On the demand pooling anomaly in inventory theory

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\textbf{A B S T R A C T}

The demand pooling anomaly of inventory theory of type F amounts to a kind of restricted order relation between the individual demands (assumed to be independent) and their average. In this paper, we present some sufficient conditions for the type F anomaly not to occur for two i.i.d. demands; furthermore we provide an asymptotic result showing whether this anomaly occurs for large \( n \) for a class of distributions containing all distributions with finite mean.

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1. Introduction

In this paper, we consider the effect of pooling on the optimal inventory levels, i.e., the relation existing between optimal stocks with aggregated random demands and the sum of optimal stocks with separate demands. It is well-known that, in contrast to the immediate intuition, a reduction in stock does not always occur and also that the difference between the stocks is often not monotone in the cost parameters (cf. [1] and the references cited therein). Define the vendor ratio by

\[ \frac{p}{c_o/c_g} = \frac{p}{c_o/c_g}, \]

where \( c_o \) and \( c_g \) are the respective unit underage and overage costs, and consider two independent random demands with respective distribution functions \( F \) and \( G \). The well-known optimal solutions to the expected-cost minimizing single period problems are to stock \( F^{-1}(p) \) and \( G^{-1}(p) \) units, respectively, whereas the solution to the aggregate demand problem is to stock \( (F \ast G)^{-1}(p) \), so that the difference of interest is

\[ D(p) = (F \ast G)^{-1}(p) - \left[ F^{-1}(p) + G^{-1}(p) \right]. \quad (1.1) \]

In [1] and in numerous other references (a comprehensive list can be found in [3]), it is shown that \( D(p) \) can have a variety of forms. There have been several attempts to explore the analytical behavior of \( D(p) \) for different types of demand distributions \( F \) and \( G \) and to extend the results to more than two distributions.

Yang and Schrage [3] introduced the term 'inventory anomaly' for the phenomenon of an inventory increase (rather than decrease) due to pooling. They actually considered two types of inventory anomaly, called 'Type F' and 'Type P'. Type F refers to the case of full substitution in which consumers demonstrate neutral preference of two (or more) alternative products so that all demand types can be pooled into a single demand type, whereas Type P is based on partial substitution, meaning that the consumers have a biased preference (for more details see [3]).

Generally speaking, the problem of demand pooling concerns the relationship between the individual demands \( X_i, \ i = 1, 2, \ldots, n \), again for simplicity assumed to be i.i.d., and their sum \( S_n = X_1 + \cdots + X_n \). For example, only considering the case that the underage cost \( c_o \) is at least as large as the overage cost \( c_g \), per unit (or equivalently, that the vendor ratio \( p \) satisfies \( p \geq 1/2 \)), Yang and Schrage [3] showed that the presence of a full ('Type F') anomaly can be characterized as follows: there exists an \( x_0 \) satisfying

\[ F_n(x_0) - F(x_0) < 0 \quad \text{and} \quad F(x_0) \geq 1/2. \quad (1.2) \]

Here \( F \) and \( F_n \) denote the distribution functions of \( X_1 \) and \( S_n \), respectively. If (1.2) holds we say that the Type F anomaly occurs at \( x_0 \). The first condition in (1.2) states that \( \mathbb{P}(S_n/n \leq x_0) < \mathbb{P}(X_1 \leq x_0) \). Accordingly, no Type F anomaly occurs if and only if:

for all \( x \) satisfying

\[ \mathbb{P}(X_1 \leq x) \geq 1/2 \quad \text{we have} \quad \mathbb{P}(S_n/n \leq x) \geq \mathbb{P}(X_1 \leq x). \]

This is a kind of restricted stochastic order relation between the average \( S_n/n \) and an individual demand \( X_i \). We present sufficient conditions for the Type F anomaly not to occur at large values of \( x_0 \) for \( n = 2 \); this is the case if \( F \) has certain exponential tails (in a sense made precise below). Furthermore we provide an asymptotic result showing whether this anomaly occurs for large \( n \) for a class of distributions containing all distributions with finite mean.
mean is larger than the median, the anomaly occurs for large \( n \); in the opposite case for every compact interval to the right of the median there is an \( n_0 \) such that the anomaly does not occur at any \( x_0 \) in this interval for all \( n \geq n_0 \).

2. Two theorems on the Type F anomaly

We start with a result on \( D(p) \) defined in (1.1) in the case \( F = G \).

**Lemma 1.** Let \( p = F(x_0), \ 0 < p < 1 \); then the following two conditions are equivalent:

\[
D(p) = (F * F)^{-1}(p) - 2F^{-1}(p) < 0
\]

and

\[
(F * F)(2x_0) - F(x_0) > 0.
\]

**Proof.** \((F * F)^{-1}(p) < 2F^{-1}(p) \Rightarrow (F * F)(F * F)^{-1}(p) < (F * F)(2x_0)\). \( \square \)

**Theorem 1.** Assume that the tails of \( F \) are of the form

\[
1 - F(x) = e^{-\alpha x}h(x)
\]

for some \( \alpha > 0 \), where \( h \) grows at most subexponentially, i.e.,

\[
\lim_{x \to \infty} h(x)e^{-\alpha x} = 0 \quad \text{for all} \ \epsilon > 0
\]

and does not decrease too fast in the following sense:

\[
\lim_{x \to \infty} e^{\epsilon x}h(x) = \infty \quad \text{for some} \ \epsilon \in (0, r).
\]

Then \( D(p) < 0 \) for all \( p \in (p_0, 1) \) for some \( p_0 < 1 \).

**Proof.** By (2.2) it is sufficient to show that

\[
(F * F)(2x) - F(x) > 0
\]

for all large \( x \). According to Theorem 7.2.1 in [2], p. 198, the characteristic function \( \varphi(z) \) of \( F \) is analytic if and only if there is an \( R \in (0, \infty) \) such that

\[
1 - F(x) = O(e^{-\alpha x}), \quad x \to \infty
\]

for all \( s \in (0, R) \), and in this case the strip of regularity contains \( |\text{Im}z| < R \). Thus it follows from (2.3) to (2.4) that \( \varphi(z) \) is analytic for \( |\text{Im}z| < \epsilon \) so that \( \varphi^2(2z), \epsilon \) the characteristic function of \((F * F)(2x))\) is analytic for \( |\text{Im}z| < 2\epsilon \). Hence, again by Theorem 7.2.1 in [2],

\[
1 - (F * F)(2x) = O(e^{-2\alpha x}), \quad x \to \infty
\]

for every \( s \in (0, \epsilon) \). Choose \( s > \epsilon + u/2 \). Then \( 2s - \epsilon > u \) and it follows from (2.5) that

\[
(F * F)(2x) - F(x) = (1 - F(x)) - (1 - (F * F)(2x))
\]

\[
\geq h(x)e^{-\alpha x} - O(e^{-2\alpha x})
\]

\[
= e^{-2\alpha x}[e^{2\epsilon x/3}\alpha x]h(x) - O(1)] > 0
\]

for sufficiently large \( x \), proving the claim. \( \square \)

**Remarks.** (1) All densities of the form \( f(x) = e^{-\alpha x}h(x) \) as \( x \to \infty \) satisfy the conditions of Theorem 1; for instance \( h \) could be a polynomial or a rational function. To prove this, let \( g(x) = f(x)e^{\alpha x} \). Then

\[
h(x) = e^{\alpha x}(1 - F(x)) = e^{\alpha x} \int_{x}^{\infty} e^{-u}g(u) \, du
\]

\[
= \int_{x}^{\infty} e^{-u}g(t + x) \, dt.
\]

Fix \( \epsilon \in (0, r) \). Then for sufficiently large \( x \),

\[
h(x)e^{-\epsilon x} \leq e^{-\epsilon x} \int_{0}^{\infty} e^{-t}e^{\epsilon x/(t + x)} \, dt
\]

\[
= (r - \epsilon/2) e^{-\epsilon x/2} \to 0, \quad \text{as} \ x \to \infty.
\]

Moreover, for \( u \in (0, r) \) and \( \epsilon \in (0, u) \),

\[
h(x)e^{\epsilon u} \geq e^{\epsilon u} \int_{0}^{\infty} e^{-t}e^{\epsilon x/(t + x)} \, dt
\]

\[
= (r + \epsilon) e^{-\epsilon x/2} \to \infty, \quad \text{as} \ x \to \infty.
\]

(2) We note that \( D(p) < 0 \) for all \( p \in (0, 1) \) cannot happen because this would mean that \( S_2/2 \) is stochastically smaller than \( X_1 \). If \( \mathbb{E}(X_1) \geq \zeta \), this would contradict \( \mathbb{E}(S_2/2) = \mathbb{E}(X_1) \). If \( \mathbb{E}(X_1) = \infty \), a simple argument communicated to us by Daoud Bshouy is as follows. Let \( \ell \) be the Laplace–Stieltjes transform (LST) of \( F \). Then \( S_{2/2} \) being stochastically smaller than \( X_1 \) would imply that

\[
\ell(t)^2 > \ell(t) \quad \text{for all} \ t > 0.
\]

However, by the Cauchy–Schwarz inequality we have

\[
1 - 2\ell(t/2) + (\ell(t/2))^2 = (1 - \ell(t/2))^2
\]

\[
\leq \left( \int_{0}^{\infty} (1 - e^{-\epsilon t}) \, df(t) \right)^2
\]

\[
\leq \int_{0}^{\infty} (1 - e^{-\epsilon t})^2 \, df(t) \int_{0}^{\infty} \, df(t)
\]

\[
= \int_{0}^{\infty} (1 - 2e^{-\epsilon t/2} + e^{-\epsilon t}) \, df(t)
\]

\[
= 1 - 2\ell(t/2) + \ell(t),
\]

yielding a contradiction. \( \square \)

The following two examples present classes of distributions for which the type F anomaly does or does not occur.

**Examples.** (1) Let us drop the nonnegativity assumption and consider general real-valued \( X_1 \). As mentioned in [3], for symmetric stable distributions with index \( \alpha \in (1, 2] \) the Type F anomaly does not occur. Indeed, in this case \( F(x) \) is \( 1/2 \) is equivalent to \( x \to 0 \) and as \( S_0 = \sum_{i=1}^{n} X_i \), we have

\[
\mathbb{P}(S_n/n \leq x) = \mathbb{P}(X_1 \leq n^{1/(\alpha - 1)/2} x) \geq \mathbb{P}(X_1 \leq x)
\]

(2.6)

for all \( x \geq 0 \). For an index \( \alpha \in (0, 1) \) the inequality in (2.6) has to be reversed and is strict for \( x > 0 \) so that the Type F anomaly occurs.

(2) Consider an arbitrary distribution \( F \) with a continuous density \( f \) which is symmetric around some \( x_0 \in \mathbb{R} \). Then the distribution \( F_n \) is symmetric around \( n x_0 \) and we have \( F_n(x_0) = 1/2, F_n(x_0) - F(x_0) = 0 \). Clearly, in this situation a sufficient condition for the Type F anomaly to occur is \( G_n(x_0) < 0 \) for \( G_n(x) = F_n(nx) - F(x) \), since in this case \( F_n(nx) - F(x) < 0 \) in a right neighborhood of \( 1/2 \). In terms of densities, this means that

\[
f(x_0) > n f_0(nx_0)
\]

where \( f_0 \) is the density of \( F_n \). However, this inequality seems hard to check in examples. \( \square \)

For \( n \) sufficiently large the question of whether the anomaly occurs can be answered for a large class of distributions. Let \( F \) be the set of all distributions \( F \) for which \( S_n/n \) converges in probability to some constant \( \mu F \in \mathbb{R} \) as \( n \to \infty \). For example, \( F \) contains all \( F \) whose expected value exists (this will be the \( \mu F \)) or, slightly more generally, whose characteristic function is differentiable at 0.

Define \( m_F = \inf x \in \mathbb{R} \mid F(x) \geq 1/2 \), i.e., \( m_F \) is the smallest median of \( F \).
Theorem 2. Let $F \in \mathcal{F}$. Then the Type F anomaly occurs for all sufficiently large $n$ if $m_F < \mu_F$. If $m_F \geq \mu_F$, then for every $u_0 > m_F$ there is an $n_0$ such that for all $n \geq n_0$ the anomaly does not occur at any $x$ in the interval $[m_F, u_0]$.

Proof. (i) First let $m_F < \mu_F$. Then fix any $x_0 \in [m_F, \mu_F)$, so that $1/2 \leq P(X_1 \leq x_0)$. As $P(S_n/n \leq x) \to 0$ for every $x < \mu_F$, it follows that $F_n(nx_0) - F_1(x_0) \to -F(x_0) < 0$. Thus the anomaly occurs for large $n$.

(ii) Now let $u_0 > m_F > \mu_F$. Then $P(S_n/n \leq m_F) \to 1$ so that for large $n$

$$\inf_{x \in [m_F, u_0]} (F_n(nx) - F_1(x)) \geq P(S_n/n \leq m_F) - P(X_1 \leq u_0) \to 1 - F(u_0) \geq 0.$$

Hence, for large $n$ the anomaly does not occur in $[m_F, u_0]$.

(iii) Finally, if $u_0 > \mu_F$, one can find a sequence $G^k \in \mathcal{F}$ of distribution functions converging to $F$ pointwise and satisfying $m_{G^k} = m_F$ and $\mu_{G^k} < \mu_F$ for all $k$. Then from (ii), again for large $n$,

$$\inf_{x \in [m_F, u_0]} (F_n(nx) - F_1(x)) = \lim_{k \to \infty} \inf_{x \in [m_F, u_0]} (G^k_n(nx) - G^k_1(x)) \geq 0. \quad \square$$

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