THE NONLINEAR CRITICAL LAYER FOR KELVIN MODES ON A VORTEX WITH A CONTINUOUS VELOCITY PROFILE

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Abstract.
We consider in this paper the propagation of neutral modes along a vortex with velocity profile \( \bar{V}(r) \), \( r \) being the radial coordinate. In the linear inviscid stability theory for swirling flows, modes that are singular at some value of \( r \) denoted \( r_c \), the critical point, are particularly significant. The singularity can be dealt with by adding viscous and/or nonlinear effects within a thin critical layer centered on the critical point. At high Reynolds numbers, the case of most interest in applications such as aeronautics and geophysical fluid dynamics, nonlinearity is the appropriate choice, although viscosity may still play a subtle role. We determine here the scaling and equations that govern the nonlinear critical layer. The method of characteristics is then employed to obtain an exact solution of the governing inviscid system comprised of four coupled PDEs, two of which are nonlinear and two are linear. Finally, solutions are obtained for the outer eigenvalue problem having no phase change across the critical layer thus permitting the existence of modes not possible in a linear theory. This result may have important implications for the short wave cooperative instability mechanism that has received so much attention in the context of aircraft trailing vortices.

Key words.

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1. Introduction. The propagation of helical perturbations to a columnar vortex seems to have been studied first by Lord Kelvin whose results were published in 1880. In cylindrical coordinates \((r, \theta, z)\), the problem involves the investigation of infinitesimal perturbations \((u_r, u_\theta, u_z)\) superimposed on a flow with velocity profile \(\{0, \bar{V}(r), 0\}\). Kelvin considered the case of a fluid in rigid rotation, \(i.e., \bar{V} = \Omega_0 r\) contained within a cylinder of radius \(a\). A single equation can be obtained for the pressure perturbation and it is Bessel’s equation of order \(m\), where \(m\) is the azimuthal wavenumber. Imposition of the boundary conditions at \(r = 0\) and \(r = a\) leads to an eigenvalue problem for the frequency \(\omega = \omega(k, m)\), where \(k\) is the axial wavenumber.

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The review article by Ash and Khorrami [1] is a convenient reference for the details.

In this paper, we are primarily interested in waves propagating on an unbounded vortex and a model that has often been employed to study this phenomenon is the discontinuous Rankine vortex with velocity profile

$$V(r) = \begin{cases} 
\Omega_0 r, & 0 \leq r \leq a \\
\frac{\Omega_0 a^2}{r}, & r > a.
\end{cases}$$

The solutions on either side of the discontinuity of vorticity at $r = a$ are matched using kinematic and pressure conditions and this leads to an eigenvalue problem for the dispersion relation. Bessel functions are again involved and the modal solutions obtained are termed Kelvin modes. The monograph by Saffman [2] details the analysis and presents dispersion curves for different azimuthal wavenumbers.

Much of the recent research on the stability of vortices is motivated by the aircraft trailing vortex problem. A pair of counter-rotating vortex filaments serves as a model that has been widely studied as representing the vortices shed from the wingtips of aircraft which if they are jumbo jets, pose a major hazard to following aircraft. Given that the strength of the trailing vortices is related to the weight of the aircraft, it is clear that any smaller aircraft attempting to land behind the new Airbus A380 is at serious risk.

A number of theoretical investigations have treated the problem of a vortex subject to an external strain, this being a way of modelling the strain induced on one member of a pair of trailing vortices by the other. A long wave instability first explained by Crow [3] initially received the most attention, but more recently the short wave cooperative instability mechanism (often termed the elliptic instability) has attracted a great deal of interest. The latter mechanism involves interacting Kelvin waves. Specifically, Moore and Saffman [4] showed that for an arbitrary strained vortex two neutral modes are coupled by the strain field if a certain resonance condition is satisfied. They derived an approximate expression for the growth rate of the resonant modes by an asymptotic analysis valid when the strain field is small. The first quantitative investigation of this instability was by Tsai & Widnall [5] who employed the discontinuous Rankine vortex in their calculations. They found that the most
unstable perturbations corresponded to a pair of Kelvin modes having zero frequency and azimuthal wavenumbers $m = \pm 1$.

Real vortices, however, have continuous profiles and the use of Rankine vortices in theoretical studies was criticized in the review article by Spalart [6] (see Section 2.2). Clearly, it is important to ask what effect the use of a profile whose vorticity is continuous might have on this instability mechanism. Sipp & Jacquin [7] have, in fact, recently done so and they concluded that the “Widnall instabilities” will not occur because of the presence of a critical layer in the continuous case. Their argument, which is correct as far as it goes, is based on linear viscous stability calculations for the Lamb-Oseen vortex $\mathcal{V}(r) = (1 - e^{-r^2})/r$ which show that the neutral Kelvin modes required for the resonant interaction discussed in [4] and [5] would be damped in the continuous case.

In this paper, we reexamine the question and investigate the effect that the addition of nonlinearity in the critical layer rather than viscosity would have. This is of interest in its own right as part of the theory of Kelvin modes and its pertinence to the cooperative elliptic instability mechanism provides further motivation. The possibility that nonlinear critical layer modes could be neutral rather than damped was, in fact, suggested in [7, p. 265]. In Section 3, we will determine a parameter that measures the relative importance of nonlinearity to viscosity. The larger the Reynolds number, the smaller the perturbation amplitude needs to be for nonlinearity to be the appropriate choice. A number sometimes cited as representative for trailing vortices behind jumbo jets is $Re = 10^7$. Clearly, this is large enough to motivate the formulation of a nonlinear approach. Even if the turbulence usually present is accounted for by introducing an effective Reynolds number, this value is still very large. Gerz and Ehret [8], in an investigation of the influence of wingtip vortices on atmospheric pollution caused by the jet exhaust, estimate the effective $Re$ as $6 \cdot 10^5$ for the wake behind a Boeing 747.

Although the foregoing discussion focused on the trailing vortex problem, the results are pertinent to other applications in engineering and in geophysical fluid dynamics. In turbomachinery, for example, the flow through a duct is sometimes
modelled by representing the flow by a superposition of a solid body rotation and a potential axial vortex [9]. And in geophysical fluid dynamics, an important application is to hurricanes. In their numerical simulations of a hurricane, Chen, Brunet and Yau [10] find that absorption of vortex Rossby waves at the critical level leads to an acceleration of the mean wind in the lower troposphere (the location of the critical layer is indicated in Figure 11 of their paper).

Before presenting the analysis, it is worth noting that the critical point singularity in a stratified shear flow has a very similar behavior to that occurring in a vortex. It is therefore possible to anticipate certain results based on those that have been demonstrated for stratified shear flows. For example, Miles [11] proved that when the local Richardson number is everywhere greater than 1/4, all singular modes must decay according to linear theory. However, when the critical layer is nonlinear and inviscid, singular neutral modes have been shown to exist (see Section 3.1.1 of the review article by the first author, Maslowe [12]). For Kelvin modes on vortices, we will show that the same is true, i.e., inviscid nonlinear modes exist in regions of parameter space where they would be damped if viscosity were used to deal with the critical layer. As a result, the possibility is revived not only of cooperative instabilities, but of other instability mechanisms that have been observed experimentally in which Kelvin modes interact to destroy vortices. In the experiments of Maxworthy, Hopfinger and Redekopp [13], for example, unstable interactions between axisymmetric and helical waves were observed, the outcome of which depended on the amplitude of the axisymmetric mode.

The reason that neutral modes with nonlinear critical layers can exist when they would be damped in a linear, viscous theory is a direct result of the absence of any phase change across the singular critical point. This means essentially that terms with branch points are written simply as absolute values, whereas in the viscous theory, a term with a branch point at \( r = r_c \) is written \( |r - r_c| e^{i \phi} \) for \( r < r_c \) and the phase change \( \phi \) is nonzero. In either case, the result must be derived by determining what outer solution can be matched to the critical layer solution.

Before proceeding to the analysis, we mention briefly that the simpler two-dimensional
case of waves propagating only in the azimuthal direction, \(i.e., k = 0\) has been investigated recently by Le Dizès [14] and by Balmforth et al. [15]. Each paper points out the relevance of the results to plasma physics, as well as to vortices. The critical layer analysis in [14] is closer to our own, one reason being that the waves are forced in [15] and may be transient, but some brief comparisons will be made after presenting our own analysis for helical modes.

2. Formulation and Outer Expansion. We consider small-amplitude helical perturbations to a vortex with azimuthal velocity profile \(\bar{V}(r)\) and a corresponding radial pressure distribution \(\bar{p}(r)\). Away from the critical layer, the perturbations are sinusoidal with phase \(\xi = k z + m \theta - \omega t\) and, because we are dealing with neutral modes, it will be convenient to use \(\xi\) as an independent variable. The momentum and continuity equations can then be written

\[
\begin{align*}
\left(\frac{m}{r} u_\theta - \omega\right) \frac{\partial u_r}{\partial \xi} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + k u_z \frac{\partial u_r}{\partial \xi} &= - \frac{\partial p^*}{\partial r} + \frac{1}{Re} \frac{\partial^2 u_r}{\partial r^2} \\
\left(\frac{m}{r} u_\theta - \omega\right) \frac{\partial u_\theta}{\partial \xi} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} + k u_z \frac{\partial u_\theta}{\partial \xi} &= - \frac{m}{r} \frac{\partial p^*}{\partial \xi} + \frac{1}{Re} \frac{\partial^2 u_\theta}{\partial r^2}
\end{align*}
\]

\[
\begin{align*}
\left(\frac{m}{r} u_\theta - \omega\right) \frac{\partial u_z}{\partial \xi} + u_r \frac{\partial u_z}{\partial r} + k u_z \frac{\partial u_z}{\partial \xi} &= - k \frac{\partial p^*}{\partial \xi} + \frac{1}{Re} \frac{\partial^2 u_z}{\partial r^2} \\
\frac{\partial (ru_r)}{\partial r} + m \frac{\partial u_\theta}{\partial \xi} + k r \frac{\partial u_z}{\partial \xi} &= 0
\end{align*}
\]

Our analysis being primarily inviscid, we have retained in the momentum equations above only those viscous terms involving second derivatives with respect to \(r\), because these terms will be the largest in the critical layer.

The scaling employed in the foregoing equations deserves some discussion because we wish to identify parameter régimes where analytical progress is possible. To begin, we denote by \(\Omega_0\), the angular velocity of the vortex at its center, and use this to scale the frequencies and time. A characteristic length scale for the vortex denoted \(a\) is used to nondimensionalize \(r\) and the wavenumber \(k\), while \(\Omega_0 a\) is employed to scale the velocities. Finally, the dimensionless pressure \(p^*\) is obtained by dividing the actual pressure by \(\rho \Omega_0^2 a^2\), where \(\rho\) is the constant density. After introducing this scaling into the momentum equations, the Reynolds number \(Re = a^2 \Omega_0 / \nu\), where \(\nu\) is the kinematic viscosity.
We next consider the linear theory because it describes the perturbation to leading order in the outer region. A separation of variables can be achieved in the linearized equations by writing

\[ u_r = \varepsilon u(r) \sin \xi \] (2.2a)
\[ u_\theta = \bar{V}(r) + \varepsilon v(r) \cos \xi \] (2.2b)
\[ u_z = \varepsilon w(r) \cos \xi \] (2.2c)
and
\[ p' = \bar{p}(r) + \varepsilon p(r) \cos \xi, \] (2.2d)

where \( \varepsilon \ll 1 \) is a dimensionless amplitude parameter. After linearizing and then substituting (2.2a-d) into Eqs. (2.1a-d), we obtain the system

\[
\begin{align*}
\gamma(r) u &= 2 \bar{V} \frac{v}{r} - p' \\
\gamma(r) v &= \frac{1}{r} (r\bar{V})' u - m \frac{r}{r} p \\
\gamma(r) w &= -kp \\
(r u)' &= m v + kr w,
\end{align*}
\] (2.3a-d)

where

\[ \gamma(r) = m \frac{\bar{V}}{r} - \omega = m \bar{\Omega} - \omega. \] (2.4)

Critical point singularities occur at any value of \( r \) for which \( \gamma(r) = 0 \).

Equations (2.3a-d) can be combined into a single second order differential equation for \( u(r) \), namely,

\[
\gamma^2 D\{SD_* u\} - \{\gamma^2 + m\gamma \frac{r^2}{r^4} \left( D[SD(r\bar{V})] - 3 \frac{S}{r} D(r\bar{V}) \right) - 2 \bar{V} k^2 \frac{S}{r} Q(r) \} u = 0,
\] (2.5)

where

\[ D = \frac{d}{dr}, \quad D_* = \frac{d}{dr} + \frac{1}{r}, \quad S = \frac{r^2}{m^2 + k^2 r^2} \quad \text{and} \quad Q(r) = \frac{D(r\bar{V})}{r}. \]

\( Q(r) \) can be recognized as the vorticity of the mean flow nondimensionalized with respect to \( \Omega_0 \), the angular velocity at the center of the vortex.
Equation (2.5) can be obtained from the equation derived by Howard and Gupta [16] for swirling flows by setting the axial velocity $W = 0$ in equation (18) of [16]. Noting the similarity of their equation (18) to the Taylor-Goldstein equation governing stratified shear flows, Howard and Gupta derived a Richardson number $1/4$ stability theorem for swirling flows employing an integral approach, as in Howard’s earlier paper on stratified flows [13]. This theorem, however, was limited to axisymmetric, i.e., $m = 0$ perturbations thus underlining the importance of perturbations with $m \neq 0$, where only a bound on the growth rate could be obtained.

The mathematical similarities with the case of a stratified shear flow are nonetheless useful and it will be seen that in our analysis the paper by Miles [18] is most pertinent. Miles used Frobenius expansions near the critical point to derive a number of important results, including the Richardson number $1/4$ theorem. Following his approach and notation, we expand all terms in (2.5) around the critical point $r_c$ to obtain a solution valid locally having the form

$$u(r) = A u_+(r) + B u_-(r),$$

(2.6)

where

$$u_{\pm}(r) = (r - r_c)^{\frac{1}{2}(1 \pm \nu)} w_{\pm}(r)$$

(2.7)

and the functions $w_{\pm}(r)$ are regular in the neighborhood of $r_c$. We have defined a local Richardson number analogous to the one arising in stratified shear flows by

$$J_c = \frac{2 k^2 \tilde{V}_c Q_c}{r_c (\gamma_c)^2}$$

(2.8)

and the parameter $\nu$ in (2.7) is related to $J_c$ by $\nu = (1 - 4 J_c)^{1/2}$.

Miles used arguments based on the variation of the Reynolds stress to prove a number of useful results that apply to singular neutral modes. For example, within the framework of linear theory, a neutral mode comprising part of a stability boundary must be proportional to one or the other of the Frobenius solutions. Even though the expression for the Reynolds stress is quite different in cylindrical coordinates, we show in Appendix A that the same conclusion applies here, i.e., on a stability boundary
either $A$ or $B$ must be zero in (2.6). This applies for $J_c < 1/4$, the case we treat in this paper.

There are two exceptional cases, however, that should be mentioned. First, when $J_c$ is small the second term in the series for $w_-(r)$ in (2.7) is very large, becoming infinite as $J_c \to 0$. That case is of interest in the vortex problem because it arises, for example, when $r_c$ is far from the center and $Q_c$ the vorticity is then small. The Frobenius solution $u_-$ in (6) can then be replaced by a linear combination of $u_+$ and $u_-$ that is well behaved as $J_c \to 0$; the associated nonlinear critical layer theory has been developed by Caillol and Maslowe [19]. The second exceptional case occurs when $J_c$ is greater than $1/4$ so that the Frobenius exponents are complex. We do not treat that case here, but it does arise for example, when there is an external forcing. Let us proceed now to the derivation of the critical layer equations for the case of $J_c \sim O(1)$ with $\nu$ real.

3. Critical layer scaling and governing equations. There are several ways to determine the scaling for the nonlinear critical layer. The most direct way is to try to get a balance between linear inertial terms and the nonlinear terms with the largest derivatives in $r$. Because we are dealing with a system, however, different equations yield different results. For the present Kelvin wave problem, either the $v$ or the $w$ momentum equation in (2.1) leads to the correct scaling, but that was not obvious a priori. That being the case, it is helpful to have an alternative. The most reliable is to examine several terms in the outer expansion which proceeds in powers of $\varepsilon$, the amplitude parameter. The power of $\varepsilon$ where the expansion first breaks down is the critical layer thickness. In a frame of reference moving with the wave, this may be when the linear perturbation becomes the same order of magnitude as the mean flow. It will be seen below [see comment after 3.4] that this in fact is the easiest way to determine the critical layer scaling.

Before examining the form of the outer expansion, it is first necessary to know the behavior of all variables near $r_c$. Let us concentrate on the case $J_c < 1/4$ corresponding to Eqs. (2.6) - (2.8). The most singular Frobenius solution is the one with the minus sign in front of $\nu$; we denote this exponent $\delta$. Because the pressure
perturbation satisfies an equation nearly identical to the one satisfied by $u$ (see p. 244 of [2]), it will have the same behavior near $r_c$. Although not a trivial exercise, consideration of the system (2.3a-d) leads to the conclusion that

$$v \sim (r - r_c)^{\delta - 1}, \quad w \sim (r - r_c)^{\delta - 1} \quad \text{and} \quad p \sim (r - r_c)^{\delta},$$

(3.1)

where to be consistent with the first of Eqs. (2.3)

$$\left( \frac{dp}{dr} - \frac{2\bar{V}}{r} v \right) \sim (r - r_c)^{1+\delta}.$$  

(3.2)

From the behavior deduced immediately above, we find that appropriate independent variables in the nonlinear critical layer are

$$\xi = k z + m \theta - \omega t \quad \text{and} \quad R = \frac{r - r_c}{\varepsilon^\beta}, \quad \text{where} \quad \beta = \frac{1}{2 - \delta}. \quad (3.3)$$

When $k = 0$, $\delta = 0$ and $\beta = 1/2$, as in [14] and [15]. The critical layer thickness, as noted above, could have been determined most easily by comparing the magnitude of the azimuthal velocity perturbation to that of the mean flow in a frame of reference moving with the wave. Denoting the azimuthal velocity in the critical layer $V(R, \xi)$, we employ the scaling

$$u_\theta - \bar{V}_c \sim \bar{V}_c '(r - r_c) + \varepsilon v(r) \cos \xi = \varepsilon^\beta [V(R, \xi) + \bar{\Omega}_c R],$$

(3.4)

where the $R$-term is included because it simplifies the governing equations. From the behavior of $v(r)$ as given by (3.1), Eq. (3.4) shows that when $(r - r_c) \sim O(\varepsilon^\beta)$, then $\varepsilon v(r)$ is the same order of magnitude as the mean flow. The remaining dependent variables in the case $J_c < 1/4$, but not small, are scaled as

$$u_r = \varepsilon^{2\beta} U(R, \xi), \quad u_z = \varepsilon^\beta W(R, \xi) \quad \text{and} \quad p^* - \frac{1}{2} \Omega_c r^2 = \varepsilon^{2\beta} P(R, \xi).$$

(3.5)

Substituting (3.3) - (3.5) now into the governing equations (2.1), the nonlinear critical layer equations to lowest order are the following:

$$r_c \frac{\partial P}{\partial R} - 2 \bar{V}_c V = 0 \quad (3.6a)$$

$$2 \frac{\bar{V}_c}{r_c} U + \frac{m}{r_c} \frac{\partial P}{\partial \xi} = -\left[ U \frac{\partial V}{\partial R} + \left( \frac{m}{r_c} V + kW \right) \frac{\partial V}{\partial \xi} \right] + \lambda \frac{\partial^2 V}{\partial R^2} \quad (3.6b)$$

$$k \frac{\partial P}{\partial \xi} = -\left[ U \frac{\partial W}{\partial R} + \left( \frac{m}{r_c} V + kW \right) \frac{\partial W}{\partial \xi} \right] + \lambda \frac{\partial^2 W}{\partial R^2} \quad (3.6c)$$

and

$$r_c \frac{\partial U}{\partial R} + m \frac{\partial V}{\partial \xi} + kr_c \frac{\partial W}{\partial \xi} = 0. \quad (3.6d)$$
The parameter \( \lambda = 1/Re \varepsilon^{3/2} \); if \( \lambda \ll 1 \), then the nonlinear critical layer thickness \( \varepsilon^3 \) is greater than that of the viscous critical layer, whose thickness is \( Re^{-1/3} \). In most applications outside of the laboratory, that condition will be satisfied easily.

The critical layer problem is highly nonlinear and the solution even at lowest order involves all the harmonics. In matching to the outer expansion, however, we can ignore higher harmonics because they decay more rapidly at large \( R \) than those terms involving the primary mode. After expanding the reduced pressure in (3.5) in a Taylor series about \( r_c \) and transforming to inner variables, we obtain the following conditions:

\[
U \sim u_0 R^\delta \sin \xi, \quad V \sim \frac{\gamma' r_c}{m} R + v_0 R^{\delta-1} \cos \xi, \quad W \sim w_0 R^{\delta-1} \cos \xi,
\]

and

\[
P \sim \frac{\gamma' v_c}{m} R^2 + p_0 R^{\delta} \cos \xi \quad \text{as} \quad R \to \infty.
\] (3.7)

Of the four constants \( u_0, v_0, w_0 \) and \( p_0 \), one is arbitrary. A convenient choice is \( p_0 = 1 \) and the system (2.3) can then be used to express the other three constants in terms of this constant.

### 3.1. The \( \lambda = 0 \) limit

As noted above, the Reynolds number is very large in most applications so that the \( \lambda = 0 \) limit is of great interest. However, it is known from previous studies of the nonlinear critical layer that viscosity still plays a subtle role. For steady flows, the arbitrary functions that arise in integrating the governing PDEs can be determined uniquely only by introducing a small viscosity. Moreover, thin viscous layers are required along streamlines separating open and closed regions in order that the vorticity and velocity components be continuous. Most of the flow field is inviscid though and an exact solution of the system (3.9) will now be presented for that case.

We begin by writing the system (3.6) in the following matrix form:

\[
A(u) \frac{\partial u}{\partial \xi} + B(u) \frac{\partial u}{\partial R} = c,
\] (3.8)
where \( A \) and \( B \) are \( 4 \times 4 \) matrices and the vectors \( u \) and \( c \) are given by

\[
\begin{align*}
\mathbf{u}(R,\xi) &= \begin{pmatrix}
P(R,\xi) \\
U(R,\xi) \\
V(R,\xi) \\
W(R,\xi)
\end{pmatrix} \\
\mathbf{c} &= \begin{pmatrix}
2\Omega_c V(R,\xi) \\
-2\Omega_c U(R,\xi) \\
0 \\
0
\end{pmatrix}.
\end{align*}
\]

(3.9)

The matrices \( A \) and \( B \) are given by

\[
A(u) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{m}{r_c} & 0 & \alpha & 0 \\
k & 0 & 0 & \alpha \\
0 & 0 & m & kr_c
\end{pmatrix}, \quad B(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & U(R,\xi) & 0 \\
0 & 0 & 0 & U(R,\xi) \\
0 & r_c & 0 & 0
\end{pmatrix}
\]

where \( \alpha(R,\xi) = \left(\frac{m}{r_c}\right)V(R,\xi) + kW(R,\xi) \).

It is clear that \( B(u) \) becomes singular when \( U = 0 \). This tells us that solving the system by the method of lines (using a spectral collocation method in the \( \xi \) direction followed by integration in the \( R \) variable) will lead to a numerically stiff system since Eqs.(3.6a-d) turn into a system of differential-algebraic equations. Indeed, when \( U = 0 \), we are led to the constraints \( V = 0, P = constant, W = constant \). This is also the reason why the small-\( \lambda \) limit of Eqns.(3.6) is difficult (in the sense of stability) to simulate numerically. We should also note that the low-rank behaviour of \( A(u) \) does not present computational difficulties. In order to deal with the possibly singular behaviour of \( B \), we now introduce an alternate solution strategy.

### 3.2. Solution by the method of characteristics.

If we denote by \( \frac{dR}{d\xi} = \frac{1}{\mu} \) the slope of the characteristics, they are given by the roots of the characteristic polynomial \( |A - \mu B| = 0 \). For the system (3.6), this condition yields

\[
det(A - \mu B) = \mu^2 r_c (\alpha - \mu U)^2 = 0,
\]

where \( \alpha = \left(\frac{m}{r_c}\right)V + kW \). Even though the system is not totally hyperbolic, \( i.e., \) there are only two characteristic directions, it develops that we can still solve (3.6) by integrating along the two characteristic directions given by

\[
\frac{dR}{d\xi} = \frac{U}{(m/r_c)V + kW} \quad \text{and} \quad \frac{d\xi}{dR} = 0.
\]

(3.10)
In accordance with the above result, we define a family of characteristics by
\[
\left( \frac{\partial R}{\partial s} \right)_\tau = U \quad \text{and} \quad \left( \frac{\partial \xi}{\partial s} \right)_\tau = \frac{m}{r_c} V + kW ,
\]
(3.11)
where \( s \) measures distance along a characteristic and \( \tau \) is a parameter identifying a particular characteristic. Eliminating \( \frac{\partial P}{\partial \xi} \) from the azimuthal and axial momentum equations in (3.6) and using the characteristic eqs. (3.11), we obtain the following integral:
\[
r_c V + 2 \bar{V} c R - \frac{m}{k} W = F(\tau) .
\]
(3.12)
A second integral can be obtained along the same family of characteristics by writing the directional derivative for the pressure, namely,
\[
\left( \frac{\partial P}{\partial s} \right)_\tau = \left( \frac{\partial P}{\partial R} \right)_{\xi} \left( \frac{\partial R}{\partial s} \right)_\tau + \left( \frac{\partial P}{\partial \xi} \right)_{R} \left( \frac{\partial \xi}{\partial s} \right)_\tau .
\]
(3.13)
Expressions for \( \frac{\partial P}{\partial R} \) and \( \frac{\partial P}{\partial \xi} \) are provided by Eqs. (3.6a) and (3.6c), i.e., the radial and axial momentum equations. The characteristic eqs. (3.11) yield the other partial derivatives and substitution of these four partial derivatives into (3.13) then allows us to integrate with respect to \( s \) to obtain an expression for the pressure, namely,
\[
P = a_0 \frac{m}{k} R W + \left( a_0 R - \frac{m}{k r_c^2} W \right) F(\tau) - a_0 \bar{V} c R^2 - \frac{1}{2} \left[ 1 + \left( \frac{m}{k r_c} \right)^2 \right] W^2 + G(\tau) ,
\]
(3.14)
where \( a_0 = 2 \bar{V} c / r_c^2 \).

The arbitrary functions \( F(\tau) \) and \( G(\tau) \) are determined by matching to the outer solution and will be specified below. First, however, two more equations are required to complete the solution. One is provided by the radial momentum equation, i.e., Eq.(3.6a) gives us a simple expression for \( \frac{\partial P}{\partial R} \) and we can integrate along the vertical lines \( \xi = \text{const.} \), because they are characteristics. The fourth relationship that we require is obtained not by integrating along a characteristic but by defining the parameter \( \tau \) in such a way that the continuity equation is satisfied automatically; the characteristics \( \tau = \text{const.} \) are then analogous to the streamlines in conventional fluid mechanics. Specifically, we require \( \tau \) to satisfy the conditions
\[
\frac{\partial \tau}{\partial \xi} = U \quad \text{and} \quad \frac{\partial \tau}{\partial R} = -(m/r_c)V - kW .
\]
(3.15)
The numerical procedure that we use is to begin the integration at a large value of \(|R|\) and then integrate toward the center of the critical layer. This allows us to determine an expression for \(\tau\) that is valid as \(R \to \infty\) that we can use to begin the integration. First, we integrate the second of Eqs. (3.15); the expression for \(\tau\) that results contains an arbitrary function of \(\xi\) which can then be determined by differentiating with respect to \(\xi\) and comparing with the asymptotic behavior of \(U\) given in (3.7). It can be verified that the following expression for \(\tau\) satisfies (3.15), the asymptotic conditions (3.7) and is consistent with the continuity equation:

\[
\tau = -\left(\frac{\gamma'_c}{2} + r_{0c}|R|^\delta\right) \cos \xi .
\] (3.16)

Noting that \(\tau \sim R^2\) for large \(|R|\), it can be seen that \(F \sim \sqrt{\tau}\) in (3.12) and that \(G \sim \tau\) in (3.14). By substituting into (3.12) and (3.14) and using the asymptotic conditions (3.7), we find more precisely that

\[
F(\tau) = (r \tilde{V})'c \sqrt{-\frac{2\tau}{\gamma'_c}} \quad \text{and} \quad G = \frac{2 \tilde{V}_c (r \tilde{V})'_c}{\gamma'_c r_{0c}^2} \tau .
\] (3.17)

### 3.3. Numerical Procedure.

As mentioned in the previous section, we begin integration of the characteristic equations for \(P, V, W\) and the radial variable \(R\) at large initial values of \(R\). We first locate \(\tau(\xi)\) for this large value of \(R\) using equation (3.16). The values of \(\tau(\xi)\) will subsequently be decreased by a fixed amount, \(d_\tau\), which in turn changes \(V, W\) and \(R\) through the second of equations (3.15). We implement a Backward Euler update

\[
d_\tau = (\tau - \tau_{old}) = -\left(\frac{m}{r_c} V + kW\right) (R_{old} - R) .
\] (3.18)

Here the subscripts _old_ refer to the values of the variable at the previous \(\tau\)-step. We also write an Euler update to compute the new values of \(P\) in terms of its current values, and use this to update \(V, W\) and \(R\). We are thus led to a nonlinear system of 4 equations comprising of equations (3.12), (3.14), (3.18) and

\[
r_c(P - P_{old}) - 2\tilde{V}_c V (R - R_{old}) = 0 .
\]

The nonlinear system needs to be solved for the new values of \((P, V, W, R)\) at each \(\tau\)-step. The manifold on which these solutions evolve seems particularly sensitive to
initial guesses in $V$ and $W$; hence, we only retain the updated $P$ and $R$ values from each iteration. We can then explicitly calculate from these the updated values of $V$ and $W$. We note here that though it may seem appealing to solve for $V$ and $W$ in terms of $P$ and $R$ using equations (3.12) and (3.14), the latter is a quadratic in $W$. This would force us to make a choice for the sign of the root (and is, indeed, the source of the numerical sensitivity to initial guesses in $V$ and $W$). Instead, we use all 4 of the equations to derive explicit expressions for $V$ and $W$ in terms of the other variables.

Care needs to be taken regarding the value of various constants as we integrate from above and below the critical layer. If we assume that in crossing the critical layer there is no phase change, then absolute values are used to deal with the branch point at $r - r_c$ in Eqs. (2.5) - (2.7), the Frobenius solution for $u$. Two limiting cases were treated in [19]; in the first, it was assumed that the vorticity $Q_c << 1$, whereas the second case assumed waves that are long in the axial direction, i.e., $k << 1$. In both limits, it was found that there was no phase change across the critical layer, so it would be surprising if that was not true for all cases with $J_c < 1/4$. Consistent with the assumption of zero phase change, we take $u_0$ in (3.6) to be the same on both sides of the critical layer, in which case $U$ will be an even function. Observing that $U$ is differentiated with respect to $R$ in the continuity equation, i.e., the last of Eqs. (3.7), it is clear that $V$ and $W$ will be odd functions of $R$ so that $u_0$ and $w_0$ in (3.6) must have opposite signs above and below the critical layer. Similar considerations lead us to conclude that $P$ is an even function.

The foregoing considerations were used to compute the solution illustrated in Figures (3.1)-(3.4) for a Lamb-Oseen vortex profile. Initial values were obtained from (3.7) and the computation was initiated at large values of $\tau$ on both sides of the critical layer, i.e. $\tau$ is even in $R$. This beginning value of $\tau$ depends on the choice of $k, r_c$ and $\gamma_c'$. In Figure (3.1), we present the characteristics $R$ and pressures $P$ obtained by the procedure described above. We repeated the procedure with several values of $d_r$, and are presenting the converged solutions. We show the results of two experiments; in
Fig. 3.1. Characteristic curves of $R$ and $P$ as functions of $\xi$, for varying $\tau$. Left figures: $r_c = 1.4, k = .36$. Right figures: $r_c = 0.88, k = .18$. Blue curves: integration from large $R$-values above the critical layer towards the critical layer. Red curves: integration from below. Particularly note the cusp-like behaviour near $R = 0$ at $\xi = \pm \pi$.

Figs. (3.1a) and (3.1c), $k = .36, m = 1, r_c = 1.4, \gamma_c' = -0.424907$. In Figs. (3.1b) and (3.1d), $k = .36, m = 1, r_c = 1.4, \gamma_c' = -0.424907$. The procedure used was to decrease $\tau$ in increments of 0.05 until a characteristic is reached that goes beyond the corners at $R = 0$ and $\xi = \pm \pi$. This occurs on the characteristic $\tau = 2.4$ in the first experiment, and $\tau = 1.0$ in the second.
For clarity, only a few of the characteristics are shown. The similarity of Figures (3.1a) and (3.1c) compared with the streamline patterns in the stratified shear flow computations in Figure 2 of Maslowe [20] is striking, even including a cusp at the corners. Although the characteristics are not streamlines in our problem, they do serve the same purpose mathematically by providing a solution of the continuity equation, so to this extent comparisons are valid.

Figure (3.2) shows that the nonlinear critical layer equations (3.8-3.9) completely eliminate the singular behavior exhibited by the linear problem. We show the fields for the azimuthal and axial velocity components $V$ and $W$, respectively, because these are the most singular according to (3.8). We again chose $k = 0.36, r_c = 1.4, \gamma'_c = -0.424960$. For this particular example, $\delta = 0.230$, so that $v \sim |r - r_c|^{-0.770}$ as $|r - r_c| \to 0$ in the outer problem and $w$ has the same behavior.

Our solution is not yet complete because we must still deal with the region of closed characteristics. In addition, there are discontinuities in $V$ and $W$ at the corners where the separatrices meet, although the radial velocity $U$ is continuous. Distortions in the mean flow, as well as thin viscous layers, would be needed to completely eliminate these discontinuities. This is not surprising in view of previous research on nonlinear critical layers in stratified shear flows. Both Haberman [21], in the small Richardson number case, and Troitskaya [22] who studied the forced wave problem with $J_c > 1/4$ found that a temperature jump took place across the critical layer and
Fig. 3.3. The characteristic solution for the pressure $P$, in $(r, \xi)$ coordinates. Here $\xi \in [-\pi, \pi]$. Recall $R = \frac{r - r_c}{\epsilon \beta}$; in this plot, $R \in [-8.29, 8.29]$. We have scaled the radial variable to enable easier reading of the graph. Note that $P$ is continuous as we move across the critical layer.

the vorticity was also discontinuous. The need for these mean flow corrections in our Kelvin wave problem is clear in Figure(3.2b), where the axial velocity component $W$ is illustrated and it can be seen that $W$ is discontinuous at the corners $(R, \xi) = (0, \pm \pi)$.

While such a vortex sheet is permitted in an inviscid problem, mean flow distortions are required to make this a valid solution in the sense of being the limit of a viscous flow as $Re \to \infty$. A comprehensive study of the $\lambda \sim O(1)$ problem would be required with consideration given to the limit as $\lambda \to 0$. Although we have not yet completed such an analysis, we can still anticipate to some extent the changes in mean flow based on the results in [21] and [22] and the small vorticity analysis of Caillol and Maslowe [19]. Before further discussing these distortions, however, let us proceed to the analysis of the closed characteristics region because related issues arise there.

3.4. The closed characteristics region. For a steady inviscid two dimensional flow, a general solution of the Euler equation is $\nabla^2 \psi = f(\psi)$, where $\psi$ is the stream-function and the function $f$ is arbitrary. In a region of closed streamlines, there is no way to determine $f(\psi)$ uniquely within the framework of inviscid theory. However, by considering a small viscous perturbation to the basic flow, Batchelor [23] proved that $\nabla^2 \psi$, the vorticity, must be a constant within such a region. The value of the constant
Fig. 3.4. \( V \) and \( W \) as functions of \( r, \theta \) and \( z \) for a fixed \( t \). Recall that \( P, V, W \) are periodic in \( \xi = m\theta + k z - \omega t \). The spirals correspond to the surfaces \( 10(\tau_c + R(\xi)), \theta, z \) for varying \( \tau \) (the radial variable is scaled to improve readability). The colors on the surfaces indicates the values of \( V \) and \( W \).

must be determined by matching in some way to the solution outside the separatrix. This is the counterpart of a difficulty here, namely, that the functions \( F(\tau) \) and \( G(\tau) \) are no longer determined by the outer region. We follow a procedure comparable to what is done in the case of a plane parallel flow, but the three dimensionality of the present problem naturally adds complications.

To begin, we can exploit the analogy between rotating and stratified flows. For a stratified flow, Grimshaw [24] extended Batchelor’s result by showing that the temperature is constant within a region of closed streamlines. This is intuitively reasonable because both the vorticity and the thermal energy equation are diffusion equations for the vorticity and heat, respectively. If we compare the first of Eqs. (3.6) with the vertical momentum equation for a stratified flow, it can be seen that the azimuthal velocity \( V \) is the equivalent of the temperature. In both cases, the pressure gradient balances some force, the buoyancy force or the linearized centrifugal force. We will therefore assume that the axial component of vorticity \( \frac{\partial V}{\partial R} + \tilde{\Omega}_c \) is constant in the region of closed flow. This was proved in [19] for the small vorticity case by projecting onto a plane \( z = \text{const.} \) and it must be nearly true in general. Integrating now with
respect to $R$, we obtain the expression below for $V(R,\xi)$ inside the separatrices

$$V(R,\xi) = (\Omega + G_0)R + g(\xi),$$  \hfill (3.19)

where $G_0$ is the axial component of vorticity in the original frame of reference. We fix $G_0$ by matching the velocity $V$ at $\xi = 0$ and determine $g(\xi)$ by matching $V$ along the separatrices. It develops that we must take $g(\xi) = 0$ in order to preserve the symmetry in our solution.

The determination of $W(\xi, R)$ is less clear cut. As an approximation, we will again assume that the vorticity, this time the azimuthal component, is constant within the separatrices. In the critical layer, to lowest order, the azimuthal vorticity is given by $-\frac{\partial W}{\partial R}$ and as a result, we obtain

$$W(R,\xi) = H_0 R + h(\xi),$$  \hfill (3.20)

where $H_0$ and $h(\xi)$ are determined in the same way as their counterparts governing $V$ and again symmetry requires $h(\xi) = 0$.

Returning now to the question of changes in the mean flow across the critical layer, it is clear that there is a strong wave-mean flow interaction. If we were to reformulate the problem to take account of this, in Eqs. (2.2), we would include $O(\varepsilon^3)$ mean flow components $\bar{V}_1(r)$ and $\bar{W}_1(r)$, say. These would then be expanded in Taylor series about the critical point $r_c$. This would lead to additional terms in the matching conditions for $V$ and $W$ in (3.7). We have included these additional terms just to investigate their effect and this was done on a trial and error basis. As expected from earlier nonlinear critical layer studies, including the mean flow jumps removes some of the discontinuities, but viscosity is required to smooth out velocity derivatives that appear in the equations defining the vorticity. The results of these experiments are not presented here because the procedure is not rigorous. It is nonetheless worth mentioning that a sign change in $\frac{\partial W}{\partial R}$ seems necessary to eliminate the discontinuity in $W$ at the corners $(R,\xi) = (0, \pm \pi)$. It turned out, surprisingly, that introducing a jump in the mean flow vorticity gradient $Q(r)$ was the most effective way to accomplish this, i.e., as a consequence of nonlinearity, a change in $\bar{V}$ can significantly modify the behavior of $W$. 


4. The Eigenvalue Problem. We now outline the procedure for solving (2.5) numerically for neutral modes with no phase change across the critical point. Results will be presented for the Lamb-Oseen vortex profile $\tilde{V}(r) = (1 - e^{-r^2})/r$. The range of integration is from $r = 0$ to $r \to \infty$ and, because (2.5) has a regular singular point at $r = 0$ and an irregular singular point at infinity, series solutions are required at both ends. A Runge-Kutta method was used to carry out the integration.

Near the origin, the solution can be represented by a Frobenius expansion having the form

$$u = u_0 r^{|m|-1} [1 + \zeta_1 r^2 + O(r^4)],$$

so that the radial perturbation velocity vanishes at the center of the vortex except in the case $m = 1$, the so called bending mode, and then it is continuous. Far from the center of the vortex, the velocity profile can be be approximated as a potential vortex so that $\tilde{V} \sim r^{-1}$. For a potential vortex, the pressure perturbation satisfies a modified Bessel equation (see pp. 341-342 of [1]); using the asymptotic expansion for $K_m$, the solution that decays exponentially and the first two equations of the system (2.3), the linearized equations of motion, we find that for large $r$

$$u \sim u_\infty \frac{e^{-kr}}{\sqrt{kr}} \left(1 + \frac{\kappa_1}{kr} + \frac{\kappa_2}{k^2 r^2} + O[(kr)^{-3}]\right).$$

Near the critical layer, we employ a linear combination of the two Frobenius solutions, as in (2.6) and (2.7). Using the conditions above to initiate the integration, we integrate toward the critical layer from either side. All variables are real and if we let $2 \eta = |r - r_c|$ and choose $B = 1$ as the arbitrary constant, then as the critical layer is approached from the vortex center, we write

$$u(\eta^-) = A u_+(\eta^-) + u_-(\eta^-) \quad \text{or else} \quad u(\eta^+) = A u_+(\eta^+) + u_-(\eta^+)$$

if the critical layer is approached from outside. Requiring $u'/u$, as well as the constant $A$ to be the same on either side of the critical layer, gives us enough conditions to determine the constants $u_0$ and $u_\infty$, as well as the dispersion relation $\omega(k)$ for a given value of $m$. 
In Fig.(4.1), a dispersion curve is illustrated for the bending mode \( m = 1 \). The local Richardson number at the critical point is also shown. The bending mode is the most important and it is clear from Figs. 5 and 11 of Leweke and Williamson [27] that this is the mode that arises naturally in their experiments. Because the \( m = 1 \) mode essentially disappears in the \( k = 0 \) limit treated in [14] and [15], the importance of generalizing the theory to helical modes is evident. The solutions that we obtained for \( m \geq 2 \) had very large values of \( k \), never being smaller than 7.80. Given that short waves are damped by viscosity this is likely the reason that they are not observed in the experiments.

To compare the results when there is a phase change with those in Fig.(4.1), we have written a program that avoids the singularity in (2.5) by indenting the contour of integration. Because \( \gamma_c' \) is negative, the integration path passes above the singularity in the complex \( r \) plane, corresponding to the viscous limit as \( Re \to \infty \) or to the initial value problem as \( t \to \infty \). The damping rate is quite large for long waves, but it is small for \( k \geq 1.20 \). This may mean that it takes very little nonlinearity to generate a neutral mode if the wavelength is not long. The reason that the frequencies in Figs. (4.1) and (4.2) are not far apart for \( k \geq 1.4 \) is that \( r_c \geq 2.8 \) and the singularity is weak that far from the center of the vortex. It is practically a potential vortex there, so how the singularity is crossed has little effect on the frequency for a given wavelength.

5. Conclusions. The differential equation governing the eigenvalue problem for helical waves propagating on a vortex has a critical point singularity if for some value of \( r \) the frequency \( \omega = m\bar{\Omega}(r) \), where \( \bar{\Omega}(r) \) is the angular velocity of the vortex. Our paper treats the class of waves for which this condition is satisfied and in addition \( \varepsilon \), a dimensionless amplitude parameter, is small enough so that linear theory is a good approximation outside a thin critical layer. At high Reynolds numbers, however, we have shown that nonlinear effects can be important for such modes in this layer even if \( \varepsilon \) is very small. Whether nonlinearity or viscosity is dominant depends on the parameter \( \lambda = 1/(Re\varepsilon^{3\beta}) \) defined below Equations (3.6), where \( 1/2 > \beta > 2/3 \). It can be recognized that \( \lambda^{1/3} \) is the ratio of the viscous to the nonlinear critical layer.
Fig. 4.1. The dispersion relation $\omega(k)$ and Richardson number at the critical point defined in eq.(2.8) for the $m=1$ bending mode.

Fig. 4.2. Variation of the frequency and decay rate $\omega_i$ as a function of wavenumber according to linear theory for a Lamb-Oseen vortex $V(r) = (1 - \exp(-r^2))/r$.

thickness so the condition for validity of the classical viscous theory is not only $\varepsilon \ll 1$ but $\varepsilon^\beta \ll Re^{-1}$, as well. This limits the role of the linear viscous theory to laboratory experiments, where the Reynolds number is much smaller than in applications such as aircraft trailing vortices.

In the foregoing analysis of the nonlinear critical layer, the similarities were noted between the helical modes on a vortex and the propagation of nonlinear waves on a parallel stratified shear flow. To some extent, we have benefited from the knowledge gained in earlier studies of stratified shear flows. In particular, we know that the
scaling and details of the matching depend very much on the value of \( J_c \), the local Richardson number at the critical point \( i.e., \) the equivalent local Richardson number defined in (8) above. There are three different régimes corresponding to \( J_c \) greater than \( 1/4 \), less than \( 1/4 \) but \( O(1) \), and finally \( J_c \sim O(\varepsilon^{1/2}) \). Even in the case of the stratified shear flow, however, questions remain to be answered about the mean flow distortion because Troitskaya [22] treated only the case of \( J_c \) greater than \( 1/4 \).

The system of four coupled partial differential equations that govern the nonlinear critical layer was derived in Section 3 and these equations are the same in all three régimes. The matching conditions, however, are different in the three cases. We have focused primarily on the Richardson number less than \( 1/4 \) case, but for some vortex profiles the case \( J_c > 1/4 \) may also be of interest. It depends not only on the velocity profile, but on the size of \( \lambda \), the parameter measuring the relative importance of nonlinearity to viscosity.

An analytical solution of the inviscid governing equations was found by the method of characteristics in Section 3.1 and this solution shows that the problem is highly nonlinear. One measure of the nonlinearity is that within the critical layer, all the higher harmonics are the same order of magnitude as the fundamental perturbation mode. Another is that there are discontinuities in the mean flow, particularly in the axial velocity induced by the wave. While such vortex sheets are employed as models in inviscid fluid dynamics, in reality a thin viscous mixing layer develops in which the velocity varies rapidly in order to smooth out the discontinuity.

An analysis of the case where \( Q_c \), the vorticity is small at the critical point [19] confirmed that both the azimuthal and axial vorticity components are different on either side of the critical layer. The complexity of that analysis made it clear that a numerical solution of Eqs. (3.6) including viscosity, \( i.e., \) with \( \lambda \sim O(1) \) would be desirable in order to avoid having to deal with higher order terms in both the inner and outer expansions. We have initiated such a study and we have been successful in obtaining solutions of (3.6) for moderate values of \( \lambda \). Dealing with small values, however, is a computational challenge that will be overcome only after we have devised a way to solve (3.6) and (3.7) as a boundary value problem. That means finding a
way to impose the asymptotic behavior below the critical layer without knowing the phase change in advance, as was done by Haberman [25] for the unstratified parallel shear flow, where the singularity being logarithmic is not as strong.

To conclude, we address the question of observability of the nonlinear waves described in this paper whether in the atmosphere, the laboratory or in numerical simulations. The mathematical similarities with the corresponding stratified shear flow problem make this the obvious place to look for some idea of what might be expected. Stratified shear flows with nonlinear critical layers have a structure resembling radar observations of what meteorologists call Kelvin-Helmholtz billows (see Section 3 of [20]). Despite the fact that these billows contain localized turbulent layers, the large scale coherence of the wave is maintained. However, in the laboratory it has not yet been possible to achieve large enough Reynolds numbers to compare with the theory. As a consequence, its greatest utility has been in numerical simulations, where structural details first revealed by the nonlinear critical layer theory appeared several years later in computational work.

Despite the mathematical analogies, the physical context is sufficiently different that we cannot extrapolate experience with stratified shear flows to the trailing vortex problem. In the latter case, even though the flow is often laminar in the vortex core, its environment and upstream history are such that it is likely to be turbulent elsewhere in an aircraft wake. Of course, nonlinear waves play a role in some turbulence theories, but it would be difficult to identify a nonlinear wave in observations because of the danger in making detailed measurements. Numerical simulations are a possibility, but the Reynolds numbers are too low in computations reported to date. It would appear, therefore, that experiments offer the most promise. The paper by Delisi and Robins [28] reports an investigation to determine the effect of stratification on the trajectory of a pair of vortices. There is a table in their paper giving the Reynolds numbers for experiments reported in several papers. These are all large enough so that with only a slight forcing, a wave could be generated satisfying the requirements of our theory, and such experiments are presently under consideration. Suppose we consider a small perturbation with $\varepsilon = 0.03$ at the value $Re = 2.5 \cdot 10^4$ of the experiments in [28]. The
calculation is not sensitive to the value of \( \beta \); using \( J_c = 0.09 \) to compute \( \beta \), we find that \( \lambda = 0.01 \), which is clearly in the nonlinear critical layer régime.

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**REFERENCES**

Appendix: Variation of the Reynolds stress. It is well known in hydrodynamic stability theory that valuable information about neutral modes can be obtained by evaluating the Reynolds stress for an unstable perturbation and then taking the limit as the growth rate tends to zero. From the energy equation for an unstable perturbation, as discussed in the review article by Stuart [26], it can be seen that the energy exchange between the mean flow and the perturbation is given by the integral

$$E = - \int_0^r \tilde{\Omega}(r) \overline{U_r C_\theta r^2} \, dr,$$
where the Reynolds stress \( \tau = -\overline{U_r U_\theta} \) and in this appendix we use capital letters for the velocity fluctuations, as in most texts on turbulence. The overline signifies an average over one wavelength in \( \xi \) and we write the radial perturbation velocity as

\[ U_r = \hat{U}_r(r) e^{i\xi} + \hat{U}_r^*(r) e^{-i\xi}, \]

where * indicates the complex conjugate.

The complex amplitude \( \hat{U}_r \) is a combination of the two Frobenius solutions

\[ \hat{U}_r = AX^+(r) + BX^-(r), \]

where \( X^+ \) and \( X^- \) are identical with the series \( u_\pm \) in (2.6) and (2.7) above. The reason for changing our notation is to agree with that used by Miles [18] in order to permit comparison with his very similar development for stratified shear flows. One small difference is that owing to the cylindrical geometry the lower limit of our domain is \( r = 0 \).

In order to compute \( \tau \), the Reynolds stress, we need an expression for \( \hat{U}_\theta(r) \). The system (2.3) yields a simple relationship between \( \hat{U}_\theta(r) \) and \( \hat{U}_r(r) \), namely,

\[ \hat{U}_\theta = S \left[ k^2 Q \frac{\gamma}{\gamma} + \frac{m}{r} D_* \right] \hat{U}_r. \]

Using this expression now for \( \hat{U}_\theta \), the Reynolds stress can be expressed in terms of the Frobenius solutions as follows:

\[ \tau = 2S(r) \frac{m}{r} \text{Im} \left[ \hat{U}_r' \hat{U}_r^* \right] = 2S(r) \frac{m}{r} \text{Im} \left[ |A|^2 X^+ X'^+ + |B|^2 X^- X'^- + A^* B X^+ X'^- + A B^* X^- X'^+ \right]. \]

The differential equation satisfied by the Reynolds stress is

\[ \frac{d\tau}{dr} = 2mD \left( \frac{S(r)}{r} \right) \text{Im} (\hat{U}_r' \hat{U}_r^*) - 2 \frac{m}{r} S(r) \text{Im} \left[ \hat{U}_r'' \hat{U}_r^* \right] = -4m \frac{S(r)}{r^2} \text{Im} \left[ \hat{U}_r' \hat{U}_r^* \right] \]

\[ = -2 \frac{\tau}{r^2} \quad \text{which is readily integrated to obtain} \quad \tau(r) = \frac{\tau_0}{r^2}. \]

The \( r^{-2} \) behavior of \( \tau \) contrasts with the case of a parallel shear flow, where \( \tau = \text{const}; \) however, in both cases we must consider the possibility of the constant being discontinuous across \( r_c \), as it is in the case of the Blasius boundary layer, for example.
For the vortices that are of primary interest in this paper, \( \tau_0 = 0 \) for \( r < r_c \), because the Reynolds stress must be finite at \( r = 0 \). And, with the exception of the limiting case \( J_c \to 0 \), the Frobenius solutions (2.7) show that \( \tau = 0 \) at the critical point. Therefore, we must also have \( \tau_0 = 0 \) for \( r > r_c \). If \( J_c \) is less than 1/4, the Reynolds stress near the critical point is given by

\[
-\overline{U_r U_\theta} = 2\nu mr_c \frac{S(r_c)}{r^2} \text{Im}[AB^*], \quad r > r_c
\]

and

\[
-\overline{U_r U_\theta} = 2\nu mr_c \frac{S(r_c)}{r^2} \text{Im}[AB^* e^{-i\pi(1+\nu)}], \quad r < r_c.
\]

It follows that either \( A = 0 \) or \( B = 0 \), as was shown by Miles for stratified shear flows.

For \( J_c > 1/4 \), on the other hand, the corresponding expressions are

\[
-\overline{U_r U_\theta} = -\mu mr_c \frac{S(r_c)}{r^2} (|B|^2 - |A|^2), \quad r > r_c
\]

and

\[
-\overline{U_r U_\theta} = -\mu mr_c \frac{S(r_c)}{r^2} (|A|^2 e^{\pi\mu} - |B|^2 e^{-\pi\mu}), \quad r < r_c,
\]

where \( \mu = (4 J_c - 1)^{1/2} \). The stress is zero when \( |B| = |A| e^{\pi\mu} \). Finally, if there is no phase change, as is the case when the critical layer is nonlinear, the constants \( A \) and \( B \) are real, so a neutral mode can be a linear combination of both Frobenius solutions.