Lexicographic max-min approach for an integrated vendor-managed inventory problem

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Abstract

Simultaneous reductions in inventory of raw materials, work-in-process, and finished items have recently become a major focus in supply chain management. Vendor-managed inventory is a well-known practice in supply chain collaborations, in which manufacturer manages inventory at the retailer and decides about the time and replenishment. In this paper, an integrated vendor-managed inventory model is presented for a two-level supply chain structured as a single capacitated manufacturer at the first level and multiple retailers at the second level. Manufacturer produces different products where demands are assumed decreasing functions of retail prices. In this chain, both the manufacturer and retailers contribute to determine their own decision variables in order to maximize their benefits. While previous research on this topic mainly included a single objective optimization model where the objective was either to minimize total supply chain costs or to maximize total supply chain benefits, in this research a fair profit contract is designed for the manufacturer and the retailers. The problem is first formulated into a bi-objective non-linear mathematical model and then the lexicographic max-min approach is utilized to obtain a fair non-dominated solution. Finally, different test problems are investigated in order to demonstrate the applicability of the proposed methodology and to evaluate the solution obtained.

Keywords: Supply chain management; Vendor managed inventory; Fair profit contract; Lexicographic max-min approach

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1. Introduction

Vendor-managed inventory (VMI) is a well-known practice for supply chain collaboration, in which manufacturer manages inventory at the retailer and decides when and how much to replenish. In recent years, there has been an increasing interest in cooperative and non-cooperative relationship between both manufacturer and retailers in the VMI program. Although the benefits of VMI to the retailer include reduction of overhead costs and, if consignment stock is adopted, transfer of inventory costs to the manufacturer, the benefits of VMI to manufacturer are not very straightforward (Lee & Ren, 2011). Meanwhile, researches mainly focus on the following three aspects of VMI programs (Guan & Zhao, 2010):

1. Investigating the benefits of VMI programs compared with normal supply modes without VMI;
2. Operational decisions in VMI programs;
3. Designing contracts for VMI programs

The literature related to this paper can be classified into those of the second category.

Yao, et al. (2007) introduced a model to explore the effects of cooperative supply chain initiatives such as VMI, first developed by Vlist, et al. (2007). In this issue, the authors showed that when the shipment sizes from a supplier to a buyer increase, inventory at the supplier goes down and inventory at the buyer goes up. Zhang, et al. (2007) presented an integrated VMI model for a single-vendor multiple-buyer supply chain problem, where the vendor first purchases and processes raw materials and then delivers finished items to multiple buyers. Investment decision, constant production, and demand were considered where the buyers’ ordering cycles might be different and that each buyer could replenish more than once in one production cycle.

Impact of the consignment inventory (CI) and VMI policies was studied by Gümüş, et al. (2008). The goal was to analyze CI in a two-party supply chain under deterministic demand and to provide some general conditions under which CI creates benefits for the vendor, for the customer,
and for the two parties together. Sari (2008) presented a comprehensive simulation model representing two popular supply chain initiatives, collaborative planning forecasting replenishment (CPFR) and VMI, in order to select an appropriate collaboration mode in business conditions. Their results showed that benefits of CPFR are always higher than VMI. Besides, an integrated production-inventory model was developed by Zavanella & Zanoni (2009), in which a particular VMI policy known as consignment stock (CS) for both the buyer and the supplier was investigated. Yu, et al. (2009) showed how the vendor can take into account the advantage of his information for increasing his own profit by using a Stackelberg game in a VMI system. Yu, et al. (2009) showed how to analyze the intrinsic evolutionary mechanism of the VMI supply chains by applying the evolutionary game theories. Darwish and Odah (2010) developed a model for a supply chain with single vendor and multiple retailers based on VMI, considering capacity constraints by selecting high penalty cost. Almehdawe & Mantin (2010) studied supply chains composed of a single capacitated manufacturer and multiple retailers. They formulated a Stackelberg game VMI framework under two scenarios: in the first, the manufacturer is the leader; in the second, one of the retailers acts as the dominant player of the supply chain. In addition, market demand was considered a function of retail price. This model was also extended by Yu, et al. (2009) when advertising investment and pricing come to the picture.

The quaternary policy towards integrated logistics and inventory aspect of the supply chain was proposed by Arora, et al. (2010). They considered a supply chain with multiple retailers and distributors, in which all distributors follow a unique policy and the VMI system is used for updating the inventory of the retailers. Yang, et al. (2010) studied the effects of the distribution centre (DC) in a VMI system comprising one manufacturer, one DC, and \( n \) retailers where the system aims to maximize the overall system profit. While Lee & Ren (2011) showed the supply chain total cost decreases under VMI, the reduction is larger when there is exchange rate uncertainty compared with the case of no exchange rate uncertainty. They considered a state-
dependent \((s, S)\) policy for the supplier. Pasandideh, et al. (2011) presented a multi-product multi-constraint economic order quantity (EOQ) model under the VMI policy for a supply chain. They developed a genetic algorithm to find the best order quantities and the maximum backorder levels such that the total inventory cost of the supply chain is minimized.

A logistics network design under VMI by considering location, transportation, pricing, and warehouse-retailer inventory replenishment decisions was presented by Shu, et al. (2012). Zanoni, et al. (2012) provided a two-level supply chain model for a single-vendor single-buyer at each level and compared different policies that the vendor might adopt to exploit the advantages offered by the VMI with consignment agreement when the vendor’s production process is subject to learning effects.

To summarize, many research works in supply chain environment assume a non-cooperative relation (such as the one in the Stackelberg game) between the manufacturer and the retailers with the manufacturer acting as the leader and the retailers as the followers (Almehdawe & Mantin 2010). In addition, most of the literature on the VMI problem only aim to optimize manufacturer's objectives and do not pay attention to retailers' objectives (Yu, et al. 2009; Almehdawe & Mantin 2010). Moreover, there has been little discussion about designing fair contracts in VMI problems so far. Besides, previous research works on this topic mainly included a single objective optimization model where the objective was either to minimize the total cost or to maximize the total benefit. However, this paper presents a two-level supply chain model by assuming a single capacitated manufacturer at the first level and multiple retailers at the second level. This chain is considered integration between the manufacturer and retailers where the manufacturer (vendor) produces multiple products, sells to retailers, and manages the retailers’ inventories under VMI. A fair profit contract between the manufacturer and his retailers is adopted in this research. Our motivation of defining a fair profit contract is that both the manufacturer and retailers are able to contribute to determine their optimal decision variables in order to maximize their benefits. In other words, the
manufacturer and his retailers maximize their benefits as close to one another as possible. Besides, the demand rate for each product in each local retail market is assumed a decreasing function of the retail price called the Cobb–Douglas demand function. Finally, this paper formulates the problem into a non-linear mathematical model with two-objectives in order to maximize both the manufacturer and retailers' benefit. It is also assumed that both the objectives are equally important and that it is a need to find a “fair” non-dominated solution by the lexicographic max-min approach. A fair non-dominated solution is a solution with all normalized objective function values as equal as possible. Following Erkut, et al. (2008), we discuss the conversion of the original lexicographic max-min problem to a lexicographic maximization problem without using the dual formulation of the LP problem.

The reminder of this paper is organized as follows. Section 2 contains problem description. The mathematical formulation of the problem is given in Section 3. Section 4 discusses the lexicographic max-min approach to solve the problem. The applicability and the performances of the proposed method are demonstrated in Section 5 using some numerical examples. Moreover, sensitivity analyses on the effects of some input parameters on the objective functions are performed in this section. Finally, we conclude the paper with a discussion of possible further research in Section 6.

2. Problem Description

Consider a two-level supply chain consisting of a single manufacturer at the first level and multiple retailers at the second. The manufacturer’s capacity is finite in producing different products with a fixed production rate. He sells the products to its retailers with a common replenishment cycle. A common replenishment cycle eliminates the influence of the variation of the replenishment cycle as well as backorder rate of every retailer. The manufacturer must sell the products to his retailers at different wholesale prices. Besides, the manufacturer and retailers are
operating in distinctive markets with no conflict of interests. Integration is established between the manufacturer and all retailers, in which manufacturer manages inventory at all levels by having access to retailers' inventory as well as his own (i.e. VMI). Moreover, each retailer pays to the manufacturer a cost of $\xi_{tec}$ per unit consumed per time unit to have his inventory managed by the manufacturer. The manufacturer’s decides on his replenishment cycle of the finished products, wholesale prices, and fraction of backlogging. Retailers’ decisions include their retail prices.

2.1. Assumptions

The followings are assumed in this paper:

1. The demand for every retailer and every product is constant over time.
2. The demand function for all retailers and all products is a convex function with respect to its retail price (see the last paragraph of page 7 for more details.)
3. Lead-time of each product at each level of the supply chain is assumed negligible compared to the common replenishment cycle time.
4. The production setup cost occurs at the beginning of each common replenishment cycle.
5. The setup cost is realized once in every replenishment cycle.
6. Planning horizon is infinite.

2.2. Indices, Parameters, and Decision Variables

The following indices, parameters, and decision variables are used throughout the paper:

Indices

- $c$: Index for retailers ($c = 1, 2, \ldots, n$)
- $i$: Index for product types ($i = 1, 2, \ldots, I$)
Input Parameters

$D_{ic}$  Retailer $c$’s demand for product $i$

$\xi_{ic}$  Inventory management cost of product $i$ for retailer $c$ ($/\text{unit/time}$)

$cm$  Production cost per unit for finished product ($/\text{unit}$)

$\Phi$  Transportation cost per unit of finished product shipped from the manufacturer to a retailer ($/\text{unit}$)

$r$  Production rate of finished products

$S_i$  Setup cost for the common cycle time for product $i$ ($)$

$SR_c$  Fixed order cost paid by the manufacturer to retailer $c$ ($$)

$\pi_{ic}$  Backorder cost paid by the manufacturer to retailer $c$ for product $i$ ($/\text{unit/time}$)

$H_i$  Holding cost at the manufacturer’s side ($/\text{unit/time}$)

$h_{ic}$  Holding cost paid by the manufacturer at retailer $c$’s side for product $i$ ($/\text{unit/time}$)

Decision Variables

$w_{ic}$  Wholesale price of finished product $i$ set by the manufacturer for retailer $c$ ($/\text{unit}$)

$P_{ic}$  Retail price charged by retailer $c$ for product $i$ ($/\text{unit}$)

$b_{ic}$  Fraction of backlogging rate of product $i$ in common replenishment cycle time for retailer $c$

$C_i$  Common replenishment cycle time for finished product $i$

About the second assumption stated in page 6, the demand faced by each retailer for each product is assumed to follow the Cobb–Douglas demand function characterized by a constant elasticity demand function of the form given in Eq. (1).

$$D_{ic} = k_c p_{ic}^{-\epsilon_c} \quad \forall i = 1, \ldots, l , \quad c = 1, \ldots, n$$  (1)
In which $k_c$ and $e_c > 1$ represent the market scale of retailer $c$ and the demand elasticity of retailer $c$ with respect to its retail price, respectively (see Almehdawi & Mantin, 2010 for more details).

3. Mathematical Model

The integrated manufacturer-retailers model can be formulated as follows:

$$\max z_1 = \sum_{i=1}^{I} \sum_{c=1}^{n} D_{ic} (w_{ic} - cm - \Phi) - \sum_{i=1}^{I} \left[ \frac{S_i}{C_i} \right] - \sum_{i=1}^{I} H_i \left[ \sum_{c=1}^{n} \left( \frac{D_{ic}^2 C_i^2}{2rC_i} \right) \right] - TC_{VM}, \quad (2)$$

$$\max z_2 = \sum_{i=1}^{I} \sum_{c=1}^{n} D_{ic} (p_{ic} - w_{ic} - \xi_{ic}), \quad (3)$$

s.t.:

$$TC_{VM} = \sum_{i=1}^{I} \sum_{c=1}^{n} \left[ S_i C_i \right] + \sum_{i=1}^{I} \sum_{c=1}^{n} h_{ic} \left( \frac{D_{ic} (1 - b_{ic})^2 C_i}{2} \right) + \sum_{i=1}^{I} \sum_{c=1}^{n} \pi_{ic} \left( \frac{D_{ic} b_{ic}^2 C_i}{2} \right) - \sum_{i=1}^{I} \sum_{c=1}^{n} \xi_{ic} D_{ic}, \quad (4)$$

$$\sum_{i=1}^{I} \sum_{c=1}^{n} D_{ic} \leq r, \quad (5)$$

$$p_{ic} > w_{ic} + \xi_{ic}, \quad \forall i = 1, ..., I, \quad c = 1, ..., n, \quad (6)$$

$$0 \leq b_{ic} \leq 1, \quad \forall i = 1, ..., I, \quad c = 1, ..., n, \quad (7)$$

$$C_i, w_{ic}, p_{ic} \geq 0, \quad \forall i = 1, ..., I, \quad c = 1, ..., n, \quad (8)$$

It is clear that the above formulation is nonlinear with two conflicting objective functions. The first objective function ($z_1$) that is given in Eq. (2) is the net profit of the manufacturer obtained by the revenue from sale of finished products to retailers at wholesale prices minus the costs including production, transportation, setup, holding, and $TC_{VM}$. The second objective function ($z_2$) that is given in Eq. (3) shows the net profit of all retailers. Eq. (4) represents $TC_{VM}$ that is defined as the total inventory cost incurred by the manufacturer to manage all retailers’ inventory.
inventory costs at each retailer’s side are the fixed inventory costs, variable inventory costs, and back-ordering costs. Inequality (5) insures that total demand faced by the manufacturer does not exceed his production capacity. Inequalities (6) show the least acceptable prices in order to assure positive net profits for all retailers. Inequalities (7) are to set limits for the fraction of backlogging rates and inequalities (8) guarantee non-negative values for all decision variables. Fig. 1 shows how one can obtain average inventories in order to derive holding and backorder costs of retailers. Besides, Fig. 2 shows the total inventory of the manufacturer for product \( i \) in a common replenishment cycle (Yu et al., 2009).

![Fig. 1. Inventory level of retailer \( c \) for product \( i \) per common replenishment cycle](image1.png)

![Fig. 2. Inventory level of the manufacturer for product \( i \) per common replenishment cycle](image2.png)

This model was originally proposed in Almehdawe and Mantin, (2010), where they formulated a Stackelberg game VMI framework with a single objective and only one product.
However, a bi-objective optimization model is derived in the current model for several products, in which $TC_{f_{M}}$ represents the total cost paid by the manufacturer to manage all retailers’ inventory. It consists of the difference between all the inventory costs he realizes and the revenue he receives from the retailers to manage their inventory.

4. The Lexicographic Max-Min Approach

A brief discussion on the lexicographic max-min (LMM) as a refinement of the standard max-min approach along with the rationale behind its use for the integrated vendor managed inventory problem at hand is presented in this section. This approach can be utilized to simultaneously maximize the smallest manufacturer's profit and the smallest retailers' profit as close to one another as possible. Taking advantage of this approach enables one to design a fair profit contract between the manufacturer and the retailers, i.e., this approach can create a good win-win scenario for both the manufacture and its retailers in the supply chain. LMM is adopted in this research because it provides solutions satisfying fairness and efficiency properties (Salles & Barria, 2008). Another advantage of the LMM approach is that it usually works on a lower dimension than the domain set, which may simplify the analysis (Salles and Barria, 2008).

The lexicographic max-min method that first was introduced by Dresher (1961) was later refined to the formal nucleolus definition by Schmeidler (1969). It has been applied for multi-period resource allocation (Klein et al., 1992), linear multiple criteria problems (Marchi & Oviedo, 1992), fair bandwidth allocation in computer networks (Salles & Barria, 2008), water resource allocation (Wang et al., 2008), and waste management (Erkut et al., 2008). Moreover, the LMM solution is known in the game theory as the nucleolus of a matrix game (Marchi and Oviedo, 1992). It is also called lexicographic max-ordering (Ehrgott, 1996) and lexicographic centers in location (Ogryczak, 1998).
The standard lexicographic approach to find max-min solutions selects a unique set of outcomes which may be a non-unique solution in the decision space but all the solutions have exactly the same distribution of outcomes (Ogryczak, 1997). Although the lexicographic leads to a Pareto efficient solution (Miettinen, 1999), this solution is not fair since it is first necessary to establish a strict precedence among all utility functions (Salles and Barria, 2008). Therefore, this simple lexicographic approach is not applicable to our research. That is why the LMM approach is taken in this research to find a fair solution for the integrated manufacturer-retailers problem modeled in Section 3.

Let \( f_i(x)(i = 1, 2, ..., P): D \rightarrow R \) be an objective function to be optimized, where \( x \) is a feasible solution and \( D \) is a set of feasible solutions. A multi-objective optimization (maximization for instance) problem, is formulated as follows:

\[
\text{Max } f_1(x), f_2(x), \ldots, f_p(x), \\
\text{ s.t.:} \\
x \in D,
\]

In these problems, a set of solutions called Pareto optimal is desired. To find the Pareto optimal solution the following order relation is first defined:

\[
x > y \iff f_i(x) \geq f_i(y), \ (\forall i = 1, 2, ..., P) \land f_i(x) > f_i(y), \ (\exists i = 1, 2, ..., P)
\]

If the above relation holds between \( x \) and \( y \), the solution \( x \) is said to dominate \( y \). Then, the Pareto optimal solution is defined as follows: A solution \( x^* \in D \) is said to be a Pareto optimal solution if there is no solution \( x \in D \) such that \( x^* < x \) (Kubotani & Yoshimura, 2003). The concept of a “fair” efficient solution is a refinement of the Pareto optimality. The fair solution is a solution with all normalized objective function values as close to one another as possible (Ogryczak, 1997). An alternative approach depends on the so-called the max-min solution concept, where the worst performance is maximized:
\[
\text{Max}\{\min_{i=1,2,\ldots,p} f_i(x) : x \in Q\} \tag{11}
\]

It should be noted that the optimal solution set of the max-min problem (11) always contains an efficient solution of the original multi-objective problem given in (9). The max-min solution concept depends on optimization of the worst outcome, and it is regarded as maintaining equity as described by the following theorem (Włodzimierz et al., 2005).

**Theorem 1:** If exists a non-dominated outcome vector \( \bar{y} \in Y \) with perfect equity \( \bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_p \) then \( \bar{y} \) is the unique optimal fair solution of the max-min problem.

\[
\text{Max}\{\min_{i=1,2,\ldots,p} y_i : y \in Y\} \tag{12}
\]

Note that the standard max-min approach depends on minimization of \( \bar{y}_1 \) and it ignores \( \bar{y}_j \) for \( j \geq 2 \). It is a reason why the Standard max-mix approach is, in general, too crude to satisfy the Pareto optimality principle (Ogryczak, 1997). Therefore, we solve a LMM problem as a refinement of this max-min problem.

Let \( \Theta(\alpha) = (a_{(1)}, a_{(2)}, \ldots, a_{(p)}) \) be a vector obtained from \( \alpha \) by rearranging its components in non-decreasing order and \( \Theta: \mathbb{R}^p \rightarrow \mathbb{R}^p \) a map, which orders the components of vectors in a non-decreasing order. That means \( a_{(1)} \leq a_{(2)} \leq \ldots \leq a_{(p)} \) where \( a_{(i)} \) is the \( i \) th component of \( \Theta(\alpha) \). Comparing lexicographically such ordered vectors \( \langle f \rangle \) one gets the so-called lex-max order. Therefore, LMM problem is:

\[
\text{lex max} \ \{\Theta(f)\} = \text{lex max} \ \{\langle f_{(1)}, f_{(2)}, \ldots, f_{(p)} \rangle : f \in A\}, \tag{13}
\]

where \( A = \{f : f = f(x), x \in D\} \).

**Theorem 2:** An optimal solution of the problem (13) is also the optimal solution of the problem (12).
In problem (13), in addition to maximize the worst (smallest) outcome, we also maximize the second smallest outcome (provided that the smallest one remains as large as possible), maximize the third smallest (provided that the two smallest remain as large as possible), and so on. The LMM solutions satisfy the principles of Pareto-optimality (efficiency) and perfect equity as described by the following theorem (Erkut et al., 2008).

**Theorem 3:** \( x^* \in D \) is Pareto-optimal with perfect equity if \( f_1(x^*) = f_2(x^*) = \cdots = f_p(x^*) \), it is an optimal fair solution of the optimization problem (13).

However, problem (13) is not a standard mathematical program. In what follows, we describe an approach to transfer the LMM problem (13) to a lexicographic maximization problem. Let \( \eta_i(y) = \sum_{j=1}^{i} y_{(j)} \) be a cumulated criteria expressing, respectively the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc (Włodzimierz Ogryczak et al., 2005). Therefore, for any given vector \( f \), the cumulated ordered value \( \eta_i(f) \) can be found as the optimal value of the following LP problem:

\[
\eta_i(f) = \max \sum_{j=1}^{P} f_j \alpha_{ij},
\]

s.t.:

\[
\sum_{j=1}^{P} \alpha_{ij} = i, \quad \forall j = 1, \ldots, P;
\]

\[
a_{ij} \in \{0, 1\} \quad : \quad \forall j = 1, 2, \ldots, P
\]

where \( \alpha_{ij} \) is a binary variable and can be relaxed to a continuous variable, i.e., \( 0 \leq \alpha_{ij} \leq 1 \). Therefore, by using the above model, lexicographic max-min problem converts to a lexicographic maximization problem. By solving the above model, one can find a fair non-dominated solution for
the integrated manufacturer-retailers problem at hand. Numerical examples are provided in the next section to demonstrate this approach.

5. Performance Evaluation and Comparison

The aim of this section is to demonstrate the applicability and to assess the performances of the proposed model using hypothetical examples having randomly generated data. These examples are solved using GAMS 23.5 software and making use of the CONOPT3 solver on an Intel(R), core (TM) i7, 3.23 GHz lap top with 512 Mb RAM.

Consider a supply chain consisting of a single manufacturer and three retailers. The manufacturer produces two products. The retailers are different in terms of market size, characterized by the market scale \( k_c \), and customer’s sensitivity to price changes, characterized by the price elasticity \( e_c \). Random examples are generated according to the information provided in Table 1, where the term “\( U \)” implies a uniform distribution. These ranges are selected based on the work by Almehdawe and Mantin (2010).

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (k_1, k_2, k_3) )</td>
<td>(3000, 2000, 2000)</td>
</tr>
<tr>
<td>( (e_1, e_2, e_3) )</td>
<td>(1.2, 1.3, 1.5)</td>
</tr>
<tr>
<td>( \xi ic )</td>
<td>( \sim U(1.2 \ , \ 2.4) )</td>
</tr>
<tr>
<td>( cm )</td>
<td>4</td>
</tr>
<tr>
<td>( \phi )</td>
<td>3</td>
</tr>
<tr>
<td>( P )</td>
<td>1000</td>
</tr>
<tr>
<td>( S_i )</td>
<td>( \sim U(10 \ , \ 30) )</td>
</tr>
<tr>
<td>( SR_c )</td>
<td>( \sim U(20 \ , \ 40) )</td>
</tr>
<tr>
<td>( \pi_{ic} )</td>
<td>( \sim U(150 \ , \ 200) )</td>
</tr>
<tr>
<td>( H_i )</td>
<td>( \sim U(2 \ , \ 5) )</td>
</tr>
<tr>
<td>( \hat{h}_{ic} )</td>
<td>( \sim U(0.5 \ , \ 3) )</td>
</tr>
</tbody>
</table>
For a randomly generated problem, the LMM approach is employed to determine which Pareto-optimal solution to be implemented in order to get a fair trade-off between manufacturer and retailers’ profits. Certainly, the lexicographic optimization may also be treated as a sequential (hierarchical) optimization process where first $g_1(x)$ is maximized on the entire feasible set, next $g_2(x)$ is maximized on the optimal set, and so on. This may be implemented as shown in the following standard sequential algorithm with predefined objective functions

$$\text{lex max}\{g_1(x), ..., g_m(x): x \in D\}$$

in which $g_s(x) = \sum_{j=1}^{p} f_j \alpha_{sj}$.

**Step 0:** Set $s := 1$

**Step 1:** Solve problem $Q_s$ defined as:

$$\max_{x \in D} \{\vartheta_s; \vartheta_s \leq g_s(x), \vartheta_j \leq g_j(x) \quad \forall j < s\}$$

and denote the optimal solution by $(x^0, \vartheta^0)$

**Step 2:** If $s = m$, then STOP ($x^0$ is the optimal solution.)

Otherwise, set $s := s + 1$ and go the **Step 1**

For example, for $s = 1$, we build the first problem $(Q_1)$ with the objective $\vartheta_1; \vartheta_1 \leq \sum_{j=1}^{p} f_j \alpha_{1j}$ being maximized and constraints shown in Eq. (15) and Eq. (16). For the next iterations ($s > 1$), the problem $Q_s$ is built by adding new constraints $\vartheta_{s-1} \leq \sum_{j=1}^{p} f_j \alpha_{s-1,j}$ where $\vartheta_{s-1}$ is the optimal objective value of the problem $Q_{s-1}$ and $\sum_{j=1}^{p} \alpha_{s-1,j} = s - 1$ and so on. For the integrated manufacturer-retailers problem at hand, inequalities (4) - (8) are considered in each iteration. The objective values obtained by both the max-min and the LMM methods along with their CPU times of reaching the solutions are given in Table 2.
Table 2: Solution comparisons of the max-min and the LMM methods

<table>
<thead>
<tr>
<th>setting</th>
<th>max-min</th>
<th>LMM</th>
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<tbody>
<tr>
<td></td>
<td>CPU time(s)</td>
<td>$z_1$ ($)</td>
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<tr>
<td></td>
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<tr>
<td>initial value</td>
<td>0.726</td>
<td>1846.479</td>
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<tr>
<td>$k_1 = 3500$</td>
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<td>2046.598</td>
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<tr>
<td>$k_1 = 2500$</td>
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<td>$k_2 = 4000$</td>
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<td>1049.722</td>
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<td>1164.608</td>
</tr>
<tr>
<td>$e_2 = 3$</td>
<td>0.193</td>
<td>968.174</td>
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</tr>
</tbody>
</table>

The results in Table 2 indicate the equality of both objectives in all cases, which means they are close to their maximum values equally. This is consistent with what was stated in Theorem 3. In other words, fair non-dominated solutions with all normalized objectives as equal as possible are obtained. Besides, LMM provides better solutions with less CPU times compared to the ones of the max-min method. In the next subsection, sensitivity analyses are performed to investigate the effects of market scale and demand elasticity on manufacturer and retailers' profits.

5.1. Sensitivity Analyses

Since $k_c$ and $e_c$ in the Cobb–Douglas demand function presented in Eq. (1) represent market scale and demand elasticity of retailer $c$ with respect to retail price, respectively, an initial value and twelve variations from the initial value are considered to observe the effects of these changes on
manufacturer’s profit \( z_1 \) and all retailers’ profits \( z_2 \). The LMM solutions for wholesale prices, retail prices, fraction backlogged, and replenishment cycles (the decision variables) based on the initial values is summarized in Table 3. The results in Table 3 show that retailer 1 with high market scale and low demand elasticity earns high retail price and demand for the products.

<table>
<thead>
<tr>
<th>Retailer</th>
<th>Product 1</th>
<th>Product 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w ) ($/unit)</td>
<td>( p ) ($/unit)</td>
</tr>
<tr>
<td>1</td>
<td>32.622</td>
<td>46.683</td>
</tr>
<tr>
<td>2</td>
<td>36.997</td>
<td>38.628</td>
</tr>
<tr>
<td>3</td>
<td>2.047</td>
<td>25.176</td>
</tr>
<tr>
<td>replenishment cycle (time)</td>
<td>1.435</td>
<td>1.582</td>
</tr>
</tbody>
</table>

Furthermore, an instance in the sensitivity analyses shows that when the market scale of retailer 1 \( k_1 \) increases from 3000 to 3500, \( z_1 \) and \( z_2 \) in the LMM and the max-min method increase from $3755.520 to $4140.088 and $1864.479 to $2046.598, respectively. Moreover, when this parameter decreases from 3000 to 2500, \( z_1 \) and \( z_2 \) in the LMM and the max-min method decrease to $3371.065 and $1487.486, respectively. In addition, when the demand elasticity faced by retailer 3 increases from 1.5, as its initial value, to 3, the LMM and the max-min solutions decrease from $3755.520 to $2725.190 and $1864.479 to $968.174, respectively. Based on the above sensitivity analyses (and the other sensitivity analyses not shown here) it can be seen that while both methods provide fair non-dominated solutions in all cases, LMM provides better solutions in terms of the objective functions than the ones of the max-min method with almost equal CPU times.

5.2. Comparison

In order to assess the performance of the proposed methodology and compare it to the one of the max-min method, different test problems with different numbers of retailers and finished products are considered in this section, where in all problems \( k_c = 2000 \) and \( e_c = 1.5 \). The results
obtained using the max-min and the LMM methods are summarized in Table 4. Besides, the values of $z_j$ obtained by LMM and max-min methods for three, five, and seven finished products are shown in Fig. 3, Fig. 4, and Fig. 5, respectively.

<table>
<thead>
<tr>
<th># of retailers</th>
<th># of products</th>
<th>Max-min</th>
<th>LMM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>CPU time(s)</td>
<td>$z_1$ ($)</td>
</tr>
<tr>
<td>$n=5$</td>
<td>3</td>
<td>0.431</td>
<td>1393.199</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.719</td>
<td>3081.036</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.965</td>
<td>6014.480</td>
</tr>
<tr>
<td>$n=7$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=9$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=11$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=15$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=17$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=19$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=21$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=23$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
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</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
<tr>
<td>$n=25$</td>
<td>3</td>
<td>0.307</td>
<td>2544.350</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.799</td>
<td>3839.617</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.229</td>
<td>4257.444</td>
</tr>
</tbody>
</table>
Fig. 3. Values of $z_1$ obtained by the two methods for a three product problem

Fig. 4. Values of $z_2$ obtained the methods for a five product problem
Paired samples $t$-test is a useful tool to test the null hypothesis that the mean of the values of $z_1$ obtained by LMM in all test problems is greater than the one obtained by the max-min method. The paired samples $t$-test (one-tailed) succeeded to reveal a statistically significant difference between the means (LMM Mean = 18158, LMM Std = 9672, max-min mean = 7255, max-min Std = 3753.). In this case, the $t$-statistic is 8.167, which is far greater than the upper 5% critical point of a $t$-student distribution with 29 degrees of freedom (1.699.) In other words, the LMM method provides better quality solutions in terms of the first objective function. Similarly, this test is implemented for the means of $z_2$ obtained by both methods and the same conclusion has been made. Moreover, a paired samples $t$-test is designed to determine if the mean CPU times are different. In this case, the $t$-statistic based on (LMM mean = 2.08, LMM Std = 2.27, max-min Mean = 2.21, max-min Std = 1.64) becomes 0.544 with a $p$-value = 0.591 that shows no significant statistical difference. In other words, the methods have the same required mean CPU time statistically. In addition, the
comparisons in terms of the objective values and CPU times reveal that, increasing the number of retailers, increases $z_1$ and $z_2$ and CPU times simultaneously. Nevertheless, it can be seen that fair solutions that are obtained by the LMM method provide better manufacturer and retailers' benefits than the ones obtained by the max-min method.

6. Conclusion and future researches

This paper proposed a bi-objective mathematical model for a VMI supply chain problem with a single manufacturer and several retailers. The formulation was shown to be a non-linear mathematical model that maximizes both the manufacturer and retailers' profits. The model application was specialized in a case with the Cobb–Douglas demand function. Moreover, we adopted a fair profit contract between the manufacturer and its retailers. Our purpose of the fair profit contract was based on the assumption that both the manufacturer and retailers contributed to determine their optimal decision variables in order to maximize their benefits. Then, the bi-objective problem was formulated as a lexicographic max-min problem in order to find a fair non-dominated solution, a solution with all normalized objectives as equal as possible. In addition, this paper discussed how to replace the original lexicographic max-min problem with the lexicographic maximum problem. Finally, the result obtained using the lexicographic maximum problem was compared to the one of the max-min method in terms of the objective functions and the required CPU time. Besides, based on some sensitivity analyses we showed that while both methods provide fair non-dominated solutions in all cases, LMM provides better objective function values than the ones of the max-min method with almost equal CPU times. This conclusion was made using paired samples $t$-tests to compare equalities of the means of the objective functions and CPU time. In other words, computational results showed that while the two methods had no statistical significant difference in the mean CPU time, the proposed method was superior to the max-min method in
terms of the two objective functions in 30 test problems with different number of retailers and different number of products. For future work extensions, the followings are recommended:

- Multi-period setting can be considered;
- Variation in the common replenishment cycle time can be assumed;
- Competition among retailers can be modeled;
- A real-world application of the proposed approach is recommended;
- It is worth utilizing simulation tools in the VMI problem with regard to fair profit contract.

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References


