On the von Staudt–Clausen’s theorem associated with $q$-Genocchi numbers

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Abstract

Recently, the von Staudt–Clausen’s theorem for $q$-Euler numbers was introduced by Kim (2013) and $q$-Genocchi numbers were constructed by Araci et al. (2013). In this paper, we give the corresponding von Staudt–Clausen’s theorem for $q$-Genocchi numbers and also get the Kummer-type congruence for $q$-Genocchi numbers.

Keywords: Genocchi numbers and polynomials, $q$-Genocchi numbers, von Staudt–Clausen’s theorem, Kummer congruence

1. Introduction

Karl von Staudt [5] and Thomas Clausen [3] first introduced a theorem including fractional part of Bernoulli numbers, which are now popularly known as the von Staudt–Clausen’s theorem (see [3,5]). Kim [7] proved that $q$-Euler numbers are $p$-adic integers and $q$-Euler numbers that can be shown in terms of the von Staudt–Clausen theorem. It can be seen that these numbers play an important role in the development of several areas of Mathematics such as Number theory, Complex analysis, Mathematical physics and so on. Some special numbers related to Bernoulli numbers, Euler numbers, Genocchi numbers, Frobenius–Euler numbers have been studied by many mathematicians (see [1–25]).

Recently, the modified $q$-Genocchi numbers and polynomials were constructed by Araci et al. [22]. They also established some new identities for the modified $q$-Genocchi numbers and polynomials.

In the complex plane, Genocchi numbers can be expressed as

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = e^{\text{e}^t} - 1, \quad (|t| < \pi),$$

where we use the notation $G^n(x) := G_n(x)$, symbolically (see [8,12,16,18,22]).

It is easy to see from (1.1) that

$$G_0 = 0 \quad \text{and} \quad (G + 1)\alpha + G_n = 2\delta_{1,n}$$

where $\delta_{1,n}$ is the Kronecker delta.
The \(q\)-Genocchi polynomials are given by

\[
\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = \left[ 2q^t \right] \sum_{m=0}^{\infty} (-1)^m e^{m+1}t^m \] (see \[22]\)).

(1.2)

In the special case, \(x = 0\) in \(1.2\), \(G_{n,q}(0) := G_n\) are called the \(q\)-Genocchi numbers. A link between the \(q\)-Genocchi numbers and the \(q\)-Genocchi polynomials is given by

\[
G_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{k-1} x^k G_k q = q^{-x} \left( [x]_q + qG_q \right)^n
\]

with the usual convention of changing \((G_n)^n\) to \(G_n\) (see \[22]\)).

Throughout this work, we shall use \(p\) as a fixed odd prime number and we shall use the following notations: \(\mathbb{Z}_p\) denotes the ring of \(p\)-adic integers, \(\mathbb{Q}\) denotes the field of rational numbers, \(\mathbb{Q}_p\) denotes the field of \(p\)-adic rational numbers, and \(\mathbb{C}_p\) denotes the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(\mathbb{N}\) be the set of natural numbers and \(\mathbb{N}^* = \mathbb{N} \cup \{0\}\). The normalized \(p\)-adic absolute value is defined by

\[
|p|_p = \frac{1}{p}.
\]

Also, we assume that \(1 - q|_p < p^{-1}\) is an indeterminate. Let \(\text{UD}(\mathbb{Z}_p)\) be the space of uniformly differentiable functions on \(\mathbb{Z}_p\) and let \(d\) be a steadied positive integer with \((p, d) = 1\), say

\[
X := X_d = \lim_{n \to \infty} \mathbb{Z}/dp^n\mathbb{Z},
\]

which leads to \(\mathbb{Z}_p\) at the value of \(d = 1\). Further

\[
X^* = \cup_{0 < a < dp^d + dp\mathbb{Z}_p} \text{ and } a + dp^n\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^n}\},
\]

\((a, p) = 1\)

where \(a \in \mathbb{Z}\) lies in \(0 < a < dp^n\). Let \(f \in \text{UD}(\mathbb{Z}_p)\) which stands for a uniformly differentiable function at a point \(a \in \mathbb{Z}_p\). To construct Kim’s \(p\)-adic \(q\)-deformed fermionic on \(\mathbb{Z}_p\), consider

\[
\sum_{0 < a < dp^n} f(a) \mu_{-q}(a + dp^n\mathbb{Z}_p) = \frac{1}{|p^n - q_{-q}|} \sum_{0 < a < dp^n} (-1)^q f(a) q^n,
\]

which represents a \(q\)-analog of Riemann sums for \(f\). The \(p\)-adic \(q\)-deformed integral for \(f\) is always known as the limit as \(n \to \infty\) of those sums, when it exists. Namely,

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-q}(a) = \lim_{n \to \infty} \frac{1}{|p^n - q_{-q}|} \sum_{a=0}^{p^n-1} (-1)^q f(a) q^n,
\]

(1.3)

which are defined by Kim (see \[7,9,10,16–19,21,22]\)). This \(p\)-adic \(q\)-deformed fermionic integral is very useful tool to construct variously generating functions based on some special polynomials and obtain new interesting properties of special polynomials.

It is trivial to get that

\[
\lim_{q \to -1} I_{-q}(f) = I_{-1}(f) = \lim_{n \to \infty} \sum_{a=0}^{p^n-1} (-1)^q f(a).
\]

(1.4)

By \(1.3\), we have the following equation: For \(f_1(x) := f(x + 1),\)

\[
qL_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).
\]

(1.5)

In \[22\], a Witt-type formula of the generating function for the \(q\)-Genocchi numbers is given by

\[
\sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} q^{-a} e^{tZ} d\mu_{-q}(a) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} q^{-a} [a]_q^n d\mu_{-q}(a) \right) \frac{t^{n+1}}{n!}.
\]

(1.6)

If we compare the coefficients in \(1.6\), we can get that

\[
\frac{G_{n+1,q}}{n+1} = \int_{\mathbb{Z}_p} q^{-a} [a]_q^n d\mu_{-q}(a) \quad \text{(see \[22\]).}
\]

(1.7)

The \(q\)-Genocchi polynomials are also introduced due to fermionic \(p\)-adic \(q\)-deformed integral as

\[
\sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} q^{-a} e^{tX} d\mu_{-q}(a) \quad \text{(see \[22\]).}
\]

(1.8)
Let \( \chi \) be the Dirichlet’s character with odd conductor \( d \in \mathbb{N} \). Let us now consider generalization of the \( q \)-Genocchi numbers as
\[
G_{n+1,q,\chi} = \int_X \chi(a) q^{-a} a^n d\mu_q(a).
\] (1.9)

We are now ready to state a distribution formula for the generalization of the \( q \)-Genocchi numbers. By (1.9), we have
\[
\frac{G_{n+1,q,\chi}}{n+1} = \int_X \chi(a) q^{-a} a^n d\mu_q(a) = \lim_{n \to \infty} \frac{1}{d_{-q}^n} \sum_{a=0}^{d_{-q}^{n-1}} \chi(a)(-1)^a [a^n_q]
\]
\[
= \frac{[d_{-q}^n]}{[d_{-q}^n]} \sum_{k=0}^{d_{-q}^{n-1}} (-1)^k \chi(k) \left( \lim_{n \to \infty} \frac{1}{d_{-q}^n} \sum_{a=0}^{d_{-q}^{n-1}} [d + k]^n d_{-q}^n (-1)^a \right) = \frac{[d_{-q}^n]}{[d_{-q}^n]} \sum_{k=0}^{d_{-q}^{n-1}} (-1)^k \chi(k) \frac{G_{n+1,q,k}}{n+1}.
\]

Therefore, we have the following theorem.

**Theorem 1.** Let \( \chi \) be the Dirichlet’s character with odd conductor \( d \in \mathbb{N} \). For \( n \in \mathbb{N}^* \), we have
\[
G_{n,q,\chi} = \frac{[d_{-q}^n]}{[d_{-q}^n]} \sum_{k=0}^{d_{-q}^{n-1}} (-1)^k \chi(k) G_{n,q,k} \left( \frac{k}{d} \right).
\]

Let \( S_{\chi,q}(t) = \sum_{n=0}^{\infty} G_{n,q,\chi} t^n \). Then
\[
S_{\chi,q}(t) = [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) e^{mt}_q. \quad \text{(1.10)}
\]

We now give a familiar theorem which is known as von Staudt–Clausen theorem.

**Lemma 1.** Let \( n \) be an even and positive integer. Then
\[
B_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z}.
\] (1.11)
For this reason, \( pB_n \) is a \( p \)-adic integer where \( p \) is arbitrary prime number, \( n \) is arbitrary integers and also \( B_n \) is Bernoulli numbers (see [1–7]).

By the same motivation of the above knowledge, we show that \( q \)-Genocchi numbers are \( p \)-adic integers, which can be written as von Staudt–Clausen’s-type theorem. Finally, we give Kummer-type congruence for the generalized \( q \)-Genocchi numbers.

2. von Staudt–Clausen’s theorem associated with weighted \( q \)-Genocchi numbers

By (1.3), (1.4) and (1.7), we easily see that
\[
G_{n+1,q} = \int_{x_q} q^{-a} a^n d\mu_q(a) = \frac{[2]_q}{2} \int_{x_q} a^n d\mu_q(a). \quad \text{(2.1)}
\]

Thus, we have
\[
\lim_{q \to 1} \frac{G_{n+1,q}}{n+1} = \int_{x_q} a^n d\mu_q(a).
\]

In [7], Kim gave the following useful inequality based on \( p \)-adic norm:
\[
\left| \sum_{j=0}^{p-1} (-1)^j [j]_q \right|_p \leq 1. \quad \text{(2.2)}
\]

Let us now consider the following equality: For \( k \geq 1 \)
\[
C_{n-1}(k) = [0]_q^{n-1} - [1]_q^{n-1} + \cdots + [p^k - 1]_q^{n-1}. \quad \text{(2.3)}
\]

Taking as the limit as \( k \to \infty \) in the both sides of (2.3), we derive that
\[
\lim_{k \to \infty} C_{n-1}(k) = \frac{2}{[2]_q} G_{n,q}.
\]

By (2.3), we have
\[ C_{n-1}(k+1) = \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} = \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{n-1} = \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{n-1} \]
\[ = \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} \left( \frac{n-1}{l} \right) [a]_q^{n-1-l} (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{l} = \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) [a]_q^{n-1-l} (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{l}. \]

As a result of the above applications, we procure
\[ C_{n-1}(k+1) \equiv \sum_{a=0}^{p^{k+1}-1} [a]_q^{n-1} (-1)^a \left( \text{mod } [p^k]_q \right). \tag{2.5} \]

By virtue of (2.4), we have
\[ \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} = \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{n-1} = \sum_{a=0}^{p^{k+1}-1} (-1)^a \sum_{j=0}^{n-1} \left( [a]_q + q^a [j]_q \right)^{n-1} \]
\[ = \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} \left( \frac{n-1}{l} \right) (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{l} = \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \]
\[ + \sum_{a=0}^{p^{k+1}-1} \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) (-1)^a \cdot j^p \left( [a]_q + q^a [j]_q \right)^{l} \equiv \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \left( \text{mod } [p]_q \right). \tag{2.6} \]

On account of (2.5) and (2.6), we have the following Lemma.

**Lemma 2.** Let \( C_{n-1}(k) = \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \). Then we have
\[ C_{n-1}(k+1) \equiv \sum_{a=0}^{p^{k+1}-1} [a]_q^{n-1} (-1)^a \left( \text{mod } [p^k]_q \right). \]

Further
\[ \sum_{a=0}^{p^{k+1}-1} [a]_q^{n-1} (-1)^a \left( \text{mod } [p^k]_q \right) \equiv \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \left( \text{mod } [p]_q \right). \]

It follows from Lemma 2 that
\[ \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \equiv \frac{2n}{[2]_q} \int_X q^{-a} [a]_q^{n-1} d\mu_q(a) \equiv \frac{2n}{[2]_q} G_{n,q} \left( \text{mod } [p]_q \right). \tag{2.7} \]

Hence, we get the following theorem.

**Theorem 2.** For \( n \geq 1 \),
\[ \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \equiv \frac{2n}{[2]_q} G_{n,q} \left( \text{mod } [p]_q \right). \]

Thanks to (2.5)–(2.7), we deduce the following.

**Corollary 1.** For \( n \geq 1 \),
\[ \frac{2n}{[2]_q} G_{n,q} + n \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^{n-1} \in \mathbb{Z}_p. \]

For \( n \geq 1 \),
\[ \left\lvert \frac{2}{[2]_q} G_{n+1,q} \right\rvert_p = 2 \left\lvert \frac{2n}{[2]_q} G_{n+1,q} \right\rvert_p = 2 \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^n + \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^n \leq \max \left\{ \frac{2}{[2]_q} G_{n+1,q} \left\lvert \frac{2}{[2]_q} G_{n+1,q} \right\rvert_p, \sum_{a=0}^{p^{k+1}-1} (-1)^a [a]_q^n \right\} \right\rvert_p. \]

Thus, by (2.2) and Corollary 1, we get the following corollary.
Corollary 2. For $n \geq 1$,
\[
\frac{2}{[2]q} \frac{G_{n-1,q}}{n+1} \in \mathbb{Z}_p.
\]
Let $\chi$ be the Dirichlet’s character with odd conductor $d \in \mathbb{N}$. The generalized $q$-Genocchi numbers attached to $\chi$ are as follows:
\[
\sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n \chi(m)e^{imq} = t \int_X \chi(a)q^{-a}e^{aq}d\mu_{-q}(a).
\]
(2.8)

Let $f = [f,p]$ be the least common multiple of the conductor $f$ of $\chi$ and $p$. From the last equation,
\[
\frac{2}{[2]q} G_{n,q} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \chi(a)q^{-a}[a]^{-1}_q d\mu_{-q}(a) = n \lim_{N \to \infty} \frac{1}{n!} \sum_{a=0}^{n} \chi(a)q^{-a}[a]^{-1}_q,
\]
which leads to
\[
\frac{2}{[2]q} G_{n,q} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \chi(a)q^{-a}[a]^{-1}_q d\mu_{-q}(a) = n \lim_{N \to \infty} \frac{1}{n!} \sum_{a=0}^{n} \chi(a)q^{-a}[a]^{-1}_q
\]
Therefore we obtain the following theorem.

Corollary 3. For $n \geq 1$,
\[
\frac{2}{[2]q} G_{n,q} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \chi(a)q^{-a}[a]^{-1}_q d\mu_{-q}(a) = n \lim_{N \to \infty} \frac{1}{n!} \sum_{a=0}^{n} \chi(a)q^{-a}[a]^{-1}_q
\]
which leads to
\[
\frac{2}{[2]q} G_{n,q} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \chi(a)q^{-a}[a]^{-1}_q d\mu_{-q}(a) = n \lim_{N \to \infty} \frac{1}{n!} \sum_{a=0}^{n} \chi(a)q^{-a}[a]^{-1}_q
\]
Therefore we obtain the following corollary.

Corollary 4. For $n \geq 1$,
\[
\frac{2}{[2]q} G_{n,q} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \chi(a)q^{-a}[a]^{-1}_q d\mu_{-q}(a) = n \lim_{N \to \infty} \frac{1}{n!} \sum_{a=0}^{n} \chi(a)q^{-a}[a]^{-1}_q
\]
which leads to
\[
\frac{2}{[2]q} G_{n,q} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} \chi(a)q^{-a}[a]^{-1}_q d\mu_{-q}(a) = n \lim_{N \to \infty} \frac{1}{n!} \sum_{a=0}^{n} \chi(a)q^{-a}[a]^{-1}_q
\]
Therefore we obtain the following corollary.
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