Permanence and existence of a positive periodic solution to a periodic stage-structured system with infinite delay

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Abstract

In this paper we consider a periodic non-autonomous competitive stage-structured system with infinite delay for the interaction between \( n \) species, the adult members of which are in competition. For each of the \( n \) species the model incorporates a time delay which represents the time from birth to maturity of that species. Infinite delay is introduced which denotes the influential effect of the entire past history of the system on the current competition interactions. We first prove by using the comparison principle that if the growth rates are sufficiently large then the solutions are uniformly permanent. Then by using Horn’s fixed point Theorem, we show that the system with finite delay has a positive periodic solution. As a consequence of this result, we prove that even the system with infinite delay admits a positive periodic solution.

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1. Introduction

In the natural world there are many species whose individual members have a life history that take them through two stages: immature and mature. For example, for many animals whose babies are raised by their parents or dependent on the nutrition from the eggs they stay in, the immature are much weaker than the mature and hardly can these immature produce any babies. In particular, we have in mind mammalian populations and some amphibious animals which exhibit these two distinct stages. For these animals population, the immature species that survives a fixed length of time \( s > 0 \), exit from the immature species and enter the mature species. The time \( s \) is called the constant time to maturity. The life history of these species is then divided into two stages: mature and immature. The mature species are characterized by their capacity to reproduce. Much research has been devoted to stage-structured population models that embodies the distinction among these different stages (see [2,4,5,22,28] and the references cited therein). Probably, the most interesting stage-structured population model is the one described by Aiello and Freedman in [1] who introduced a single...
species model of population growth consisting of immature and mature individuals where the stage structure is modeled by the introduction of a constant time delay (see also [2,3] in which the time delay is modeled another way). The case of multispecies with competition interaction (only between mature individuals) was considered in [23,24], where the authors established some criteria giving the permanence, extinction and the existence of a global positive equilibrium.

In view of the fact that in real-life competition interactions, instantaneous responses are rare or weak relatively to delayed responses, more realistic models should consist of delay differential systems instead of systems with instantaneous feedback. Many authors consider continuously distributed delays as ecologically more realistic than discrete delays to model the delayed feedback responses in competition among the mature systems with instantaneous feedback. In this respect, Liu et al. introduced in [26] distributed delay terms into the more realistic than discrete delays to model the delayed feedback responses in competition among the mature species (see [33,18,8,13]).

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In this paper we try to answer these questions when we take into account both the periodicity of the environment and the time delay. For the sake of generality, we consider the periodic competitive stage-structured system with infinite delay

\[
\begin{align*}
\frac{dx_i}{dt} &= b_i(t - \tau_i)e^{\int_{t-\tau_i}^{t} d_j(s) ds} x_i(t) - x_i(t) \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\
&\quad -x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{-\infty}^{0} g_{ij}^+(s)x_j(t+s)ds + x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{-\infty}^{0} g_{ij}^-(s)x_j(t+s)ds, \\
\frac{dy_i}{dt} &= b_i(t)x_i(t) - d_i(t)y_i(t) - b_i(t - \tau_i)e^{\int_{t-\tau_i}^{t} d_j(s) ds} x_i(t)
\end{align*}
\] (1.2)

for \( t > 0 \) and \( i = 1, \ldots, n \), with the initial values

\[ x_i(s) = \varphi_i(s), \quad y_i(s) = \xi_i(s), \quad s \leq 0. \]

In the sequel we will denote by \( B_i(t) \) the expression \( b_i(t - \tau_i)e^{\int_{t-\tau_i}^{t} d_j(s) ds} \). Use will be made of the assumptions

(H1) The functions \( b_i(t), d_i(t), a_{ij}(t), k_{ij}(t) \) are assumed to be positive, continuous and periodic with common period \( \omega > 0 \), \( i, j = 1, \ldots, n \). The constants \( \tau_i \) are positive for \( i = 1, \ldots, n \).

(H2) The functions \( g_{ij}^+(s), g_{ij}^-(s) \), \( i, j = 1, \ldots, n \), are non-negative, integrable on \(( -\infty, 0)\) and there exists a positive and continuous function \( h_0 \) on \(( -\infty, 0)\) such that \( g_{ij}^+(s) \leq h_0(s) \) for \( s \in ( -\infty, 0) \) and \( \int_{-\infty}^{0} h_0(s)ds < \infty \).

(H3) The initial functions \( \varphi_i, \xi_i \) are assumed to be positive, continuous and bounded on \(( -\infty, 0)\). For the positivity of the solution \( y_i(t) \) we will assume that:

(H4) \( y_i(0) = \int_{-\tau_i}^{0} b_i(s)\varphi_i(s)e^{\int_{s}^{0} d_j(u)du}ds \) for any \( i = 1, \ldots, n \).

In this model \( x_i \) represents the density of the mature species and \( y_i \) the density of the immature one of type \( i \). The term \( b_i(t - \tau_i)e^{\int_{t-\tau_i}^{t} d_j(s) ds} x_i(t - \tau_i) \) represents the number of immature species that where born at time \( t - \tau_i \) which still survive at time \( t \) and are transferred from the immature stage to the mature stage at time \( t \). The function \( d_i(t) \) denotes the death rate of the immature species of type \( i \) at time \( t \). \( a_{ij}(t) \) are the interspecific competition coefficients, \( a_{ij}(t) \) is the intraspecific competition coefficient of mature species of type \( i \). We have introduced in the dynamics of the model the terms \( \sum_{j=1}^{n} k_{ij}(t) \int_{-\infty}^{0} g_{ij}^-(s)x_j(t+s)ds \), which suggest that the interspecific and intraspecific competition in the competitive system \((1.2)\) involve the response delays to resource limitations extending over the entire past (for more details of the biological significance of the infinite delay, we refer to [10,33]). Thus, system \((1.2)\) extends the model \((1.1)\) in [26] to the case of infinite delay. In [26] we have proved that if the maximum effects of the distributed delay terms are small and that the lower bounds of the interspecific competition coefficients \( a_{ij}^\infty \) (1 \( i = 1, \ldots, n \)) are large compared to the upper bounds of the interspecific competition coefficients \( a_{ij}^\infty \), \( 1 \leq j \leq n, \) then the positive solutions are uniformly permanent. In this paper we extend this result to the infinite delay case. Furthermore, we prove, by using Horn’s fixed point Theorem, that the system with finite delay has a positive periodic solution. As a consequence of this last result we show that the system with infinite delay also has a positive periodic solution.

The paper is organized as follows: In the next section, sufficient conditions are established for the existence, the positivity and the uniform permanence of solutions of system \((1.2)\). In Section 3, we show the existence of a positive periodic solution in the case of finite delay. This is followed by a similar result in the infinite delay case. A brief discussion is given at the end of the paper to conclude this work.

2. Uniform permanence

We begin this section by establishing an existence result for solutions of system \((1.2)\). If \( f(t) \) is a real valued and periodic function with period \( \omega > 0 \), we use the following notations:

\[ f^\prime = \inf_{t \in [0,\omega]} f(t), \quad f^m = \sup_{t \in [0,\omega]} f(t). \]

Put $C_{ij}^+ = \int_0^\infty g_{ij}^+(s)ds$, $C_{ij}^- = \int_0^\infty g_{ij}^-(s)ds$ for $i, j = 1, \ldots, n$. In order to overcome the difficulties caused by the infinite delay, we take as a phase space for system (1.2) the Banach space $C$ of continuous functions $\phi = (\varphi, \zeta) : (-\infty, 0) \to \mathbb{R}^r \times \mathbb{R}^r$, $s \mapsto (\varphi(s), \zeta(s))$ equipped with the norm $||\phi||_C = \int_{-\infty}^0 h_0(s)\sup_{s \leq t \leq 0} |\phi(t)|ds < \infty$, where $h_0$ is the function given in (H2). Define the positive cone in $C$ by

$$C^+ = \{ \phi = (\varphi, \zeta) \in C : \varphi(s) > 0, \ \zeta(s) > 0 \ \text{for} \ s \leq 0 \}.$$  

For each $\phi = (\varphi, \zeta) \in C$, denote by $(x(t, \phi), y(t, \phi))$, where $x(t, \phi) = (x_1(t, \phi), \ldots, x_n(t, \phi))$ and $y(t, \phi) = (y_1(t, \phi), \ldots, y_n(t, \phi))$, $t > 0$, the solution of system (1.2) corresponding to the initial value $\phi \in C$.

**Proposition 2.1.** Assume that assumptions (H1)–(H4) hold, then for each $\phi = (\varphi, \zeta) \in C^+$, system (1.2) has a unique positive solution $(x(t, \phi), y(t, \phi))$ defined for all $t > 0$. Further, if in addition the assumption

$$(H5) \ a_i^l > \sum_{j=1}^n k_{ij}^m C_{ij}^- \ \text{for} \ i = 1, \ldots, n,$$

holds, then the solution $(x(t, \phi), y(t, \phi))$ is bounded.

**Proof.** In [35], Wang and Huang proved that the Banach space $C$ defined above satisfies all the axioms described by Hale and Kato [15] and Sawano [29] for functional differential equations with infinite delay. Define now two functionals $f_j = (f_j^1, \ldots, f_j^n), \ j = 1, 2$, $f_j : \mathbb{R} \times C \to \mathbb{R}$, $(t, \phi) \mapsto f_j^l(t, \phi)$, where for each $i = 1, \ldots, n$

$$f_j^1(t, \phi) = B_i(t)\varphi_i(-\tau_i) - \varphi_i(0)\sum_{j=1}^n a_{ij}^l(t)\varphi_j(0) - \varphi_i(0)\sum_{j=1}^n k_{ij}^l(t)\int_{-\infty}^0 g_{ij}^+(s)\varphi_j(s)ds$$

$$+ \varphi_i(0)\sum_{j=1}^n k_{ij}^l(t)\int_{-\infty}^0 g_{ij}^-(s)\varphi_j(s)ds$$

and

$$f_j^2(t, \phi) = b_i(t)\varphi_i(0) - d_i(t)\zeta_i(0) - B_i(t)\varphi_i(-\tau_i)$$

for $t > 0$ and $\phi = (\varphi, \zeta)$. Let the new functional $f : \mathbb{R} \times C \to \mathbb{R}^r \times \mathbb{R}^r$ be defined by $f(t, \phi) = (f^1(t, \phi), f^2(t, \phi))$. It is easy to check that $f$ is continuous in $(t, \phi)$ and locally Lipschitzian with respect to $\phi$. Therefore, by the theory of functional differential equations with infinite delay (see [15,29]), system (1.2) has a unique solution $(x(t, \phi), y(t, \phi))$ for each $\phi \in C$ which is defined for all $t > 0$. Further, by the positivity property of the semiflow of the first equation of system (1.2) (see [26]) we have $x(t, \phi) > 0$, for $\phi \in C^+$ and $t > 0$, which gives with assumption (H4) that $y(t, \phi) > 0$ for any $\phi \in C^+$, $t > 0$ (see [25]).

We now prove the boundedness of the solutions. Let $\phi = (\varphi, \zeta) \in C^+$ and choose $M$ such that

$$M > \max_{1 \leq i \leq n} \left\{ \frac{B_i^m}{a_i^l - \sum_{j=1}^n k_{ij}^m C_{ij}^-}, \sup_{s \leq -\infty, 0} \varphi_i(s) \right\}. \quad (2.1)$$

We claim that

$$x_i(t, \phi) \leq L \quad (2.2)$$

for $t > 0$ and $i = 1, \ldots, n$. Otherwise by continuity there would exist a $\bar{t} > 0$ and $i_0 \in \{1, \ldots, n\}$ such that $x_{i_0}(\bar{t}, \phi) = M$, $\frac{dx_{i_0}}{dt}(\bar{t}, \phi) > 0$ and $x_i(t, \phi) \leq M$ for $0 < t \leq \bar{t}$ and $i = 1, \ldots, n$. We have from the first equation of system (1.2) and the positivity of $x_i$ that

$$\frac{dx_{i_0}}{dt}(\bar{t}, \phi) \leq B_{i_0}(\bar{t})x_{i_0}(\bar{t} - \tau_{i_0}, \phi) - a_{i_0i_0}(\bar{t})x_{i_0}^2(\bar{t}, \phi) + \sum_{j=1}^n x_{i_0}(\bar{t}, \phi)\sum_{j=1}^n k_{i_0j}(\bar{t})\int_{-\infty}^0 g_{i_0j}^-(s)x_j(\bar{t} + s, \phi)ds.$$  

Since $x_i(\bar{t} + s, \phi) \leq M$ for all $s \leq 0$, it appears from (2.1) that

$$\frac{dx_{i_0}}{dt}(\bar{t}, \phi) \leq M \left\{ B_{i_0}^m - M \left( a_{i_0i_0} - \sum_{j=1}^n k_{i_0j}^m C_{i_0j}^- \right) \right\} < 0,$$
which is a contradiction. Now integrating both sides of the second equation of system (1.2) on $(0, t)$ and using (H4) we obtain (see [25])

$$y_i(t) = \left( \int_{t-\tau}^{t} b_i(s)\xi_i(s) e^{\int_{0}^{s} d_i(u)du} \right) \times e^{-\int_{0}^{t} d_i(s)ds}, \quad t > 0. \tag{2.3}$$

We can easily see from (2.2) and (2.3) that the $y_i$’s are also bounded. In the sequel we denote a common upper bound of $x_i$ and $y_i$ by $M$. □

**Remark 2.1.** We can prove by using Razumikhin theorem for uniform ultimate boundedness [14], that the $x_i$’s and the $y_i$’s are uniformly ultimately bounded i.e. there exists $M > 0$ such that $\limsup_{t \to +\infty} x_i(t, \phi) \leq M$ and $\limsup_{t \to +\infty} y_i(t, \phi) \leq M$ for any $\phi \in C^+$. Let $r > 0$ and denote by $B(0, r)$ the ball of center 0 and radius $r$ in $C$. From (2.2) and (2.3) we deduce that there exists a constant $R > 0$ (depending on $r$) such that

$$\forall \phi \in B(0, r) \cap C^+, |x(t, \phi)|, |y(t, \phi)| \leq R, \quad t \in \mathbb{R}, \tag{2.4}$$

where $| \cdot |$ denotes the norm in $\mathbb{R}^n$. This fact will be used later in Section 3.

We next state the following comparison result.

**Lemma 2.1 [26].** Let $a, b, c$ and $d$ be positive constants and let $x(t)$ be a continuously differentiable function such that

$$\begin{align*}
\frac{dx}{dt}(t) &\leq bx(t - \tau) - cx(t) + dx(t) - ax^2(t), & t > 0, \\
x(t) &= \varphi(t), & -\tau \leq t \leq 0,
\end{align*}$$

where the initial function $\varphi$ is assumed to be in

$$C_+^* := \{ \phi \in C([0, 0]; \mathbb{R}) : \phi > 0 \}.$$

(i) If $b > c - d$, then for any $\varepsilon > 0$ sufficiently small there exists $T_{\varepsilon} > 0$ such that $x(t) < \frac{b - c + d}{a} + \varepsilon$ for $t > T_{\varepsilon}$.

(ii) Further, if $\frac{dx}{dt}(t) \geq bx(t) - cx(t) + dx(t) - ax^2(t)$ for $t > 0$, then for any $\varepsilon > 0$ (sufficiently small) there exists $T_{\varepsilon} > 0$ such that $x(t) > \frac{b - c + d}{a} - \varepsilon$ for $t > T_{\varepsilon}$.

To prove our result on uniform permanence we will need the following lemma.

**Lemma 2.2.** Assume that (H5) holds. Let $(\gamma_i^{(k)})_{k \geq 0}$ be a decreasing sequence such that $\gamma_i^{(k)} > 0$ for any $k \geq 0$ and $\lim_{k \to +\infty} \gamma_i^{(k)} = 0$. Choose $\gamma_i^{(0)}$ such that $\gamma_i^{(0)} > \frac{b_i}{d_i - \sum_{j=1}^{n} k_{ij}^m C_{ij}}$, $i = 1, \ldots, n$ and define the sequence $(\gamma_i^{(k)})_{k \geq 0}$ by

$$\begin{align*}
\gamma_i^{(0)} &= \gamma_i^{(0)}, \\
\gamma_i^{(k)} &= \frac{1}{d_i} \left( B_i^n + \sum_{j=1}^{n} k_{ij}^m C_{ij} \gamma_j^{(k-1)} \right) + \varepsilon^{(k-1)}
\end{align*} \tag{2.5}$$

for $k \geq 1$ and $i = 1, \ldots, n$. If $\varepsilon^{(0)}$ is sufficiently small then the sequence $(\gamma_i^{(k)})_{k \geq 0}$ is decreasing in $k$ and converges as $k \to +\infty$ to $\gamma_i > 0$, $i = 1, \ldots, n$ where $\gamma_i$ is the unique solution of

$$\gamma_i = \frac{1}{d_i} \left( B_i^n + \sum_{j=1}^{n} k_{ij}^m C_{ij} \gamma_j \right), \quad i = 1, \ldots, n. \tag{2.6}$$

**Proof.** We have from (2.5) with $k = 2$ and $i = 1, \ldots, n$

$$\gamma_i^{(k)} - \gamma_i^{(k-1)} = \frac{1}{d_i} \sum_{j=1}^{n} k_{ij}^m C_{ij} \left( \gamma_j^{(k-1)} - \gamma_j^{(k-2)} \right) + \varepsilon^{(k-1)} - \varepsilon^{(k-2)}. \tag{2.7}$$

Further, if we choose $\varepsilon^{(0)} > 0$ sufficiently small and since $\gamma_i^{(0)} > \frac{b_i}{d_i - \sum_{j=1}^{n} k_{ij}^m C_{ij}}$, we obtain

Taking the limit in (2.12) as $c$ from (2.6) we obtain that 

$$\lim_{c \to +\infty} \gamma_i = \gamma_i(0) = \frac{1}{d_{ii}} \left[ B^m + \sum_{j=1}^{n} k_{ij} C_{ij}^{-1} - d_{ii} \right] \gamma_i(0) + a_i I_{\gamma_i(0)} < 0. \quad (2.8)$$

Using (2.7) and (2.8), the fact that $(c(k))_{k \geq 0}$ is decreasing and an induction argument we obtain that the sequence $(\gamma_i(k))_{k \geq 0}$ is decreasing in $k$ for all $i = 1, \ldots, n$. From (2.5) and the fact that $\gamma_i(0), c(k) > 0$, we deduce, again by induction, that $\gamma_i(k) > 0$ for all $k \geq 0$ and $i = 1, \ldots, n$. Therefore the sequence $(\gamma_i(k))_{k \geq 0}$ converges to some $\gamma_i \geq 0$ i.e. 

$$\lim_{k \to +\infty} \gamma_i(k) = \gamma_i, \quad i = 1, \ldots, n. \quad (2.9)$$

Taking the limit in (2.5) as $k \to +\infty$, having in mind that $c(k) \to 0$, we find that $(\gamma_i(k))$ converges to $\gamma_i > 0$, $i = 1, \ldots, n$ with $\gamma_i$ satisfying (2.6). Suppose that for some $i_0$, $\gamma_{i_0} = 0$, then from (2.6) we obtain that $B^m_{i_0} = 0$ which is a contradiction. The uniqueness of $\gamma_i$ follows from (H5). \textbf{□}

**Theorem 2.1.** Assume that (H1)–(H5) hold and

$$B^m_i > \sum_{j=1, j \neq i}^{n} a_{ij}^m \gamma_j + \sum_{j=1}^{n} k_{ij}^m C_{ij}^{-1} \gamma_j, \quad i = 1, \ldots, n, \quad (2.9)$$

where $\gamma_i$ is defined in (2.6). Then the positive solutions of system (1.2) are uniformly permanent, i.e. there are $\delta_i, \xi_i > 0$ such that

$$\delta_i \leq \lim \inf_{t \to +\infty} x_i(t, \phi) \leq \lim \sup_{t \to +\infty} x_i(t, \phi) \leq \xi_i, \quad \forall \phi \in C^+, \quad i = 1, \ldots, n. \quad (2.10)$$

**Proof.** Let $\phi = (\varphi, \xi) \in C^+$. By Proposition 2.1 there is an $M > 0$ such that

$$0 < x_i(t + s, \varphi), \quad y_i(t + s, \varphi) \leq M, \quad s \leq 0, \quad t > 0. \quad (2.11)$$

Select $\gamma_i(0)$ such that $\gamma_i(0) > \max \left( \frac{B^m_i}{a_{ii} - \sum_{j=1}^{n} k_{ij}^m C_{ij}^{-1}}, M \right)$ and let $(c(k))_{k \geq 0}$ be a positive and decreasing sequence such that $c(k) \to 0$ as $k \to +\infty$. Define now the sequence $(\gamma_i(k))_{k \geq 0}$ as in (2.5). We have from the first equation of system (1.2) that

$$\frac{dx_i}{dt} \leq B^m_i x_i(t - \tau_i, \phi) - a_{ii}^m \gamma_i(0) + x_i(t, \phi) \sum_{j=1}^{n} k_{ij}^m \int_{-\infty}^{0} g_{ij}^{-}(s)x_j(t + s, \phi)ds$$

$$\leq B^m_i x_i(t - \tau_i, \phi) - a_{ii}^m \gamma_i(0) + x_i(t, \phi) \sum_{j=1}^{n} k_{ij}^m C_{ij}^{-1} \gamma_j.$$ 

for any $t > 0$. By Lemma 2.1 and if we choose $\gamma_i(0)$ sufficiently small there is a large time $T_{\gamma_i(0)}^{(1)} > 0$ such that

$$x_i(t, \phi) \leq \frac{1}{\delta_i} \left( B^m_i + \gamma_i(0) \sum_{j=1}^{n} k_{ij}^m C_{ij}^{-1} \gamma_j \right) + \varepsilon(0) = \gamma_i^{(1)} > 0$$

for $t > T_{\gamma_i(0)}^{(1)}$. Repeating this process $k$ times for $\gamma_i^{(1)} > \gamma_i^{(2)} > \cdots > \gamma_i^{(k)}$ we find a sequence $T_{\gamma_i(0)}^{(1)} < T_{\gamma_i^{(1)}}^{(2)} < \cdots < T_{\gamma_i^{(k-1)}}^{(k)}$ such that

$$x_i(t, \phi) \leq \frac{1}{\delta_i} \left( B^m_i + \sum_{j=1}^{n} k_{ij}^m C_{ij}^{-1} \gamma_i^{(k-1)} \right) + \varepsilon^{(k-1)} = \gamma_i^{(k)} \quad (2.12)$$

for $t > T_{\gamma_i^{(k-1)}}^{(k)}$, $k \geq 1$ and $i = 1, \ldots, n$. Now by Lemma 2.2, $\gamma_i^{(k)} \to \gamma_i > 0$ as $k \to +\infty$ for any $i = 1, \ldots, n$. From (2.11) we deduce that for $k \geq 1$

$$\lim_{t \to +\infty} x_i(t, \phi) \leq \gamma_i^{(k)}. \quad (2.12)$$

Taking the limit in (2.12) as $k \to +\infty$ we obtain

$$\lim_{t \to +\infty} x_i(t, \phi) \leq \gamma_i \quad (2.13)$$
for \( i = 1, \ldots, n \). It is clear from the assumption (2.9) of the theorem that we can select \( \varepsilon > 0 \) such that
\[
B_i^j - \sum_{j=1, j \neq i}^n a_{ij}^m (\gamma_j + \varepsilon) - \sum_{j=1}^n k_{ij}^m C_{ij}^+ (\gamma_j + \varepsilon) > 0, \quad i = 1, \ldots, n.
\] (2.14)

For this \( \varepsilon \), we can deduce from (2.13) that there exists \( T_\varepsilon > 0 \) such that
\[
x_i(t, \phi) \leq \gamma_i + \varepsilon
\] (2.15)
for \( t > T_\varepsilon \). Define \( H = \sup \{ \sum_{i=1}^n x_i(t + s, \phi) : t > 0, s \leq 0 \} \). Since \( \int_{-\infty}^0 g_{ij}^+(s)ds < \infty \), for any \( \eta > 0 \) arbitrarily small, we can choose \( \sigma > \max_{1 \leq i \leq n}(\tau_i) \) sufficiently large such that
\[
H \int_{-\infty}^{-\sigma} g_{ij}^+(s)ds < \eta.
\] (2.16)

Therefore from the first equation of system (1.2) we have that
\[
\frac{dx_i}{dt} \geq B_i^j x_i(t - \tau_i, \phi) - x_i \sum_{j=1, j \neq i}^n a_{ij}^m x_j - a_{ii}^m x_i^2 - x_i \sum_{j=1}^n k_{ij}^m t \int_{-\sigma}^{0} g_{ij}^+(s) x_j(t + s, \phi)ds
\]
for \( t > T_\varepsilon + \sigma \). Thus,
\[
\frac{dx_i}{dt} \geq B_i^j x_i(t - \tau_i, \phi) - x_i \sum_{j=1, j \neq i}^n a_{ij}^m (\gamma_j + \varepsilon) - a_{ii}^m x_i^2 - x_i \sum_{j=1}^n k_{ij}^m C_{ij}^+ (\gamma_j + \varepsilon) - x_i \sum_{j=1}^n k_{ij}^m H \int_{-\infty}^{-\sigma} g_{ij}^+(s)ds.
\] (2.17)

Using Lemma 2.1 and the inequality (2.17), for any \( \varepsilon > 0 \) there exists a \( T_\varepsilon' > T_\varepsilon + \sigma \) such that
\[
x_i(t, \phi) > \frac{1}{a_{ii}^m} \left( B_i^j - \sum_{j=1, j \neq i}^n a_{ij}^m (\gamma_j + \varepsilon) - \sum_{j=1}^n k_{ij}^m C_{ij}^+ (\gamma_j + \varepsilon) - \eta \sum_{j=1}^n k_{ij}^m \right) - \varepsilon
\]
for all \( t > T_\varepsilon' \). Now since \( \varepsilon, \varepsilon', \eta \) are arbitrarily small, we deduce that
\[
\liminf_{t \to +\infty} x_i(t, \phi) \geq \frac{1}{a_{ii}^m} \left( B_i^j - \sum_{j=1, j \neq i}^n a_{ij}^m \gamma_j - \sum_{j=1}^n k_{ij}^m C_{ij}^+ \gamma_j \right) = \delta_i^* > 0
\] (2.18)
for \( i = 1, \ldots, n \). Thus \( x_i \) are uniformly permanent. From (2.3) we obtain that
\[
0 < b_i^\tau \delta_i e^{-\tau \delta_i^*} \leq \liminf_{t \to +\infty} y_i(t, \phi) \leq \limsup_{t \to +\infty} y_i(t, \phi) \leq b_i^m \tau \gamma_i.
\]

If we take \( \delta_i = \min(b_i^\tau \delta_i e^{-\tau \delta_i^*}, \delta_i^*) > 0 \) and \( z_i = \max(b_i^m \tau \gamma_i, \gamma_i) \), then the second relation (2.10) is satisfied. 

3. Periodic solution

In this section we are concerned with the existence of a positive periodic solution to system (1.2). A standard approach to derive \( \omega \)-periodic solutions is to consider the Poincaré map associated to the system which is defined as a function which maps an initial function \( \phi \) (or value) in the phase space \( \omega \)-units along the unique solution \( u_\omega(\phi) \) corresponding to the initial function \( \phi \) (see [6]). Then, by using the uniqueness property of the solution, one can prove that a fixed point of the Poincaré map leads to a periodic solution. The idea is then to find suitable conditions and use an appropriate fixed point theorem. To that end, many authors [7,21] used different fixed point theorems such as Brouwer’s, Schauder’s, Horn’s, etc. Many fixed point theorems require
that the operator (the Poincaré map) maps compact sets into compact sets (for example Brouwer fixed point Theorem in the case of finite dimension), or that the operator itself be compact (Schauder fixed point theorem). In the case of ordinary differential equations (without delay) the phase space is a finite dimensional space and the closure of bounded sets are compact sets, hence Brouwer fixed point theorem is applicable. In the case of retarded differential equations the phase space is an infinite dimensional space (space of continuous functions) and the closure of bounded sets are not necessarily compact sets, so Brouwer fixed point theorem cannot be used. On the other hand, in the case of infinite delay the time interval in the phase space is \((-\infty,0]\), then under the Poincaré operator, an initial function on \((-\infty,0]\) becomes a restriction of a function defined on \((-\infty,\omega]\), \(\omega > 0\). That is, the history represented by the initial function on \((-\infty,0]\) is still carried over under the Poincaré map. Thus, it is possible that, under the Poincaré map, a bounded set gets mapped into a non-compact set. The Poincaré map cannot be compact in such a case and consequently Schauder fixed point theorem is useless. Recently, Teng and Chen [32] proved (under some assumptions such as uniform permanence) that the Poincaré map of a Kolmogorov type system with finite delay has a fixed point. Their method is based on Horn’s fixed point Theorem (cf. [16]). They constructed some equicontinuous sets (in the phase space) the compactness of which is derived from Ascoli-Arzela theorem. Unfortunately, this technique does not apply to the case of infinite delay. The reason is that an equicontinuous set in the phase space \(C((-\infty,0],\mathbb{R}^n)\) is not necessarily compact bearing in mind that Ascoli-Arzela theorem would require the compactness of the interval \((-\infty,0]\). To the present time few works have been devoted to the existence of positive periodic solutions for infinite delay systems (see [11,12,34,38,39]). In this paper we prove, by adapting the technique used in [32], that system (1.2) has a positive periodic solution. Our idea is to derive a periodic solution of the infinite delay system (1.2) through a periodic solution of an associated system with finite delay. The periodic solution of the finite delay system is obtained by modifying and adapting the technique in [32] to stage-structured systems and combining it with a suitable use of Horn’s fixed point theorem. Let \(\sigma > 0\) be fixed and consider the following system with finite delay

\[
\begin{align*}
\frac{dx_i}{dt} &= b_i(t-\tau_i)e^{-\int_{t-\tau_i}^{t} d_i(s)ds} x_i(t-\tau_i) - x_i(t) \sum_{j=1}^{n} a_{ij}(t)x_j(t) - x_i(t) \\
&\quad \times \sum_{j=1}^{n} k_{ij}(t) \int_{0}^{\sigma} g_{ij}(s)x_j(t+s)ds + x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{0}^{\sigma} g_{ij}(s)x_j(t+s)ds, \\
\frac{dy_i}{dt} &= b_i(t)x_i(t) - d_i(t)y_i(t) - b_i(t-\tau_i)e^{-\int_{t-\tau_i}^{t} d_i(s)ds} x_i(t-\tau_i).
\end{align*}
\]  

Before giving our main theorem in this section, we need the following compactness lemma.

**Lemma 3.1** [12]. Let

\[
\Omega = \{ \phi \in C : \| \phi \|_C \leq h, |\phi(s_1) - \phi(s_2)| \leq L|s_1 - s_2|, \ s_1, s_2 \in (-\infty, 0) \},
\]

where \(h\) and \(L\) are positive constants. Then \(\Omega\) is compact in \(C\).

We also need the following fixed point theorem.

**Lemma 3.2** [16]. Let \(S_0 \subset S_1 \subset S_2\) be convex subsets of a Banach space \(X\) with \(S_0\) and \(S_2\) compact and \(S_1\) open in \(S_2\). Let \(f : S_2 \rightarrow X\) be a continuous mapping such that, for some integer \(m > 0\),

\[
\begin{align*}
&f^j(S_1) \subset S_2, \quad 1 \leq j \leq m - 1, \\
&f^j(S_1) \subset S_0, \quad m \leq j \leq 2m - 1.
\end{align*}
\]

Then \(f\) has a fixed point in \(S_0\).

From Lemma 3.1 we derive the following result.

**Lemma 3.3.** Assume that the hypotheses of Theorem 3.2 hold and let \(0 < \delta^*_i < \alpha^*_i, \ H > 0\). Consider the set

\[
S = \{ \phi = (\phi, \xi) \in C : \delta^*_i \leq \phi_i(s), \xi_i(s) \leq \alpha^*_i, |\phi(r) - \phi(s)| \leq H|r - s|, r, s \in [-\infty, 0], \ i = 1, \ldots, n \}.
\]

Then there are \(0 < m_0 < \delta^*_i\) and \(M_0 > \alpha^*_i, \ (i = 1, \ldots, n)\) such that

where \( x(t, \phi) = (x_1(t, \phi), \ldots, x_n(t, \phi)) \), \( y(t, \phi) = (y_1(t, \phi), \ldots, y_n(t, \phi)) \) denotes the solution of system (3.1) for \( t > 0 \).

**Proof.** Observe first that \( S \) is not empty. Since \( S \subset C^+ \) and is bounded, by (2.4) there is \( M_0 > \max_{1 \leq i \leq n}(x_i^+) \), such that

\[
0 < x_i(t, \phi) \leq M_0 \quad \forall t > 0 \quad \forall \phi \in S.
\]

Select \( \gamma^{(0)} \) such that \( \gamma^{(0)} > \max \left( \frac{b_i}{a_i} - \sum_{j=1}^{n} \kappa_j C_{ij}^{\phi} M_0 \right) \) and let \((\psi^{(k)})_{k \geq 0} \) be a positive and decreasing sequence such that \( \psi^{(k)} \to 0 \) as \( k \to +\infty \). Define \( \gamma_i^{(k)} \) as in (2.4) for \( k \geq 0 \) and let \( u_i^{(k)}(t) \) be the solution of the problem

\[
\begin{aligned}
\frac{du_i^{(k)}}{dt} &= B_i^m u_i^{(k)}(t - \tau_i) - a_i^m \left( u_i^{(k)}(t) \right)^2 + u_i^{(k)}(t) \sum_{j=1}^{n} k_{ij} C_{ij}^{\phi} \gamma_i^{(k)}, \\
u_i^{(k)}(s) &= M_0, \quad s \in [-\infty, 0]
\end{aligned}
\]

for \( t > 0 \). We have from the first equation of system (3.1) that

\[
\begin{aligned}
\frac{dx_i}{dt} (t, \phi) &\leq B_i^m x_i(t - \tau_i, \phi) - a_i^m (x_i(t))^2 + x_i(t, \phi) \sum_{j=1}^{n} k_{ij} C_{ij}^{\phi} \gamma_i^{(0)}, \\
x_i(s, \phi) &= \phi_i(s) \leq M_0, \quad s \in [-\infty, 0].
\end{aligned}
\]

By the comparison principle (see [30])

\[
x_i(t, \phi) \leq u_i^{(0)}(t), \quad t > 0, \quad \phi \in S.
\]

From Lemma 2.1, if we choose \( \psi^{(0)} \) sufficiently small, there is \( T_{\psi^{(0)}} > 0 \) (independent on \( \phi \)) such that

\[
u_i^{(0)}(t) \leq \frac{1}{a_i^m} \left( B_i^m + \sum_{j=1}^{n} k_{ij} C_{ij}^{\phi} \gamma_i^{(0)} \right) + \psi^{(0)} = \gamma_i^{(1)}, \quad t > T_{\psi^{(0)}}.
\]

Therefore, (3.3) and (3.4) imply that

\[
x_i(t, \phi) \leq \gamma_i^{(1)}(t) \quad \forall t > T_{\psi^{(0)}} \quad \forall \phi \in S.
\]

Repeating this process \( k \)-times for \( \psi^{(1)} > \psi^{(2)} > \cdots > \psi^{(k)} \) we find a sequence \( T_{\psi^{(0)}} < T_{\psi^{(1)}} < \cdots < T_{\psi^{(k-1)}} \) independent on \( \phi \) such that

\[
x_i(t, \phi) \leq \frac{1}{a_i^m} \left( B_i^m + \sum_{j=1}^{n} k_{ij} C_{ij}^{\phi} \gamma_i^{(k-1)} \right) + \psi^{(k-1)} = \gamma_i^{(k)}, \quad \forall t > T_{\psi^{(k-1)}}
\]

for all \( \phi \in S \). From the assumptions (of Theorem 2.1) and the fact that \( \gamma_i^{(k)} \to \gamma_i \) as \( k \to +\infty \), we can select \( k_0 \geq 1 \) such that

\[
B_i^m > \sum_{j=1, j \neq i}^{n} a_i^m \gamma_j^{(k_0)} + \sum_{j=1}^{n} k_{ij} C_{ij}^{\phi} \gamma_j^{(k_0)}, \quad i = 1, \ldots, n.
\]

For this \( k_0 \) the relation (3.5) implies that, for any \( t > T_{\psi^{(k_0-1)}} \) and \( \phi \in S \), we have

\[
\frac{dx_i}{dt} (t, \phi) \geq B_i^m x_i(t - \tau_i, \phi) - x_i \sum_{j=1, j \neq i}^{n} a_i^m \gamma_j^{(k_0)} - a_i^m (x_i(t))^2 - x_i \sum_{j=1}^{n} k_{ij} C_{ij}^{\phi} \gamma_j^{(k_0)}.
\]

From Lemma 3.1, \( S \) is a compact set in \( C \), we then see that

\[
\beta_i = \min \{ x_i(t, \phi) : (t, \phi) \in [T_{\psi^{(k_0-1)}} - \sigma, T_{\psi^{(k_0-1)}}] \times S \} > 0.
\]
Define $u_i$ as the solution of the problem
\[
\begin{aligned}
\frac{d u_i(t)}{d t} &= B_i^i u_i(t - \tau_i) - u_i \sum_{j=1, j \neq i}^n a_{ij}(u_j) - a_{ii}(u_i)^2, \\
- u_i \sum_{j=1}^n t^{(j)} C_{ij}(u_j), & \quad t > T_{e, (i-1)} \\
\end{aligned}
\]
where $T_{e, (i-1)}$ is the solution of the problem $u_i(t) = \beta_i, s \in [T_{e, (i-1)} - \sigma, T_{e, (i-1)}]$.

By the comparison principle (see [30]), $m_i$ is the solution of system (3.1) such that for $t > T_{e, (i-1)}$ and $\phi \in S$. Now by Lemma 2.1, there is $\varepsilon' > 0$ and $T_{e'} > T_{e, (i-1)}$ such that for $t > T_{e'}$, we have
\[
u_i(t) \geq \frac{1}{a_i}(B_i - \sum_{j=1, j \neq i}^n a_{ij}(u_j) - \sum_{j=1}^n k_{ij} C_{ij}(u_j)) - \varepsilon' = \delta_i > 0.
\]

Therefore, $x_i(t, \phi) \geq \delta_i \forall t > T_{e'} \forall \phi \in S$, where $T_{e'}$ is independent on $\phi$. Again by the compactness of the set $[0, T_{e'}] \times S$ we can define $z_i = \min_{0 \leq t \leq T_{e'}} x_i(t, \phi) > 0$, and hence
\[
x_i(t, \phi) \geq m_i \forall t > 0 \forall \phi \in S, \quad i = 1, \ldots, n,
\]
where $m_i = \min_{0 \leq i \leq n\{m_i\}}$. Using the comparison principle for ordinary differential equations we prove that there is $m_i > 0$ such that $y_i(t, \phi) \geq m_i, \forall t > 0$ and $\phi \in S$. We can then define $m_0 > 0$ as $m_0 = \min_{i=1}^n(m_i, m_i)$ with $m_i = \min_{0 \leq i \leq n\{m_i\}}$ and $m_i = \min_{0 \leq i \leq n\{m_i\}}$.

We are now in position to state our main result in this section.

**Theorem 3.1.** Assume that assumptions (H1)–(H5) hold. If (2.9) is satisfied, then system (3.1) has a positive and $\omega$-periodic solution.

**Proof.** From Theorem 2.1 the solutions of system (3.1) are uniformly permanent with the coefficients $\delta_i$ and $\alpha_i$. From this theorem, we can select $0 < \delta_i < \delta_i$ and $\alpha_i > \alpha_i$ such that
\[
\begin{aligned}
\delta_i < \liminf_{t \to +\infty} x_i(t, \phi) &\leq \limsup_{t \to +\infty} x_i(t, \phi) < \alpha_i, \\
\delta_i < \liminf_{t \to +\infty} y_i(t, \phi) &\leq \limsup_{t \to +\infty} y_i(t, \phi) < \alpha_i \quad \forall \phi \in C^+.
\end{aligned}
\]

Define the set $S^* := \{\phi = (\varphi, \xi) \in C : \delta_i^* < \varphi_i(s), \xi_i(s) < \alpha_i, s \in [-\sigma, 0]\}.$

By (3.6) and since the coefficients of system (3.1) are bounded, we can define $H^* > 0$ and finite by
\[
H^* = \sup \{[f(t, \Psi)] : \Psi = (\psi, \chi) \in C, \delta_i^* < \psi_i(s), \chi_i(s) < \alpha_i, s \in [-\sigma, 0], i = 1, \ldots, n, t > 0\},
\]
where $f$ is the functional whose components are the right hand sides of (3.1). Define $S^{**}$ as
\[
S^{**} := \{\phi = (\varphi, \xi) \in C : \delta_i^* < \varphi_i(s), \xi_i(s) < \alpha_i, s \in [-\sigma, 0], \phi(r) - \phi(s) \leq H^* - s, r, s \in [-\sigma, 0], i = 1, \ldots, n\}.
\]

Now according to Lemma 3.1 $S^{**}$ is compact, therefore by Lemma 3.2, there are $0 < m_0 < \delta_i^*$ and $M_0 > \alpha_i$ such that
\[
m_0 \leq x_i(t, \phi) \leq M_0, \quad m_0 \leq y_i(t, \phi) \leq M_0, \quad t > 0 \forall \phi \in S^{**}.
\]

Again, since the coefficients of system (3.1) are bounded we can define
\[
H_0 = \sup \{[f(t, \Psi)] : \Psi = (\psi, \chi) \in C, m_0 < \psi_i(s), \chi_i(s) < M_0, s \in [-\sigma, 0], i = 1, \ldots, n, t > 0\}.
\]

Notice that $H_0 \geq H^*$ but we can choose $H_0$ such that $H_0 > H^*$. We set...
\[ S_0 = \{ \phi = (\varphi, \xi) \in C : m_0 \leq \varphi_i(s), \xi_i(s) \leq M_0, s \in [-\sigma, 0] | \phi(r) - \phi(s)| \leq H_0|r-s|, \ r, s \in [-\infty, 0], \ i = 1, \ldots, n \}. \]

From relation (2.4) there is \( M_1 > M_0 \) such that
\[ 0 < x_i(t, \phi), y_i(t, \phi) < M_1, t > 0 \quad \forall \phi \in S_0. \]

Next, we consider \( H_1 > H_0 \) such that
\[ H_1 > \sup \{ |f(t, \Psi)| : \Psi = (\psi, \chi) \in C, 0 < \psi_i(s), \chi_i(s) < M_1, s \in [-\infty, 0], \ i = 1, \ldots, n, \ t > 0 \} \]
and define \( S_1 \) by
\[ S_1 = \{ \phi = (\varphi, \xi) \in C : 0 < \varphi_i(s), \xi_i(s) < M_1, s \in [-\sigma, 0] | \phi(r) - \phi(s)| \leq H_1|r-s|, \ r, s \in [-\infty, 0], \ i = 1, \ldots, n \}. \]

Once again by (2.4), there exists \( M_2 > M_1 \) such that
\[ 0 < x_i(t, \phi), y_i(t, \phi) < M_2, t > 0 \quad \forall \phi \in S_1 \]
and we consider \( H_2 > H_1 \) such that
\[ H_2 = \sup \{ |f(t, \Psi)| : \Psi = (\psi, \chi) \in C, 0 < \psi_i(s), \chi_i(s) < M_2, s \in [-\infty, 0], \ i = 1, \ldots, n, \ t > 0 \}. \]

Define \( S_2 \) by
\[ S_2 = \{ \phi = (\varphi, \xi) \in C : 0 < \varphi_i(s), \xi_i(s) < M_2, s \in [-\sigma, 0] | \phi(r) - \phi(s)| \leq H_2|r-s|, \ r, s \in [-\infty, 0], \ i = 1, \ldots, n \}. \]

Clearly \( S_0 \subset S_1 \subset S_2 \) and \( S_0, S_1, S_2 \) are bounded and convex sets in \( C \). Denote by \( U \) the set
\[ U := \{ \phi \in C : 0 < \varphi_i(s), \xi_i(s) < M_1 \quad \text{for} \ s \in [-\infty, 0] \}. \]

Then \( U \) is open in \( C \) and \( S_1 \subset U \). Further, we need to prove that \( S_1 \) is open in \( S_2 \). It suffices to prove that the set \( \overline{C_{S_1}} \) is closed in \( S_2 \). To that end let \( (\psi_k)_{k \geq 1} \) be a sequence such that \( \psi_k \subset S_2, \psi_k \in S_1 \) for any \( k \geq 1 \) and \( \psi_k \rightarrow \psi \) in \( S_2 \). Since \( S_1 \) is a closed set, then \( \psi \in S_2 \). Assume now that \( \psi \in S_1 \). So \( \psi \in U \) and since \( U \) is open in \( C \) there is then \( N \geq 1 \) such that \( \psi_k \in U \) for any \( k \geq N \). By this fact and since \( \psi_k \in S_1 \), there are two sequences \((s_k)_{k \geq 1}, (r_k)_{k \geq 1}, s_k, r_k \in [-\infty, 0] \) for \( k \geq 1 \) such that
\[ |\psi_k(s_k) - \psi_k(r_k)| \geq H_1|s_k - r_k|, \quad k \geq 1. \]

From \((s_k)_{k \geq 1}\) and \((r_k)_{k \geq 1}\), we can extract two sequences denoted again by \((s_k)_{k \geq 1}\) and \((r_k)_{k \geq 1}\) such that \( s_k \rightarrow s \) and \( r_k \rightarrow r \) as \( k \rightarrow +\infty \). Taking the limit in (3.13) as \( k \rightarrow +\infty \), we obtain that \( |\psi(s) - \psi(r)| \geq H_1|s - r| \), that is \( \psi \notin S_1 \) which is a contradiction. So \( S_1 \) is an open set in \( S_2 \). By definition \( S_0 \) and \( S_2 \) are closed and equicontinuous sets in \( C \). Hence, by Ascoli-Arzelà Theorem, \( S_0 \) and \( S_2 \) are compact sets in \( C \). Define now the Poincaré map
\[ P_{\omega} : S_2 \rightarrow C, \phi \mapsto (x_{\omega}(., \phi), y_{\omega}(., \phi)), \]
where \( x_{\omega}(s, \phi) = (x_1(s + \omega, \phi), \ldots, x_n(s + \omega, \phi)) \), \( y_{\omega}(s, \phi) = (y_1(s + \omega, \phi), \ldots, y_n(s + \omega, \phi)) \) for \( s \in [-\infty, 0] \) and \( \omega \) is the period of system (3.1). By the continuous dependence of the solution \((x_{\omega}(t, \phi), y_{\omega}(t, \phi))\) on the initial data \( \phi \) (see [8]), \( P_{\omega} \) is a continuous map from \( S_2 \) into \( C \). For each \( j \geq 1 \), let us define \( P_{\omega}^j = P_{\omega} \circ \cdots \circ P_{\omega} \)-times, that is \( P_{\omega}^j(\phi)(s) = (x(s + j\omega, \phi), y(s + j\omega, \phi)) \) for \( s \in [-\infty, 0] \) and \( \phi \in S_2 \).

We would like to show that \( P_{\omega}^j(S_1) \subset S_2 \) for any \( j \geq 1 \). To see this, let \( \phi \in S_1, j \geq 1 \) and \( s, r \in [-\infty, 0] \). There are three cases to distinguish:

(i) \( s + j\omega, r + j\omega \leq 0 \). By the definition of \( S_1 \) we have
\[ |x_i(s + j\omega, \phi) - x_i(r + j\omega, \phi)| = |\varphi_i(s + j\omega) - \varphi_i(r + j\omega)| \leq H_1|s - r| < H_2|s - r|, \]
\[ |y_i(s + j\omega, \phi) - y_i(r + j\omega, \phi)| = |\xi_i(s + j\omega) - \xi_i(r + j\omega)| \leq H_1|s - r| < H_2|s - r|. \]

(ii) $r + j\omega \geq 0$, $s + j\omega \leq 0$. By (3.11) and (3.12) we have

$$0 < x_i(r + j\omega, \phi), \quad y_i(r + j\omega, \phi) \leq M_2, \quad \left| \frac{dx_i(t, \phi)}{dt} \right|, \quad \left| \frac{dy_i(t, \phi)}{dt} \right| \leq H_2, \quad t > 0$$

(3.15)

for $i = 1, \ldots, n$. Thus, by the definition of $S_1$ and the second relation in (3.15), we clearly have (recall that $x_i(0, \phi) = \varphi_i(0)$)

$$|x_i(r + j\omega, \phi) - x_i(s + j\omega, \phi)| = |x_i(r + j\omega, \phi) - \varphi_i(s + j\omega)|$$

$$\leq |x_i(r + j\omega, \phi) - x_i(0, \phi)| + |\varphi_i(0) - \varphi_i(s + j\omega)|$$

$$\leq H_2(r + j\omega) + H_1(-s - j\omega) < H_2|r - s|$$

and by the same way we prove that

$$|y_i(r + j\omega, \phi) - y_i(s + j\omega, \phi)| \leq H_2|r - s|. \quad (3.17)$$

(iii) $r + j\omega, s + j\omega \geq 0$. Again by (3.11), (3.12) we have for $i = 1, \ldots, n$

$$0 < x_i(r + j\omega, \phi), x_i(s + j\omega, \phi), y_i(r + j\omega, \phi), y_i(s + j\omega, \phi) \leq M_2, \quad \left| \frac{dx_i(t, \phi)}{dt} \right|, \quad \left| \frac{dy_i(t, \phi)}{dt} \right| \leq H_2, \quad t > 0$$

(3.18)

and hence from the second relation in (3.18) we deduce that

$$|x_i(s + j\omega, \phi) - x_i(r + j\omega, \phi)| \leq H_2|s - r|, \quad s, r \in [-\infty, 0]. \quad (3.19)$$

So by (3.14), (3.16), (3.17) and (3.19) we obtain that $P^j_{\omega}(\phi) \in S_2$. Hence $P^j_{\omega}(S_1) \subset S_2$ for any $j \geq 1$. We now claim that there is $N \geq 1$ such that

$$P^j_{\omega}(\overline{S_1}) \subset S_0 \quad \forall j \geq N. \quad (3.20)$$

We first prove that for any $\phi \in \overline{S_1}$, there is $N(\phi) \geq 1$, $(N(\phi)\omega > \sigma)$ such that

$$(x_i(., 0, 0, \phi), y_i(., 0, 0, \phi)) \in S^* \quad \forall t \geq N(\phi)\omega - \sigma. \quad (3.21)$$

Let $\phi \in \overline{S_1}$. By (3.6), since the interval $[-\sigma, 0]$ is bounded (from below), there is $N(\phi) \geq 1$, $(N(\phi)\omega > \sigma)$ such that we have for any $t \geq N(\phi)\omega - \sigma$

$$\delta^*_i < x_i(s + t, \phi), y_i(s + t, \phi) < x^*_i, s \in [-\sigma, 0]. \quad (3.22)$$

Hence from (3.22) the property (3.21) is satisfied. Let us define

$$\tilde{\phi}(s) = (x_{N(\phi)\omega}(0, 0, \phi), y_{N(\phi)\omega}(0, 0, \phi)) = (x(s + N(\phi)\omega, 0, \phi), y(s + N(\phi)\omega, 0, \phi))$$

for $s \in [-\sigma, 0]$. Note that $\tilde{\phi} \in S^*$ because of (3.21). Also, the definition (3.7) allows us to write that

$$\left| \frac{dx_{N(\phi)\omega}(., 0, 0, \phi)}{dt} \right|, \quad \left| \frac{dy_{N(\phi)\omega}(., 0, 0, \phi)}{dt} \right| \leq H^*. $$

Thus,

$$\tilde{\phi} \in S^{**}. \quad (3.23)$$

On the other hand we have for any $t > N(\phi)\omega$

$$(x(t, 0, \phi), y(t, 0, \phi)) = (x(t, N(\phi)\omega, \tilde{\phi}), y(t, N(\phi)\omega, \tilde{\phi}))$$

$$= (x(t - N(\phi)\omega, 0, \tilde{\phi}), y(t - N(\phi)\omega, 0, \tilde{\phi})) \quad (3.24)$$

(see the note before Theorem 3.1). Bearing in mind (3.23), (3.24) and (3.9), we entail from (3.8) that

$$(x_{t+\sigma}(., 0, \phi), y_{t+\sigma}(., 0, \phi)) \in S_0 \quad \forall t \geq N(\phi)\omega. \quad (3.25)$$
For any $\phi \in \overline{S}_1$, let us designate by $B_{N(\phi)}$ the expression

$$B_{N(\phi)} = \bigcap_{j > |N(\phi)| + \sigma} \left( P_{\omega}^{-1}(S_0) \right).$$

By (3.25) we have $\phi \in B_{N(\phi)}$ and then $\overline{S}_1 \subset \bigcup_{\phi \in \overline{S}_1} B_{N(\phi)}$. We claim that $B_{N(\phi)}$ is a neighborhood of $\phi$ in $S_2$. Otherwise there would exist a sequence $(\phi_k)_{k \geq 1} \subset S_2$ and $t_k = j_k \omega > 0$, $k \geq 1$, $\phi_k \to \phi$, $t_k \to +\infty$ as $k \to +\infty$ such that

$$(x_k(., 0, \phi_k), y_n(., 0, \phi_k)) \notin S_0 \quad \forall k \geq 1.$$  \tag{3.26}$$

By (3.21) there is an $N(\phi) \geq 1$ such that

$$(x_1(., 0, \phi), y_1(., 0, \phi)) \in S^* \quad \forall t \geq N(\phi) \omega - \sigma.$$  \tag{3.27}$$

From the continuous dependence of the solutions on the initial value (see [8]), since $S^*$ is open in $C_\sigma$ and $\phi_k \to \phi$, $t_k \to +\infty$ as $k \to +\infty$, there is $k_0 \geq 1$ such that for any $k \geq k_0$

$$t_k > N(\phi) \omega + \sigma, (x_k(., 0, \phi_k), y_1(., 0, \phi_k)) \in S^*, \quad t \in [N(\phi) \omega - \sigma, N(\phi) \omega].$$  \tag{3.28}$$

From (3.7) and (3.28) we have $\frac{dy_1(0,0,\phi)}{dt} < H^*$ for $t \in [N(\phi) \omega - \sigma, N(\phi) \omega]$, which gives again with (3.28)

$$x_k(., 0, \phi_k), y_1(., 0, \phi_k)) \in S^*, \quad t \in [N(\phi) \omega - \sigma, N(\phi) \omega].$$  \tag{3.29}$$

Setting

$$\psi_k(s) = \left( x_{N(\phi) \omega}(s, 0, \phi_k), y_{N(\phi) \omega}(s, 0, \phi_k) \right) = (x(s + N(\phi) \omega, 0, \phi_k), y(s + N(\phi) \omega, 0, \phi_k)),$$

for $s \in [-\sigma, 0]$, we deduce from (3.29) that $\psi_k \in S^*$ for any $k \geq k_0$. We have now for any $t > N(\phi) \omega$

$$(x(t, 0, \phi_k), y(t, 0, \phi_k)) = (x(t, N(\phi) \omega, \psi_k), y(t, N(\phi) \omega, \psi_k)) = (x(t - N(\phi) \omega, 0, \psi_k), y(t - N(\phi) \omega, 0, \psi_k)).$$  \tag{3.30}$$

As $\psi_k \in S^*$ for $k \geq k_0$, we conclude from (3.8) that

$$(x_{t+\sigma}(., 0, \phi_k), y_{t+\sigma}(., 0, \phi_k)) \in S_0 \quad \forall t > N(\phi) \omega.$$  \tag{3.31}$$

If we pick $t = t_k - \sigma > N(\phi) \omega$ in (3.31), we obtain that

$$(x_k(., 0, \phi_k), y_n(., 0, \phi_k)) \in S_0$$

for any $k \geq k_0$. This contradicts (3.26). Thus $(B_{N(\phi)})_{\phi \in \overline{S}_1}$ is a cover of $\overline{S}_1$. Because $\overline{S}_1$ is compact in $S_2$, we can extract a finite cover $B_{N_1}, \ldots, B_{N_q}$ such that

$$\overline{S}_1 \subset \bigcup_{i=1}^{q} B_{N_i}.$$  

Put $N = \max_{1 \leq i \leq q}(N_i)$. For any $\phi \in \overline{S}_1$, it appears then that $P_{\omega}(\phi) \in S_0$ for any $j \geq N$, and (3.20) is satisfied. Horn’s fixed point Theorem (Theorem 3.1) allows us to conclude that $P_{\omega}$ has a fixed point $\phi^* = (\phi^*, \zeta^*) \in S_0$. That is

$$P_{\omega}(\phi^*) = \phi^*,$$  \tag{3.32}$$

so $x(s + \omega, \phi^*) = \phi^*(s)$, $y(s + \omega, \phi^*) = \zeta^*(s)$, $\forall s \in [-\sigma, 0]$. Since the coefficients of system (3.1) are periodic in $t$, $(x(t + \omega, \phi^*), y(t + \omega, \phi^*))$ is also a solution of system (3.1) for $t > 0$. By the uniqueness property for system (3.1) (see Proposition 2.1) we conclude that

$$x(t + \omega, \phi^*) = x(t, \phi^*),$$
$$y(t + \omega, \phi^*) = y(t, \phi^*), \quad t > 0,$$

which means that $(x(t, \phi^*), y(t, \phi^*))$ is $\omega$-periodic. As $\phi^* > 0$, we see from Proposition 2.1 that $(x(t, \phi^*), y(t, \phi^*))$ is a positive solution. This completes the proof of the theorem. $\square$

Theorem 3.1 leads to the following result.

Lemma 3.4. Assume that the assumptions (H1), (H3) and (H4) hold and that the functions \(h^+_i(s), h^-_i(s)\) are non-decreasing over \((-\tau, 0)\) with \(\tau = \max_{1 \leq i \leq n} \{\tau_i, \sigma_{ij}\}\) and \(a_{ij}^\prime > \sum_{j=1}^{n} h^+_i(s), i = 1, \ldots, n\). If the assumption

\[
B_i^m > \sum_{j=1, j \neq i}^{n} a_{ij}^m \gamma_j + \sum_{j=1}^{n} \int_{-\tau}^{0} h^+_i(s) \gamma_j, \quad i = 1, \ldots, n
\]

holds, then system (1.1) has a positive \(\omega\)-periodic solution, where \(\gamma_i > 0\) is given by \(\gamma_i = \frac{1}{a_{ii}^m}(B_i^m + \sum_{j=1}^{n} \int_{-\tau}^{0} h^+_i(s) \gamma_j), i = 1, \ldots, n\).

We now turn to prove the existence of a positive periodic solution in the infinite delay case.

Theorem 3.2. Assume that the assumptions (H1)-(H5) hold and

\[
B_i^m > \sum_{j=1, j \neq i}^{n} a_{ij}^m \gamma_j + \sum_{j=1}^{n} k_{ij}^m C_{ij}^m \gamma_j, \quad i = 1, \ldots, n,
\]

where \(\gamma_i\) are defined in (2.6). Then system (1.2) has a positive and \(\omega\)-periodic solution.

Before giving the proof of this theorem, we state a lemma which can be found in Hale and Kato [15]. In the sequel we denote by \(C\) the Banach space defined in Section 2.

Lemma 3.4 [15]. Let \(f_k, k \geq 1, f\) be continuous functions from \(\Omega \subset \mathbb{R} \times C\) into \(\mathbb{R}^n\), where \(\Omega\) is an open set in \(\mathbb{R} \times C\) and \(x^k(t) : (-\infty, A] \to \mathbb{R}^n\) the solution of the delayed functional differential equation

\[
\frac{dx}{dt} = f_k(t, x_t), \quad t \in [0, A],
\]

where \(A\) is a positive real number. Assume further that there exists a function \(x\) defined and continuous on \((-\infty, A]\) such that \(x^k \to x, x^k_t \to x_t\) uniformly on \([0, A]\) and \(f_k(t, \phi) \to f(t, \phi)\) uniformly on a set \(\Omega_0 \subset \Omega\), which contains \(\{(t, x^k_t); t \in [0, A], k \geq 1\}\) as \(k \to +\infty\). Then \(x(t)\) is a solution of the delayed functional differential equation

\[
\frac{dx}{dt} = f(t, x_t), \quad t \in [0, A].
\]

Proof of Theorem 3.3. For each \(m \geq 1\), we consider the following system with finite delay:

\[
\begin{align*}
\frac{dx}{dt} & = b_i(t - \tau_i) e^{-\int_{t-\tau_i}^{t} d_{i}(s)ds} x_i(t - \tau_i) - x_i(t) \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\
& \quad - x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{-m_0}^{0} g^+_i(s)x_j(t + s)ds \\
& \quad + x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{-m_0}^{0} g^-_i(s)x_j(t + s)ds,
\end{align*}
\]

(3.33)

with \(t > 0\). As we can find \(m_0 \geq 1\) such that \(m_0 \omega \geq \max_{1 \leq i \leq n} \{\tau_i\}\), system (3.33) can be treated as system (3.1) with \(\sigma = m_0\), \(m \geq m_0\). From Theorem 3.2, for any \(m \geq m_0\), there is \(\phi_m \in C_{m_0} = C([-m_0, 0]; \mathbb{R}^n)\), \(\phi_m > 0\) on \([-m_0, 0]\) and a positive \(\omega\)-periodic solution \((x_m(t, \phi_m), y_m(t, \phi_m))\) of system (3.33) such that

\[
(x_m(s + \omega, \phi_m), y_m(s + \omega, \phi_m)) = \phi_m(s), \quad s \in [-m_0, 0].
\]

(3.34)

Since the solutions of system (3.33) are uniformly permanent with the coefficients \(d_i\) and \(\sigma_i\) (see Theorem 2.1) then from (2.10) and since \((x_m(t, \phi_m), y_m(t, \phi_m))\) is periodic in \(t\), we have

\[
\delta_i \leq x_m(t, \phi_m), y_m(t, \phi_m) \leq \alpha_i, \quad t \in [0, \omega].
\]

(3.35)

The relations (3.34) and (3.35) yield that

\[
\delta_i \leq \phi_m(s), \quad \xi_m(s) \leq \alpha_i, \quad s \in [-\omega, 0],
\]

(3.36)
where \( \phi_m(s) = (\varphi(s), \zeta(s)) \) and \( \varphi(s) = (\varphi_m(s), \ldots, \varphi_{mn}(s)), \zeta(s) = (\zeta_m(s), \ldots, \zeta_{mn}(s)) \). From (3.34) we easily see that \( \phi_m \) is periodic in \([-m0, 0]\), thus, combining this fact with (3.36) we obtain

\[
\delta_i \leq \phi_m(s), \tilde{\phi}_m(s) \leq \alpha_i, \quad s \in [-m0, 0].
\]

(3.37)

To work in the phase space \( C(((-\infty, 0); \mathbb{R}^n \times \mathbb{R}^n) \) we define the new functions \( \tilde{\phi}_m \) as follows:

\[
\tilde{\phi}_m \text{ is } \omega\text{-periodic in } (-\infty, 0] \quad \text{and} \quad \tilde{\phi}_m(s) = \phi_m(s), \quad s \in [-\omega, 0]
\]

(3.38)

and denote by \( \chi_m \) the function defined by

\[
\chi_m(s) = \begin{cases} 1, & s \in [-m0, 0], \\ 0, & s < -m0. 
\end{cases}
\]

Consider the following system with infinite delay:

\[
\begin{aligned}
\frac{dx}{dt} &= b_i(t - \tau_i) e^{-\int_{t-\tau_i}^{t} d_i(s) \, ds} x_i(t - \tau_i) - x_i(t) \sum_{j=1}^{n} a_{ij}(t) x_j(t) \\
&\quad - x_i(t) \sum_{j=1}^{n} \int_{-\infty}^{0} \chi_m(s) g_{ij}^+(s) x_j(t + s) \, ds \\
&\quad + x_i(t) \sum_{j=1}^{n} \int_{-\infty}^{0} \chi_m(s) g_{ij}^-(s) x_j(t + s) \, ds, \\
\end{aligned}
\]

(3.39)

\[
\begin{aligned}
\frac{dy}{dt} &= b_i(t) x_i(t) - d_i(t) y_i(t) - b_i(t - \tau_i) e^{-\int_{t-\tau_i}^{t} d_i(s) \, ds} x_i(t - \tau_i). 
\end{aligned}
\]

If we denote by \( (x_m(t, \tilde{\phi}_m), y_m(t, \tilde{\phi}_m)) \) the solution of system (3.39) corresponding to the initial value \( \tilde{\phi}_m \), then we have for each \( m \geq m_0 \),

\[
(x_m(t, \tilde{\phi}_m), y_m(t, \tilde{\phi}_m)) = (x_m(t, \phi_m), y_m(t, \phi_m)), \quad t \in [0, \omega],
\]

(3.40)

which combined with (3.34) and (3.38) gives

\[
(x_m(s + \omega, \tilde{\phi}_m), y_m(s + \omega, \tilde{\phi}_m)) = \tilde{\phi}_m(s), \quad s \in [-\omega, 0].
\]

(3.41)

The function \( (x_m(t, \tilde{\phi}_m), y_m(t, \tilde{\phi}_m)) \) is then a positive and \( \omega \)-periodic solution of system (3.39). Moreover, from the relations (3.35), (3.37), (3.40) and (3.41) we entail that

\[
\delta_i \leq \chi_m(t, \tilde{\phi}_m), \chi_m(t, \tilde{\phi}_m) \leq \alpha_i, \quad t \in [0, \omega],
\]

(3.42)

where \( \phi_m = (\tilde{\varphi}, \tilde{\zeta}) \) with \( \tilde{\varphi} = (\varphi_{m1}, \ldots, \varphi_{mn}) \) and \( \tilde{\zeta} = (\zeta_{m1}, \ldots, \zeta_{mn}) \). For any \( m \geq m_0 \), let us designate by

\[
f_m(t, \phi) = (f_{m1}^1(t, \phi), f_{m1}^2(t, \phi), \ldots, f_{mn}^1(t, \phi), f_{mn}^2(t, \phi)),
\]

where \( t \in [0, \omega] \) and \( \phi \in C \), the functional in the right hand side of system (3.39) and by

\[
f(t, \phi) = ((f_{i1}^1(t, \phi), f_{i2}^1(t, \phi), \ldots, f_{in}^1(t, \phi), f_{in}^2(t, \phi)) \]

the one in the right hand side of system (1.2). We also define the set

\[
\Omega_0 = \{(t, (x,y)) \in C : \delta_i \leq x_i, y_i \leq \alpha_i, t \in [0, \omega], m \geq m_0 \},
\]

it is clear from (3.42) that \( \{(t, (x_m(\cdot, \tilde{\phi}_m), y_m(\cdot, \tilde{\phi}_m))) : t \in [0, \omega] \} \subset \Omega_0 \) where \( (x_m(\cdot, \tilde{\phi}_m), y_m(\cdot, \tilde{\phi}_m)) \) are the positive \( \omega \)-periodic solutions of system (3.39). Further, we have after use of (3.42), for any \( (t, (x,y)) \in \Omega_0, t \in [0, \omega], m \geq m_0, i = 1, \ldots, n \)

\[
|f_{mi}^1(t, X) - f_{mi}^1(t, X)| = \left| x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{-\infty}^{0} g_{ij}^+(s) x_j(s + t) \, ds + x_i(t) \sum_{j=1}^{n} k_{ij}(t) \int_{-\infty}^{0} g_{ij}^-(s) x_j(s + t) \, ds \right|
\]

\[
\leq |x_i| \sum_{j=1}^{n} k_{ij}^m |x_j| \int_{-\infty}^{0} g_{ij}^+(s) \, ds + |x_i| \sum_{j=1}^{n} k_{ij}(t) |x_j| \int_{-\infty}^{0} g_{ij}^-(s) \, ds \to 0
\]
and
\[ |f_m^2(t, X) - f_n^2(t, X)| \to 0, \]
as \( m \to +\infty \), where \( X = (x, y) \). This means that \( f_m(t, \psi) \to f(t, \psi) \) uniformly as \( m \to +\infty \) on \( \Omega_0 \). Taking into account the fact that the coefficients of system (3.39) are bounded uniformly in \( m \), and by (3.42) we infer that there is a constant \( H > 0 \), independent on \( m \) such that
\[
\left| \frac{dx_m}{dt}(t, \tilde{\phi}_m), \frac{dy_m}{dt}(t, \tilde{\phi}_m) \right| \leq H, \quad t \in [0, \omega].
\]
The sequence of functions \((x_m(\cdot, \tilde{\phi}_m), y_m(\cdot, \tilde{\phi}_m))_{m \geq 1}\) is then equicontinuous on the compact interval \([0, \omega]\). By Ascoli-Arzela theorem we can find a subsequence \((x_{m_k}(\cdot, \phi_{m_k}), y_{m_k}(\cdot, \phi_{m_k}))_{k \geq 1}\) and a continuous function \((x^*(t), y^*(t))\) such that
\[
x_{m_k} \to x^*, y_{m_k} \to y^* \text{ uniformly as } k \to +\infty \quad \text{in } [0, \omega].
\]
The relation (3.41) allows us to derive a continuous and \( \omega \)-periodic function \( \phi^* = (\varphi^*, \xi^*) \) in \((-\infty, 0]\) such that
\[
\tilde{\phi}_{m_k} \to \phi^* \text{ uniformly as } k \to +\infty \quad \text{in } [-\omega, 0].
\]
we define the positive \( \omega \)-periodic function
\[
(x(t, \phi^*), y(t, \phi^*)) := \begin{cases} (x^*(t), y^*(t)), & t \in [0, \omega], \\ (\varphi^*(s), \xi^*(s)), & s \leq 0. \end{cases}
\]
From (3.43),(3.44) we can see that \((x_{m_k}(t), y_{m_k}(t)) \to (x_t, y_t)\) uniformly on \([0, \omega]\). Applying Lemma 3.2, we conclude that \((x(t, \phi^*), y(t, \phi^*))\) is a positive and \( \omega \)-periodic solution of system (1.2). This completes the proof of the theorem. \( \Box \)

Finally, we give an example to illustrate the feasibility of our main results.

**Example.** Consider the following system which is a particular case of system (1.2)
\[
\begin{align*}
\frac{d^0x(t)}{dt} &= (2 + \sin(t - 0.1))e^{-0.1}x_1(t - 0.1) - 15x_1^2 - (1.2 + \cos 2t)x_1x_2 \\
&\quad - x_1 \int_{-\infty}^0 e^s x_2(t + s) ds, \\
\frac{d^0y(t)}{dt} &= (2 + \cos(t - 0.2))e^{-0.2}x_2(t - 0.2) - 20x_2^2 - (1.3 + \sin 2t)x_1x_2 \\
&\quad - x_2 \int_{-\infty}^0 e^s x_1(t + s) ds.
\end{align*}
\]
(3.45)

It is easy to see that the coefficients of system (3.45) are positive continuous and periodic with period \( 2\pi \). Further, we have the following values
\[
\begin{align*}
da^0_1 &= 1, & da^m_1 &= 1, & da^0_2 &= 1, & da^m_2 &= 1, \\
b^0_1 &= 1, & b^m_1 &= 3, & b^0_2 &= 1, & b^m_2 &= 3, \\
d^0_{a1} &= 15, & da^m_{a1} &= 15, & da^0_{a2} &= 20, & da^m_{a2} &= 20, \\
d^0_{a1} &= 0.2, & da^m_{a1} &= 2.2, & da^0_{a2} &= 0.3, & da^m_{a2} &= 2.3, \\
C^0_{11} &= 0, & C^m_{11} &= 0, & C^0_{21} &= 0, & C^m_{21} &= 0, \\
C^0_{12} &= 0, & C^m_{12} &= 1, & C^0_{22} &= 1, & C^m_{22} &= 0. \\
\gamma_1 &= 0.18, & \gamma_2 &= 0.12.
\end{align*}
\]

It is easy to see that condition (2.3) in Theorems 2.1 and 3.3 is satisfied. If we take as initial data the constant values
\[
x_1(t) = 0.2, \quad x_2(t) = 0.1, \quad -\infty \leq t \leq 0,
\]
then a numerical simulation with MATLAB gives the graphics in Fig. 1. Let us consider now the new initial values
\[
x_1(t) = 0.02, \quad x_2(t) = 0.04, \quad -\infty \leq t \leq 0.
\]
A new simulation gives the graphics in Fig. 2. Figs. 1 and 2 show that the solutions of system (3.45) are uniformly permanent and the system has a positive periodic solution with period $2\pi$.

4. Discussion

In this paper we have established, by using some comparison techniques, sufficient conditions leading to uniform permanence of solutions of the infinite delay system (1.2). These assumptions can be interpreted as follows: If the lower bound of the birth rate of the mature species are sufficiently large compared to the delays coefficients, and if the lower bound of the intraspecific coefficients are large enough compared to the interspecific interaction coefficients, then the solutions of the infinite delay system (1.2) are uniformly permanent. Further, by a suitable use of Horn’s fixed point Theorem we have proved that the system with finite delay has a positive periodic solution. As a consequence of this last result, we have shown that the system with infinite delay has also a positive periodic solution. In [32], Teng and Chen proved that any periodic Kolmogorov system with finite delay which is uniformly permanent has a positive periodic solution. Theorem 3.2 extends this result to stage-structured systems as these systems are not of Komogorov’s type. This paper allows us to conjecture that this result is true for any uniformly permanent system with infinite delay which has a positive invariant semiflow and satisfy the property stated in Lemma 3.1. This work allows us to conclude that:

(i) Periodic stage-structured systems with finite or infinite time delay may have a positive periodic solution.

The existence of the periodic solution is independent of the length of the stage-structure. From a biological point of view, this strongly suggests that:

(ii) When seasonal forces are in action, the effects on stage-structured population models often lead to synchronous solutions regardless of the time delay (finite or infinite).

Finally, it seems to us that an interesting and probably challenging problem associated with system (1.2) would be to study the uniqueness and global attractivity of the positive periodic solution. We leave this for a future work.

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