th>
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Table 3
Optimal \( w' \) and \( B(w', T, L, \{ P_k \}) \) based on cumulative repair-cost limit \( L \) for a system subject to shocks, under case (ii) of Main Theorem. \( \alpha = 1300, \beta = 1000, L = 1000, W_i \sim \exp(\lambda = 1/1000), r(t) = \alpha(x - 2), P_k \neq q' \), where the values of \( c_0 \) and \( c_1 \) are referred to the case considered by Sheu (1993). The optimal solutions for several cases are presented in Tables 1–6.

From the numerical results, some important observations are obtained.

- From Tables 1–5, the minimal \( s \)-expected cost per unit time will be reduced when the probability of minimal repair (type-I failure) is increased. This is intuitive as when type-I failures are more likely, the shocks are more likely repairable. Thus, the more expensive replacement happens less frequently so that expected regeneration cycle becomes longer and the average operating cost is reduced. If we consider the time from the replacement to the type-II failure as the lifetime of the machine, then longer the lifetime of the machine is, the lower the operating cost is under the multi-criteria replacement policy.
- From Tables 1 and 2, by comparing Cases 1–3, we find that the minimal \( s \)-expected cost per unit time can be either reduced or increased when the probability of minimal repair depends on the number of shocks. In fact, when \( \delta = 0.5 \) (Case 2 in Tables 1 and 2), the probability of minimal repair is increasing in \( k \), the average cost is lower than that of \( \delta = 1 \) of Case 1 where the probability of minimal repair is independent of \( k \). Note that Case 2 represents a situation where as the number of shocks increases, the next shock is more likely to be a type-II failure again. On the other hand, when \( \delta = 1.5 \) (Case 3 in Tables 1 and 2), the probability of minimal repair is decreasing in \( k \), the average cost is now higher than that of \( \delta = 1 \) of Case 1. Case 3 represents a situation, where as the number of shocks increases, the next shock is more likely to be a type-II failure. These cases may represent different stages of the lifetime of a machine (or a system). During the early stage of the lifetime, the probability of type-II failure (call for replacement) may be decreasing with the number of shocks. After this stage, the machine’s type-II failure probability becomes independent of the number of repairs. This may be called the middle stage of the life cycle. Case 3 can represent the late stage of the life cycle as the older machine’s type-II failure probability becomes increasing in shocks. Therefore, the order of “cost of Case 2 < cost of Case 1 < cost of Case 3” reflects the fact that newer machine has lower average cost under the optimal replacement policy. Fig. 2 is given for illustration of the \( \{ P_k \} \) cases from Table 1.
- From Table 4, the optimal cumulative repair-cost limit \( L' \) can be obtained along with the optimal replacement schedule \((n,T')\) when the cumulative repair-cost limit is not pre-determined.

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Table 4
Optimal \((n,T')\) and \(B(n, T, L, \{ P_k \})\) for a system subject to shocks, under the cumulative repair-cost limit \( L \) being adopted or not. \( c_0 = 1000, c_1 = 1400, L = \exp(\lambda = 1/400), r(t) = \alpha(x - 2), P_k \neq q' \), where the values of \( c_0 \) and \( c_1 \) are referred to the case considered by Sheu (1993). The optimal solutions for several cases are presented in Tables 1–6.

From the numerical results, some important observations are obtained.

- From Tables 1–5, the minimal \( s \)-expected cost per unit time will be reduced when the probability of minimal repair (type-I failure) is increased. This is intuitive as when type-I failures are more likely, the shocks are more likely repairable. Thus, the more expensive replacement happens less frequently so that expected regeneration cycle becomes longer and the average operating cost is reduced. If we consider the time from the replacement to the type-II failure as the lifetime of the machine, then longer the lifetime of the machine is, the lower the operating cost is under the multi-criteria replacement policy.
- From Tables 1 and 2, by comparing Cases 1–3, we find that the minimal \( s \)-expected cost per unit time can be either reduced or increased when the probability of minimal repair depends on the number of shocks. In fact, when \( \delta = 0.5 \) (Case 2 in Tables 1 and 2), the probability of minimal repair is increasing in \( k \), the average cost is lower than that of \( \delta = 1 \) of Case 1 where the probability of minimal repair is independent of \( k \). Note that Case 2 represents a situation where as the number of shocks increases, the next shock is more likely to be a type-II failure again. On the other hand, when \( \delta = 1.5 \) (Case 3 in Tables 1 and 2), the probability of minimal repair is decreasing in \( k \), the average cost is now higher than that of \( \delta = 1 \) of Case 1. Case 3 represents a situation, where as the number of shocks increases, the next shock is more likely to be a type-II failure. These cases may represent different stages of the lifetime of a machine (or a system). During the early stage of the lifetime, the probability of type-II failure (call for replacement) may be decreasing with the number of shocks. After this stage, the machine’s type-II failure probability becomes independent of the number of repairs. This may be called the middle stage of the life cycle. Case 3 can represent the late stage of the life cycle as the older machine’s type-II failure probability becomes increasing in shocks. Therefore, the order of “cost of Case 2 < cost of Case 1 < cost of Case 3” reflects the fact that newer machine has lower average cost under the optimal replacement policy. Fig. 2 is given for illustration of the \( \{ P_k \} \) cases from Table 1.
- From Table 4, the optimal cumulative repair-cost limit \( L' \) can be obtained along with the optimal replacement schedule \((n,T')\) when the cumulative repair-cost limit is not pre-determined.

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From Table 5, the minimal s-expected cost per unit time is reduced when the cumulative repair-cost limit policy is adopted. Clearly, using the historical cost information to determine the replacement policy has improved the cost efficiency.

Table 6 shows that the cost advantage of optimal policy \((n, T)\) over either the optimal \(n^{*}\) only or the optimal \(T^{*}\) only policy. This result implies that combining both planned and unplanned replacement actions will improve the performance of the system.

5. Conclusions

In this paper, a multi-criteria replacement policy was analyzed for a system subject to shocks. The long-term s-expected cost per unit time for the system was developed, incorporating costs related to minimal repair and different forms of replacements. We have shown the existence and the structural properties of the optimal replacement policy based on number of minimal repair \(n\) and optimal replacement age \(T\) for a given cumulative repair cost limit \(L\). Furthermore, the optimal \(L^{*}\) can be obtained along with the optimal replacement schedule \((n, T)\). The model provided a general framework for analyzing the maintenance policies for the system subject to shocks. Many previous models in the literature become the special cases of our model. We also presented some numerical examples to demonstrate the properties of the optimal policies when multiple control variables were considered.

However, a limitation of our model is that the repairs are specified to be minimal (i.e., the repaired system is as bad as old). In some real situations, it would seem more practical to consider the imperfect repairs, namely, the repaired system state is between “as bad as old” (minimal repair) and “as good as new” (replacement). Incorporating this realistic factor into the proposed policy can be a direction for the future research.

Acknowledgement

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which greatly enhanced the clarity of the article. All of the suggestions were incorporated directly in the text. This research was supported by the National Science Council of Taiwan, ROC, under Grant No. NSC 98-2221-E-126-012-MY3. The last author wishes to thank the financial support from NSERC Grant RGPIN197319 of Canada.

Appendix A. Proof of Lemma 1

(i) First, we need to show that

\[
\int_0^T e^{-A(T)} A(t)^n r(t) dt / \int_0^T e^{-A(T)} A(t)^n dt \tag{A.1}
\]

is increasing in \(n\) when the conditions of Lemma 1 are satisfied. Let

\[
J(T) = \left\{ \int_0^T e^{-A(t)} A(t)^n r(t) dt \times \int_0^T e^{-A(t)} A(t)^n dt \right\} - \left\{ \int_0^T e^{-A(t)} A(t)^n r(t) dt \times \int_0^T e^{-A(t)} A(t)^n dt \right\}.
\]

Differentiating \(J(T)\) with respect to \(T\), we get

\[
f(T) = e^{-A(T)} A(T)^n \times \int_0^T [r(T) - r(t)][A(T) - A(t)] e^{-A(t)} A(t)^n dt > 0,
\]

since \(r(t)\) is increasing in \(t\). We have \(J(0) = 0\) and \(f(T) > 0\). Thus, \(J(T) > 0\) for all \(T > 0\), and hence (A.1) is increasing in \(n\).

Next, we show that

\[
\lim_{n \to \infty} \frac{\int_0^T e^{-A(T)} A(t)^{n-1} r(t) dt}{\int_0^T e^{-A(T)} A(t)^n dt} = r(T). \tag{A.2}
\]

Evidently,

\[
\frac{\int_0^T e^{-A(T)} A(t)^n r(t) dt}{\int_0^T e^{-A(T)} A(t)^n dt} \leq r(T), \tag{A.3}
\]

for \(n = 1, 2, 3, \ldots\). On the other hand, for any \(T_1 \in (0, T)\), we have

\[
\frac{\int_0^T e^{-A(T)} A(t)^n r(t) dt}{\int_0^T e^{-A(T)} A(t)^n dt} = \frac{\int_0^{T_1} e^{-A(T)} A(t)^n r(t) dt + \int_{T_1}^T e^{-A(T)} A(t)^n r(t) dt}{\int_0^{T_1} e^{-A(T)} A(t)^n dt + \int_{T_1}^T e^{-A(T)} A(t)^n dt} \leq \frac{\int_0^{T_1} e^{-A(T)} A(t)^n dt}{\int_0^{T_1} e^{-A(T)} A(t)^n dt} + 1.
\]

Further, for \(T_2 \in (T_1, T)\), the bracket of the denominator is

\[
\frac{\int_0^{T_2} e^{-A(T)} A(t)^n dt}{\int_0^{T_2} e^{-A(T)} A(t)^n dt} \leq \frac{\int_0^{T_1} e^{-A(T)} A(t)^n dt}{\int_0^{T_2} e^{-A(T)} A(t)^n dt},
\]

\[
\leq \frac{\int_0^{T_1} e^{-A(T)} dt}{\int_0^{T_2} e^{-A(T)} dt} \to 0 \text{ as } n \to \infty. \tag{A.4}
\]

Thus, from (A.3)–(A.5), we have

\[
r(T) = \lim_{n \to \infty} r(T) \geq \lim_{n \to \infty} \frac{\int_0^T e^{-A(T)} A(t)^n r(t) dt}{\int_0^T e^{-A(T)} A(t)^n dt} \geq r(T),
\]

which implies (A.2) because \(T_1\) is arbitrary in \((0, T)\).

(ii) Since \(A(T) > A(t)\) for \(T > t\), we have the inequality

\[
\frac{e^{-A(T)} A(T)^n}{\int_0^T e^{-A(T)} A(t)^n dt} > \frac{e^{-A(T)} A(t)^n}{\int_0^T e^{-A(T)} A(t)^n dt} \times \int_0^T [A(T) - A(t)] e^{-A(t)} A(t)^n dt > 0
\]

Fig. 2. Comparison of the reported cases.
and
\[ \int_0^T e^{-\lambda(T)} A(T)^k \, dt = \frac{e^{-\lambda(T)}}{\lambda(T)^k} \to \infty \quad \text{as } n \to \infty. \]

### Appendix B. Proof of Theorem 1

First, it is evident that \( Q(0; n) \leq L(P_n) = 0 \) and
\[
Q(\infty; n) = \lim_{T \to -\infty} Q(T; n) = \lim_{T \to -\infty} \int_{-\infty}^T \Pi_r(t) \, dt = \left[ (c_1 - c_0) + \sum_{i=0}^k c_{w_{i+1}} \right] \int_0^T e^{-\lambda(T)} A(T)^k \, dt + c_{w_0} r(T)
\]
\[ = \left[ (c_1 - c_0) + \sum_{i=0}^k c_{w_{i+1}} \right] \int_0^T e^{-\lambda(T)} A(T)^k \, dt + c_{w_0} r(T)
\]
\[ = \left[ (c_1 - c_0) + \sum_{i=0}^k c_{w_{i+1}} \right] \lim_{T \to -\infty} \int_0^T e^{-\lambda(T)} A(T)^k \, dt + c_{w_0} r(T)
\]
\[ = \left[ (c_1 - c_0) + \sum_{i=0}^k c_{w_{i+1}} \right] \frac{e^{-\lambda(T)}}{\lambda(T)^k} \to \infty, \quad \text{as } n \to \infty.
\]

since \( \lim_{T \to -\infty} \Pi_r(T) = 0 \) and \( \lim_{T \to -\infty} R(T) \to \infty. \)

Next, from Lemma 1 we have
\[
A(\infty; T; L(P_n)) = \lim_{n \to \infty} A(n; T; L(P_n))
\]
\[
= \lim_{n \to \infty} \int_0^T \Pi_r(t) \left\{ (c_{w_0} + c_0 - c_1) e^{-\lambda(T)} A(T)^k \right\} \, dt + c_{w_0} r(T)
\]
\[ = \left[ (c_1 - c_0) + \sum_{i=0}^k c_{w_{i+1}} \right] \int_0^T e^{-\lambda(T)} A(T)^k \, dt + c_{w_0} r(T)
\]
\[ \to \infty, \quad \text{as } n \to \infty.
\]

(iii) Suppose that \( c_1 \leq c_0 \). Then from Lemma 1, (15) and (A.7), we have that \( A(n; T; L(P_n)) \to \infty \) as \( n \to \infty \). Thus, there exists a finite and unique \( n' \) which satisfies (13) for all \( \lambda > 0 \). Further, from (A.6), there exists a finite \( \bar{T} \) which satisfies (16) for all \( n \). Hence, there exists finite \( \bar{n} \) and \( \bar{T} \) which satisfy (13) and (16), respectively.

### References


