Alternative Stability Conditions for Hybrid Systems

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Abstract—In this paper, we consider stability of autonomous hybrid dynamical systems without inputs. We provide conditions to guarantee asymptotically stability that are similar to the known conditions in the literature, but have certain advantages that we explain here. Moreover, our proofs are different and more elegant from the proofs of the similar results in the literatures.

I. INTRODUCTION

Hybrid systems are dynamical systems exhibiting both continuous and discrete dynamics. This is a reason why the word `hybrid` is appended. Hybrid modelling is widely presented in many modern real world applications such as robots controlling [10], [23], computer science [22], control systems [20], [8], biological and medical systems [29], [21], [4], [1]. Moreover, hybrid phenomena have been modelled in many different frameworks since last few decades or more. Those frameworks include hybrid automata [27], [18], impulsive differential equations (or inclusions) [3], [2], [15], [28] and switched systems [17].

To work with hybrid systems, we use the framework developed in [24], [11] and [7]. For the most part, there are some differences from [27], [18], [26] and [2] due to not only their structure but also solution’s definition. The most considerable advantages of the frameworks developed in [24], [11] and [7] are results on robust asymptotic stability and extended classical stability analysis tools. In addition, models such as hybrid automata, impulsive differential equations and switching systems can be translated to the framework developed in [24], [11] and [7]. One of all benefits of translations is that the stability theorems can be applied to other classes of systems, e.g., invariance principles for switching systems [12].

Although sufficient conditions for asymptotic stability have been well given by hybrid Lyapunov theorems and hybrid LaSalle’s invariance principles in [24] and [25], we aim to provide alternative conditions being comfortable to apply, in some cases, than previous works in the literature. We consider a Lyapunov candidate function as the energy function. Instead of losing energy by both of continuous and discrete dynamics, we allow energy to be loosed by only one of them when all of our conditions, introduced later in the Section IV, hold for the hybrid systems. We do not require neither the strict inequalities in the hybrid Lyapunov theorems nor a consideration of hybrid invariant principles. Furthermore, the existence of attractive complete discrete ,or continuous, solutions to a hybrid system does not need to verify in our conditions. For simplicity, we just consider stability of the origin instead of stability of a compact set.

The rest of this work is organized as follows. Section II gives some notation and a description of hybrid systems and their solutions. Section III presents stability and invariance notions. Main results and proofs are given in Section IV. Section V gives some examples of applications, and conclusions are given in Section VI.

II. PRELIMINARIES

We begin by listing some basic definitions and notation. Denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space, $\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$, and $\varnothing$ denotes the empty set. Denote by $x^T$ the transpose of vector $x$. Denote by $|\cdot|$ a vector norm, and $\langle \cdot, \cdot \rangle$ denotes the scalar product.

Given $A \subset X \subset \mathbb{R}^n$, denote by $\overline{A}$ the closure of $A$. A set $A$ is relatively closed in $X$ if $A = \overline{A} \cap X$; when $X$ is open, then $A$ is relatively closed in $X$ if and only if $X \setminus A$ is open [7]. Given a mapping $f : \mathbb{R}^m \to \mathbb{R}^n$, the domain and range of $f$ is denoted by $\text{dom} f$ and $\text{rge} f$ respectively. For any $y \in \text{rge} f$, $f^{-1}(y) := \{ x \in \text{dom} f : f(x) = y \}$. The standard gradient of $f$ at $x$ is denoted by $\nabla f(x)$, and if $f$ is locally Lipschitz then $\nabla f(x)$ exists almost everywhere. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$, $\alpha \in \mathcal{K}$, if $\alpha$ is continuous, zero at zero and strictly increasing. It belongs to class-$\mathcal{K}_{\infty}$ if, in addition, it is unbounded.

Solution $x : [a, b] \to \mathbb{R}^n$ to differential equation $\dot{x}(t) = f(x(t))$ will be understood in the Carathéodory sense [24], i.e., a function $f$ is required to be absolutely continuous, and $\dot{x}(t) = f(x(t))$ is required to hold for almost all $t \in [a, b]$. For an absolutely continuous $x : [a, b] \to \mathbb{R}^n$, the derivative $\dot{x}(t) = \frac{dx}{dt}(t)$ exists for all $t \in [a, b]$ except a set of measure zero.

Description of hybrid systems, the solutions concept and all of assumptions are given in the following.

A. Hybrid Systems

A hybrid system $\mathcal{H}$ has a vector state space $X$, possibly including both continuous and discrete components, which takes values in a state space given by a non-empty open set $X \subset \mathbb{R}^n$. The continuous behavior, called flow, of hybrid system $\mathcal{H}$ is described by a differential equation $\dot{x} = f(x)$, which is called flow map. While the discrete behavior, called jump, is modelled by a difference equation $x^+ = g(x)$,
which is called jump map. The conditions allowing flows and/or jumps that are given by the flow set $C$ and the jump set $D$, respectively, subsets of a state space $X$. Then, a hybrid system $H = (X, f, C, g, D)$ can be written as follows

$$
H: \quad x ∈ X \implies \begin{cases} 
    \dot{x} = f(x) & x ∈ C \\
    x^+ = g(x) & x ∈ D.
\end{cases}
$$

Moreover, the function $f$ maps every point in the vector state space $X$ to $\mathbb{R}^n$ while the function $g$ maps from $X$ to $X'$. The structure and concept of the hybrid system (1) are taken from [24] and [11]. The assumptions on the flow map and jump map presented in (1) are introduced in Section II-C.

B. Solutions to Hybrid Systems

The concept of solutions was presented in [24] and [11]. Their results suggest that solutions will stay in $X$. At points in $C \cap D ≠ \emptyset$, solutions can be nonunique, i.e., they can either flow or jump. This is a reason why we do not need the flow map $f$ and jump map $g$ to satisfy Lipschitz. Solutions can not be continued at points in $C$ where flow is not possible. At points in $X \setminus (C \cup D)$, no solution exists.

Solutions to (1) defined in [24] and [11] are given by two-parameter functions: continuous time $t$ and discrete time $j$. The variable $t$ keeps track of the time that the system flows, and the other variable $j$ counts the number of jumps. Almost all of the following definitions are taken from [24].

**Definition 2.1 (Hybrid Time Domain):** A subset $E ⊂ \mathbb{R}_{≥0} \times X$ is a hybrid time domain if it is a union of finite or infinite sequence of intervals $[t_j, t_{j+1} × \{j\}$ such that $0 = t_0 \leq t_1 \leq ... \leq t_j \leq t_{j+1} \leq ...$

**Definition 2.2:** Let $E$ contain points $(t_1, j_1)$ and $(t_2, j_2)$, $(t_1, j_1) ≤ (t_2, j_2)$ means that $t_1 + j_1 \leq t_2 + j_2$.

**Definition 2.3:** Given a hybrid time domain $E$, $sup E = (sup_E, sup_j E)$, and $length(E) = sup_E + sup_j E$, where $sup_E = sup \{t ∈ \mathbb{R}_{≥0} | \exists j ∈ N, (t, j) ∈ E\}$, and $sup_j E = sup \{j ∈ N | \exists t ∈ \mathbb{R}_{≥0}, (t, j) ∈ E\}$.

**Definition 2.4 (Hybrid Arc):** A function $x : E → \mathbb{R}^n$ is a hybrid arc if $E$ is a hybrid time domain and if for each $j ∈ N$ the function $t → x(t, j)$ is locally absolutely continuous.

**Definition 2.5:** Given a hybrid arc $x : dom x → \mathbb{R}^n$, define $rge x := \{y ∈ \mathbb{R}^n : x(t, j) = y, (t, j) ∈ dom x\}$.

**Definition 2.6 (Types of Hybrid Arcs):** A hybrid arc $x$ is

1) for all $j ∈ N$ such that $P^j$ has non-empty interior, where $P^j \times \{j\} = dom x \cap ([0, ∞) × \{j\}), x(t, j) ∈ C$ for all $t ∈ int P^j, x(t, j) = f(x)$ for almost all $t ∈ P^j$.

2) for all $(t, j) ∈ dom x$ such that $(t, j + 1) ∈ dom x, x(t, j + 1) ∈ D, x(t, j) = g(x(t, j))$.

**Definition 2.8 (Maximal Solutions):** A solution $x$ to (1) is maximal if there is no another solution $x'$ to (1) such that $dom x ⊂ dom x'$ and $x(t, j) = x'(t, j)$ for all $(t, j) ∈ dom x$. The set $S_H(\xi)$ denotes the set of all maximal solutions $x$ to $H$ with $x(0, 0) = \xi$.

C. ASSUMPTIONS

Let the following assumptions hold for the hybrid system $H = (X, f, C, g, D)$:

(A1) $X$ is open.

(A2) $C$ and $D$ are relatively closed in $X$.

(A3) $f : X → \mathbb{R}^n$ is continuous on $C$.

(A4) $g : X → X$ is continuous on $D$.

They are arranged by combination of essential assumptions in continuous and discrete time systems to guarantee the existence of solutions to the hybrid system $H$. We refer readers to [11] and also [13] for more details.

III. STABILITY AND INVARIANCE

A Lyapunov theorem and an invariance principle for hybrid systems are presented in this section. The concepts of stability and invariance for hybrid systems (1) were proposed in [24] and [11]. Lyapunov stability and hybrid invariance principles were also introduced in [30], [5], [6] and [25]. The following definitions and theorems in this section are taken from [24] and [25]. We introduce them for the case of only one equilibrium point, i.e., the stability of the origin $\{0\} ⊂ X ⊂ \mathbb{R}^n$.

**Definition 3.1:** For a hybrid system $H = (X, f, C, g, D)$, the origin is

- pre-stable if for each $ε > 0$, there exist $δ > 0$ such that any solution $x$ to $H$ with $|x(0, 0)| ≤ δ$ implies $|x(t, j)| ≤ ε$ for all $(t, j) ∈ dom x$;
- attractive if $x(t, j) → 0$ as $t + j → ∞$;
- pre-asymptotically stable if it is both pre-stable and attractive.

The prefix pre can be dropped if all maximal solutions to $H = (X, f, C, g, D)$ are complete. The following theorem is hybrid Lyapunov theorem which has been published in [24].

**Theorem 3.1:** For the hybrid system $H = (X, f, C, g, D)$, (i) suppose that $U ⊂ X$ is a neighborhood of the origin, $V : X → \mathbb{R}^n$ is continuous on $X$, locally Lipschitz on a neighborhood of $C$, and there exist $ψ_1$ and $ψ_2$ belonging to class-K∞ such that $ψ_1(|x|) ≤ V(x) ≤ ψ_2(|x|)$ for all $x ∈ C \cup D$. If for all $x ∈ U$,

$$
(\nabla V(x), f(x)) ≤ 0,
$$

$$
V(g(x)) − V(x) ≤ 0,
$$

then the origin is pre-stable.
(ii) Suppose additionally that
\[ \langle \nabla V(x), f(x) \rangle < 0, \]  
\[ V(g(x)) - V(x) < 0, \]  
for all \( x \in U \setminus \{0\} \). Then the origin is pre-asymptotically stable.

The pre-asymptotic stability of the origin, for a hybrid system \( \mathcal{H} = (\mathcal{X}, f, C, g, D) \), can be verified by the above theorem. In many cases, the origin is still pre-asymptotically stable, even though the condition (ii) does not hold. Some examples are also shown in Section V. When the condition (i) is satisfied, invariance principles for hybrid systems can be applied to guarantee the pre-asymptotic stability of the origin.

**Definition 3.2:** For a hybrid system \( \mathcal{H} = (\mathcal{X}, f, C, g, D) \), the set \( M \subset \mathcal{X} \) is said to be

- **weakly forward invariant** if for each \( \xi \in M \), there exist at least one solution \( x \in S_{\mathcal{H}}(\xi) \) with \( x(t, j) \in M \) for all \( (t, j) \in \text{dom} x \);
- **weakly backward invariant** if for each \( q \in M, N > 0 \), there exist \( \xi \in M \) and at least one solution \( x \in S_{\mathcal{H}}(\xi) \) such that for some \( (t^*, j^*) \in \text{dom} x, t^* + j^* \geq N \), the solution satisfies \( x(t^*, j^*) = q \) and \( x(t, j) \in M \) for all \( (t, j) \leq (t^*, j^*), (t, j) \in \text{dom} x \);
- **weakly invariant** if it is both weakly forward invariant and weakly backward invariant;
- **strongly forward invariant** if for each \( \xi \in M \) and each \( x \in S_{\mathcal{H}}(\xi) \), then \( x(t, j) \in M \) for all \( (t, j) \in \text{dom} x \).

**Theorem 3.2:** For a hybrid system \( \mathcal{H} = (\mathcal{X}, f, C, g, D) \), suppose that (i) holds, and there exists \( \epsilon > 0 \) such that for all \( r \in (0, \epsilon) \) the largest weakly invariant subset in
\[ V^{-1}(r) \cap U \cap [u_C^{-1}(0) \cup (u_D^{-1}(0) \cap g(u_D^{-1}(0)))] \]  
is empty, where \( u_C(x) := \langle \nabla V(x), f(x) \rangle \) for all \( x \in C \) and \( u_D(x) := V(g(x)) - V(x) \) for all \( x \in D \). Then the origin is pre-asymptotically stable.

The above theorem has been called **hybrid Krasovskii** which was introduced in [24]. The following theorem is given in [25] and says the conditions of asymptotic stability which either \( u_C \) or \( u_D \), defined in Theorem 3.2, is not strictly negative.

**Theorem 3.3:** For a hybrid system \( \mathcal{H} = (\mathcal{X}, f, C, g, D) \), suppose that (i) holds. If either
(a) \( u_C(x) < 0 \) for all \( x \in C \setminus \{0\} \),
(b) any complete discrete solution \( x \) to \( \mathcal{H} \) with \( \text{rge} x \in U \) converges to the origin; or
(a′) \( u_D(x) < 0 \) for all \( x \in D \setminus \{0\} \),
(b′) any complete continuous solution \( x \) to \( \mathcal{H} \) with \( \text{rge} x \in U \) converges to the origin;
is satisfied, then the origin is asymptotically stable.

Even though Theorem 3.2 and 3.3 are consequences of the **hybrid LaSalle’s invariance principle**, the similar conditions can also be obtained without any application of hybrid invariance principles. They are introduced in the next section.

### IV. MAIN RESULTS

We neither require strict inequalities in conditions (ii) of Theorem 3.1 to show the asymptotic stability nor apply any application of hybrid invariance principles. Moreover, we do not need to check that there exist a complete discrete, or continuous, solution and if it converges to the origin.

**Theorem 4.1:** Let assumptions (A1) - (A4) hold for the hybrid system \( \mathcal{H} = (\mathcal{X}, f, C, g, D) \). Suppose every solution \( x \in S_{\mathcal{H}}(\xi) \) is complete for all \( \xi \in C \cup D \). There exists a smooth function \( V : \mathcal{X} \to \mathbb{R}^+ \), and there exist \( \psi_1 \) and \( \psi_2 \) belonging to class-\( K_{\infty} \) such that \( \psi_1(|x|) \leq V(x) \leq \psi_2(|x|) \) for all \( x \in C \cup D \). If the following conditions are satisfied:
(G1) For any nontrivial solution \( x \in S_{\mathcal{H}}(\xi), x \) is not eventually continuous,
(G2) \( \langle \nabla V(x), f(x) \rangle \leq 0 \) for all \( x \in C \setminus \{0\} \),
(G3) \( V(g(x)) - V(x) < 0 \) for all \( x \in D \setminus \{0\} \),
then the origin is asymptotically stable.

**Proof:** Please note that
\[ \frac{\text{d}}{\text{d}t} V(x) = \langle \nabla V(x), f(x) \rangle . \]

Given \( \epsilon > 0 \), choose \( r \in (0, \epsilon) \) such that
\[ B_r := \{ x \in \mathcal{X} : |x| \leq r \} \subset C \cup D. \]

Let \( \alpha := \min V(x) \), then \( \alpha > 0 \). For \( \beta \in (0, \alpha) \), define
\[ \Omega_\beta := \{ x \in B_r : V(x) \leq \beta \}. \]

Firstly, we need to show that \( \Omega_\beta \) is in the interior of \( B_r \). Suppose by contradiction that \( \Omega_\beta \) is not in the interior of \( B_r \), then there is a \( b \in \Omega_\beta \) that lies on the boundary of \( B_r \). Thus, \( V(b) \geq \alpha > \beta \), but \( V(b) \leq \beta \) for all \( b \in \Omega_\beta \).

Let us show that \( \Omega_\beta \) is strongly invariant set. Without loss of generality, let us suppose \( x(0, 0) \in \Omega_\beta \cap C \). Since the condition (G1) hold, there exists \( t_1 > 0 \) such that \( x(t_1, 0) \in D \). We get \( x(t_1, 0) \in \Omega_\beta \) because \( V(x(t_1, 0)) \leq V(x(0, 0)) \leq \beta \).

If there is no \( j \in \mathbb{N} \) such that \( x(t_1, j) \in C \), we can conclude that \( x(t, j) \in \Omega_\beta \) for all \( (t, j) \in \text{dom} x \) because

![Fig. 1. Representation of sets in the proof of Theorem 4.1 and Theorem 4.2](image-url)
For any nontrivial solution $x(t_j+1) < V(x(t_j)) \leq V(x(t_j, 0))$ for all $j \in \mathbb{N}$. If there exists an integer $j_1$ such that $x(t_1, j_1)$ lies in $\mathcal{C}$, then we get $x(t_1, j_1) \in \Omega_\beta$ since $V(x(t_1, j_1)) < V(x(t_1, j_0))$ for all $j_0 \in \{0, 1, 2, ..., j_1 - 1\}$. Moreover, we still get $x(t, j_1) \in \Omega_\beta$ for all $t \geq t_1$ because of the condition (G2). With this procedure, we can obtain a result that any solutions starting from $\Omega_\beta$ always lie in $\Omega_\beta$.

Since the continuity of $V$ on $\mathcal{C}$ and $V(0) = 0$, there exists $\delta > 0$ such that $|x| < \delta$ implies $V(x) < \beta$. Then, $B_\delta \subset \Omega_\beta \subset B_r$.

Furthermore, we need to show that the origin is stable. Suppose $x(0, 0) \in B_3$. It follows that $(0, 0) \in B_3 \Rightarrow (0, 0) \in \Omega_\beta \Rightarrow x(t, j) \in \Omega_\beta \Rightarrow x(t, j) \in B_r$ for all $(t, j) \in \text{dom } x$. Therefore, $|x(0, 0)| < \delta$ implies $|x(t, j)| < \varepsilon$ for all $(t, j) \in \text{dom } x$.

Finally, we show that the origin is attractive. It is sufficient to show that $V(x(t, j)) \rightarrow 0$ as $t + j \rightarrow \infty$. Suppose a contradiction that $V(x(t, j))$ does not converge to zero. Since (G1) - (G3) hold and $V$ is bounded from below by zero, suppose $V(x(t, j))$ converge to $c > 0$ as $t + j \rightarrow \infty$. Because $V$ is continuous and $V(0) = 0$, there exists a positive $q$ such that $B_q \subset \Omega_\beta$. Therefore, $x(t, j)$ lies outside $B_q$ as $t + j \rightarrow \infty$ since $V(x(t, j)) \rightarrow c$. Define

$$-\gamma := \max_{q \leq |x(t, j)| \leq r} \nabla V(x(t, j), f(x(t, j)))$$

and

$$-\sigma := \max_{q \leq |x(t, j)| \leq r} \left[V(x(t, j + 1)) - V(x(t, j))\right].$$

We get $-\gamma \leq 0$ and $-\sigma < 0$ due to the conditions (G2) and (G3) respectively. Let $(t, \eta)$ denote the least time $t$ such that $(t, \eta) \in \text{dom } x$ and $j(\tau)$ denote the least number $j$ such that $(t, j) \in \text{dom } x$. It follows that

$$V(x(t, j)) = V(x(0, 0)) + \int_0^t \frac{d}{d\tau} V(x(\tau, j(\tau))) d\tau$$

$$+ \sum_{\eta = 1}^j [V(x(t(\eta), \eta)) - V(x(t(\eta), \eta - 1))]$$

$$\leq V(x(0, 0)) - \gamma t - \sigma j.$$

Since $V(x(t, j))$ eventually becomes negative, we obtain a contradiction.

Our next result is proposed in the following theorem. We consider complete maximal solutions to $\mathcal{H} = (\mathcal{C}, f, \mathcal{C} \cup \mathcal{D})$ which are not eventually discrete.

**Theorem 4.2:** Let assumptions (A1) - (A4) hold for the hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{C} \cup \mathcal{D})$. Suppose every solution $x \in S_{\mathcal{H}}(\xi)$ is complete for all $\xi \in \mathcal{C} \cup \mathcal{D}$. There exists a smooth function $V : \mathcal{C} \rightarrow \mathbb{R}^n$, and there exist $\psi_1$ and $\psi_2$ belonging to class-$\mathcal{K}_\infty$ such that $\psi_1(|x|) \leq V(x) \leq \psi_2(|x|)$ for all $x \in \mathcal{C} \cup \mathcal{D}$. If the following conditions are satisfied:

(F1) For any nontrivial solution $x \in S_{\mathcal{H}}(\xi)$, $x$ is not eventually discrete.

(F2) $\langle \nabla V(x), f(x) \rangle < 0$ for all $x \in \mathcal{C} \setminus \{0\}$.

(F3) $V(g(x)) - V(x) \leq 0$ for all $x \in \mathcal{D} \setminus \{0\}$.

then the origin is asymptotically stable.

**Proof:** As we have already known from Theorem 4.1 that $\Omega_\beta$ is in the interior of $B_r$. Suppose $x(0, 0) \in \Omega_\beta$. We will show that $\Omega_\beta$ is strongly forward invariant when conditions (F1) - (F3) hold. Without loss of generality, let us suppose that the initial point lie in $\Omega_\beta$. Then there exists $j_1 \in \mathbb{N}$ such that $x(0, j_1) \in \mathcal{C}$ due to the condition (F1). Thus, $x(0, j_1) \in \Omega_\beta$ because $V(x(0, j_1)) < V(x(0, j_0)) \leq V(x(0, 0)) - \beta$ for all $j_0 \in \{0, 1, 2, ..., j_1\}$. Moreover, we can conclude that $x(t_1, j_1) \in \Omega_\beta$ for all $t_1 > 0$ because, from (F2), $V(x(t_1, j_1)) < V(x(0, j_1)) \leq \beta$ for all $t_1 > 0$ and $x(t_1, j_1) \in \mathcal{C} \setminus \{0\}$. If there is no positive $t$ such that $x(t, j_1) \in \mathcal{D}$, then $x(t, j) \in \Omega_\beta$ for all $(t, j) \in \text{dom } x$.

If there exists $t_2 > t_1$ such that $x(t_2, j_1) \in \mathcal{D}$, then $(t_2, j_1)$ will still be in $\Omega_\beta$ since $V(x(t_2, j_1)) < V(x(t_1, j_1))$. Since (F1) holds, there exist $j_2 \in \mathbb{N}$ such that $x(t_2, j_2) \in \mathcal{C}$. Further $x(t_2, j_2) \in \Omega_\beta$ because $V(x(t_2, j_2)) < V(x(t_2, i)) \leq V(x(t_2, j_1)) \leq \beta$ for all $i \in \{j_1, j_1 + 1, j_1 + 2, ..., j_2\}$. By the above procedure, we can conclude that any solution $x$ starting form $\Omega_\beta$ will lie on $\Omega_\beta$ for all $(t, j) \in \text{dom } x$.

Since $V$ is continuous, and $V(0) = 0$, there exists $\delta > 0$ such that $|x| < \delta$ implies $V(x) < \beta$. Then, $B_\delta \subset \Omega_\beta \subset B_r$. For any $x(0, 0) \in B_3$, it follows that $x(0, 0) \in \Omega_\beta$. Since $\Omega_\beta$ is strongly forward invariant Then, $x(t, j) \in \Omega_\beta \subset B_r$ for all $(t, j) \in \text{dom } x$. Therefore, $|x(0, 0)| < \delta$ implies $|x(t, j)| < r \leq \varepsilon$ for all $(t, j) \in \text{dom } x$, i.e., the origin is stable.

Finally, we show that the origin is attractive. Suppose by a contradiction that $x(t, j)$ does not converge to the origin. Since (F1) - (F3) hold and $V$ is bounded from below by zero, then $V$ converges to $c > 0$. Therefore, there exists $q > 0$ such that $B_q \subset \Omega_\beta$ due to the continuity of $V$. As $t + j \rightarrow \infty$, $x(t, j)$ lies outside $B_q$ since $V(x(t, j)) \rightarrow c > 0$. It follows that

$$V(x(t, j)) \leq V(x(0, 0)) - \gamma t - \sigma j.$$

Where $\gamma$ and $\sigma$ are two constant defined in the proof of Theorem 4.2. Under the conditions (F2) and (F3), we get $-\gamma < 0$ and $-\sigma \leq 0$ respectively. If $x$ is not a Zeno solution, then we get a contraction since $V(x(t, j))$ eventually becomes negative as $t + j \rightarrow \infty$. If $x$ is Zeno, there exists a nonnegative constant $T := \sup_{x \in \text{dom } x} t$. Since $x$ is not eventually discrete, we get $T > 0$. Let $x(t^*, j^*)$ be a point lying outside $B_q$ such that $V(x(t^*, j^*)) = c$, thus

$$x(t^*, j^*) \in (\mathcal{C} \setminus \{0\}) \cup (\mathcal{D} \setminus \{0\}).$$

Suppose, with no loss of generality, $x(t^*, j^*) \in \mathcal{D} \setminus \{0\}$. There exists $\eta > 0$ such that $x(t^*, j^* + \eta) \in \mathcal{C} \setminus \{0\}$ due to the condition (F1). Furthermore, when $x(t^*, j^* + \eta) \in \mathcal{C} \setminus \{0\}$, then there exists $\tau > 0$ such that $(t^* + \tau, j^* + \eta) \in \text{dom } x$. Therefore, a contradiction is obtained since $t^* + \tau > T$.

Our proofs are extended from [16]. If at least one solution to a hybrid systems $\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})$ is not complete, then the contradiction in our proofs can not be obtained. In our main results, the hybrid Lyapunov function cannot increase neither during any flows nor jumps. If it is allowed to
increase in one of those cases, then the dwell-time conditions can be used to verify the global asymptotic stability, see [15] for more detail.

V. EXAMPLES OF APPLICATIONS

One of classical hybrid phenomena is a bouncing ball, i.e., a ball is dropped from some height above the floor. It flows by some initial velocity and gravitation force until a collision with the floor happens. After touching the floor, the velocity jumps, i.e., it changes sign instantaneously and reduce its magnitude by a restitution factor \( \lambda \in [0, 1) \).

Example 5.1: Denote the height of the bouncing ball by \( x_1 \) and its velocity by \( x_2 \). Let the state be

\[
x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \in \mathcal{X} := \mathbb{R}^2.
\]

The flow map \( f \) and the flow set \( C \) are defined as

\[
f(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \text{and} \quad C := \{ x \in \mathbb{R}^2 : x_1 \geq 0 \}
\]

where \( \gamma \) represents the gravitational constant. The jump map and the jump set are respectively defined by

\[
g(x) := \begin{bmatrix} x_1 \\ -\lambda x_2 \end{bmatrix} \quad \text{and} \quad D := \{ x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0 \}
\]

where \( \lambda \in [0, 1) \) denotes the restitution factor between the ball and the floor.

Consider the continuously differentiable function

\[
V(x) = \frac{1}{2} x_2^2 + \gamma x_1.
\] (7)

Since the inequalities (2) and (3) hold, the origin is stable. Note that the hybrid Lyapunov theorem, Theorem 3.1, can not be applied to check the asymptotic stability because the strict inequality (4) does not hold.

Consider the following

\[
\langle \nabla V(x), f(x) \rangle = 0 \quad \text{for all} \quad x \in \mathcal{C} \setminus \{0\},
\]

and

\[
V(g(x)) - V(x) = -\frac{1}{2}(1 - \lambda^2) x_2^2 < 0 \quad \text{for all} \quad x \in \mathcal{D} \setminus \{0\},
\]

then (G2) and (G3) are satisfied.

Moreover, such nontrivial solutions to \( \mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D}) \) is perfect since

\[
\mathcal{H} = (\mathcal{X}, f, \mathcal{C}, g, \mathcal{D})
\]

Thus, the conditions (F2) and (F3) are satisfied. Since each nontrivial solution is perfect, the condition (F1) is also satisfied. We finally conclude that the origin is asymptotically stable by an application of Theorem 4.2.

Example 5.2: With the example of a bouncing ball, we allow a restitution factor between the ball and the floor to equal to one. And the air resistance, or drag, is considered to the system. Flows and jumps of the state can be described as follows:

\[
f(x) := \begin{bmatrix} x_2 \\ -\gamma - k x_2 \end{bmatrix} \quad \text{and} \quad g(x) := \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}.
\]

The positive number \( k \) indicates a linear drag constant.

To guarantee the stability of the origin, Let us consider the following with the equation (7).

\[
\langle \nabla V(x), f(x) \rangle = -k x_2^2 < 0 \quad \text{for all} \quad x \in \mathcal{C} \setminus \{0\},
\]

and

\[
V(g(x)) - V(x) = 0 \quad \text{for all} \quad x \in \mathcal{D} \setminus \{0\}.
\]

Thus, the conditions (F2) and (F3) are satisfied. Since each nontrivial solution is perfect, the condition (F1) is also satisfied. We finally conclude that the origin is asymptotically stable by an application of Theorem 4.2.

Example 5.3: With the same flow set \( \mathcal{C} \) and jump set \( \mathcal{D} \) as in the previous example, let us redefine

\[
f(x) := H \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad g(x) := \begin{bmatrix} \rho_1(|x_1|) \\ \rho_2(|x_2|) \end{bmatrix},
\]
where $H$ is a Hurwitz matrix, i.e., every eigenvalue of $H$ has negative real part. Denote $\text{id}$ the identity map, and define $\rho_i \in \mathbb{K}_{\infty}$ with $\rho_i \leq 1$. There exists a positive definite symmetric matrix $P$ \(^1\) and a Lyapunov function in a form \(^2\)
\[ V(x) = x^T P x \]
such that
\[ \langle \nabla V(x), f(x) \rangle = -x^T (PH + H^T P)x < 0 \]
for all $x \in \mathcal{C} \setminus \{0\}$. Furthermore, we also get
\[ V(g(x)) - V(x) \leq 0 \text{ for all } x \in \mathcal{D} \setminus \{0\}. \]

We have already shown that (F2) and (F3) are satisfied. Moreover, (F1) also holds since $g(\mathcal{D}) \subset \mathcal{C}$. Thus, we can directly conclude that the origin is asymptotically stable from Theorem 4.2.

VI. CONCLUSIONS

Alternative conditions to guarantee the asymptotic stability of autonomous hybrid systems are provided. The conditions are based on hybrid Lyapunov functions defined in Theorem 4.2 and Theorem 4.1. If the assumptions (A1) - (A4) are satisfied, and either (F1) - (F3) or (G1) - (G3) hold, then the origin is asymptotically stable. To directly conclude the asymptotic stability of the origin, we do not require the strict dissipation for hybrid Lyapunov function along both flows and jumps. Although our conditions are similar to Theorem 3.3, the advantages of our results can be described as follows. Firstly, instead of an application of any invariance principles as in [24] or [25], our results are directly obtained by simple trajectory-based proofs. Secondly, we do not need to check the conditions (b) and (b’) in Theorem 3.3 that are possibly difficult to check in some cases. Some examples of application are illustrated by a classical hybrid phenomena and its extension.

VII. ACKNOWLEDGMENTS

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REFERENCES


\(^1\)Given a positive definite symmetric matrix $Q$, $P$ is the unique solution of $PH + H^T P = -Q$.

\(^2\)See [19], Theorem 10.1 and [16], Page 135-136.