Stabilization of the wave equation with variable coefficients and boundary condition of memory type

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Abstract. We consider the stabilization of the wave equation with space variable coefficients in a bounded region with a smooth boundary, subject to Dirichlet boundary conditions on one part of the boundary and linear or nonlinear dissipative boundary conditions of memory type on the remainder part of the boundary. Our stabilization results are mainly based on the use of differential geometry arguments, on the multiplier method and the introduction of suitable Lyapounov functionals.

Keywords: wave equation, variable coefficients, memory boundary conditions, stabilization

1. Introduction

The aim of this paper is to study the wave equation with variable coefficients subject to Dirichlet boundary conditions on one part of the boundary and dissipative boundary conditions of memory type on the remainder part of the boundary. More precisely, let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain with a smooth boundary \( \Gamma \). We assume that \( \Gamma \) is divided into two closed and disjoint parts \( \Gamma_0 \) and \( \Gamma_1 \), i.e., \( \Gamma = \Gamma_0 \cup \Gamma_1 \) and \( \Gamma_0 \cap \Gamma_1 = \emptyset \). Moreover we assume that the measure of \( \Gamma_0 \) is positive.

In this domain \( \Omega \), we consider the initial boundary value problem

\[
\begin{align*}
  u_{tt} + A u &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
  u &= 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \\
  \frac{\partial u}{\partial \nu_A}(t) + \int_0^t k(t - s) u_t(s) ds + b u_t(t) &= 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \\
  u(x,0) &= u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega,
\end{align*}
\]

where the operator \( A \) is in the form

\[
A u = - \text{div}(A \nabla u)
\]
when $A$ is a symmetric matrix
\begin{equation}
A(x) = (a_{ij}(x))_{1 \leq i,j \leq n} \tag{1.6}
\end{equation}
with coefficients $a_{ij} \in C^1(\overline{\Omega})$ and satisfying
\begin{equation}
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \tag{1.7}
\end{equation}
for some constant $\alpha > 0$.

In condition (1.3), $k : [0, +\infty) \to \mathbb{R}$ is a function of class $C^2$, $b$ is a positive constant and $\frac{\partial u}{\partial \nu_A}$ is the co-normal derivative
\begin{equation}
\frac{\partial u}{\partial \nu_A} = \langle Au, \nu \rangle, \tag{1.8}
\end{equation}
where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$ and $\nu(x)$ denotes the outward unit normal vector to the point $x \in \Gamma$. In the sequel we will also use $v \cdot w$ to denote the usual inner product between two vectors $v, w$. Moreover, we will use $u_t$ to denote the time derivative of $u$.

We consider $b$ constant only for the sake of simplicity. See Remark 4.5 for $b = b(x)$ with $b(\cdot)$ continuous and $b(x) > 0$, $\forall x \in \Gamma_1$. For the case $b(x) = (x - x_0) \cdot \nu(x)$, see also Remark 4.6 and [13].

System (1.1)–(1.4) comes, for instance, from linear models for evolution of sound in a compressible fluid. The boundary condition (1.3) describes the reflection of sound at surfaces of some materials with memory of interest in engineering practice. It is quite general and covers a fairly large variety of physical configurations. We refer to [18] for some discussions about this model and to [16] for related acoustic models.

Another approach giving a boundary memory condition like (1.3) is the modeling of the boundary of $\Omega$ as the surface of a viscoelastic material. For more details see [19] and the references given there. Frictional dissipative boundary condition (i.e., the case $k = 0$ in (1.3)) for the wave equation was studied by many authors, see [11–13] and the references cited there. On the contrary for boundary condition with memory, only a few number of papers exists [1,2,4,8,10,20–22]. In these papers, the authors consider the wave equation with constant coefficients operator and (mainly) linear dissipative boundary conditions of memory type and prove the exponential or polynomial decay of the energy by combining the multiplier method with the use of a suitable Lyapunov functional or integral inequalities.

The main goal of the present work is to extend the previous results to the case of variable coefficients and to linear or nonlinear dissipative boundary conditions of memory type by using the approach from differential geometry initiated in [24] and by introducing suitable Lyapunov functionals.

To obtain our stability results we assume that the kernel $k(\cdot)$ in the boundary condition (1.3) satisfies one of the following sets of assumptions:
\begin{equation}
k(t) \geq 0, \quad k'(t) \leq -\gamma_0 k(t), \quad k''(t) \geq -\gamma_1 k'(t) \tag{1.9}
\end{equation}
for positive constants $\gamma_0$, $\gamma_1$; or
\begin{equation}
k(t) \geq 0, \quad k'(t) \leq -\gamma_0 [k(t)]^{1 + \frac{1}{p}}, \quad k''(t) \geq \gamma_1 [-k'(t)]^{1 + \frac{1}{p+1}} \tag{1.10}
\end{equation}
for positive constants $\gamma_0$, $\gamma_1$ and for some $p > 1$. 
Note that assumptions (1.9) imply that the function \( k \) and \(-k'\) are exponentially decaying to 0; while assumptions (1.10) imply that \( k \) and \(-k'\) are polynomially decaying to 0 as \( 1/(1 + t)^p \) and \( 1/(1 + t)^{p+1} \) respectively.

Furthermore, we consider the stabilization for problem (1.1), (1.2), (1.3) with a nonlinear boundary feedback

\[
\frac{\partial u}{\partial \nu_A}(t) + \int_0^t k(t - s)u_t(s)\,ds + bg(u(t)) = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty),
\] (1.11)

where \( g \in C(\mathbb{R}) \) is a nondecreasing function such that

\[
g(0) = 0 \quad \text{and} \quad g(s) > 0, \quad \forall s \neq 0. \tag{1.12}
\]

Moreover, we assume that

\[
|g(s)| \leq C_1 |s|, \quad \forall s \in \mathbb{R},
\] (1.13)

for some \( C_1 > 0 \).

Indeed, without the memory term it is well known (see, e.g., [26]) that under some assumptions on \( g \) it is possible to give polynomial or, under more restrictive assumptions, exponential stability results. So, it is natural to look for analogous results in the case of a boundary condition of memory type.

To get the exponential stability, we will require that \( g \) satisfies

\[
|g(s)| \geq C_2 |s|, \quad \forall s \in \mathbb{R},
\] (1.14)

for some \( C_2 > 0 \) (see for instance [26]). Note that the two properties (1.13) and (1.14) are equivalent to

\[
|s|^2 + |g(s)|^2 \leq Cg(s)s, \quad \forall s \in \mathbb{R},
\] (1.15)

for some \( C > 0 \).

To prove the polynomial stability, we need to assume that \( g \) satisfies (1.13), while (1.14) is replaced by

\[
|g(s)| \geq C_2 \inf\{|s|, |s|^q\}, \quad \forall s \in \mathbb{R},
\] (1.16)

for some \( C_2 > 0 \) and \( q > 1 \) (see for instance [26]).

In order to deal with variable coefficients we need some technical assumptions.

According to [24] (see also [14]) let us introduce the Riemannian metric generated by the spatial operator. Let

\[
G(x) = (g_{ij}(x))_{1 \leq i,j \leq n} = A^{-1}(x).
\]

For any \( x \in \mathbb{R}^n \) define the inner product and the norm on the tangent space \( \mathbb{R}_x^n = \mathbb{R}^n \) by

\[
g(X,Y) = \langle X, Y \rangle_g = \sum_{i,j=1}^n g_{ij}(x)\alpha_i\beta_j, \quad \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}_x^n; \tag{1.17}
\]
Assumptions on $A$. As in [24] we assume that there exists a $C^1$ vector field $H$ in the Riemannian metric $(\mathbb{R}^n, g)$ such that
\begin{equation}
\langle DXH, X \rangle_g \geq a_0 |X|^2_g, \quad \forall x \in \overline{\Omega}, \forall X \in \mathbb{R}^n_x, \quad (1.19)
\end{equation}
for some positive constant $a_0$. Here $D$ denotes the Levi-Civita connection in the Riemannian metric $g$ and $DXH$ is the covariant derivative of $H$ with respect to $X$ (see Section 3 for more details).

Moreover we assume that
\begin{equation}
\sup_{\Omega} \text{div} H < \inf_{\overline{\Omega}} \text{div} H + 2a_0, \quad (1.20)
\end{equation}
and
\begin{equation}
H \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad H \cdot \nu \geq \delta \quad \text{on} \quad \Gamma_1, \quad (1.21)
\end{equation}
for a constant $\delta > 0$. For other conditions between $\Gamma_0$ and $\Gamma_1$, see Remark 4.6 and Section 7.

Remark 1.1. Assumption (1.19) has been introduced by Yao in [24] to extend to the case of variable coefficients the standard identity with multiplier. Obviously it holds in the case of constant coefficients taking as $H$ the standard multiplier $m(x) = x - x_0$. Note that in this case also (1.20) is verified. We refer to [24, 14, 9] for examples of function $H$ verifying this assumption in the nonconstant case. We also refer to Section 8 for examples verifying the assumptions (1.19) and (1.20). In general it is not true that for any variable coefficients and domain $\Omega$ a function $H$ verifying this assumption exists.

Nevertheless the assumption (1.19) on the coefficients is quite natural and is often a necessary condition for control results. We refer to the papers of Yao [24] and Macià and Zuazua [15] for examples of regular coefficients not satisfying this property and for which the corresponding wave equation is not controllable or stabilizable, since there exist turning waves that never hit the exterior boundary.

We define the energy of the problem (1.1)–(1.4) by
\begin{equation}
E(t) := \frac{1}{2} \int_{\Omega} \left\{ u_t^2 + \langle A \nabla u, \nabla u \rangle \right\} \, dx + \frac{1}{2} \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma - \frac{1}{2} \int_{\Gamma_1} \int_0^t k'(t - s) [u(t) - u(s)]^2 \, ds \, d\Gamma. \quad (1.22)
\end{equation}

Note that this energy is nonincreasing (see Lemma 2.3 below).

In the following we will call regular solution of problem (1.1)–(1.4) or of problem (1.1), (1.2), (1.4) and (1.11), a solution corresponding to an initial datum $(u_0, u_1) \in (H^2(\Omega) \cap H^{1+\epsilon}_0(\Omega)) \times H^1_{\nu}(\Omega)$ which satisfies the compatibility condition
\begin{equation}
\frac{\partial u_0}{\partial \nu_A} + bu_1 = 0 \quad \text{on} \quad \Gamma_1. \quad (1.23)
\end{equation}
or

\[
\frac{\partial u_0}{\partial \nu_A} + bg(u_1) = 0 \quad \text{on } \Gamma_1
\]  

(1.24)

respectively. In the first case, \( u \) will be in \( C^2(\mathbb{R}^+; L^2(\Omega)) \cap C^1(\mathbb{R}^+; H^1_0(\Omega)) \cap C(\mathbb{R}^+; H^2(\Omega)) \), while in the second case, it will have a similar, but weaker, regularity (see Theorem 2.1 and Corollary 6.3 below).

Now we are able to state the main results of our paper. For the linear problem (1.1)–(1.4) or problem (1.1), (1.2), (1.4) with the nonlinear boundary condition (1.11), we can give an exponential stability result when the kernel \( k \) is exponentially decaying to 0, while we obtain a polynomial stability result when \( k \) is polynomially decaying to 0.

**Theorem 1.2.** Assume that \( \Gamma = \Gamma_0 \cup \Gamma_1 \) with \( \Gamma_0, \Gamma_1 \) closed sets with \( \Gamma_0 \cap \Gamma_1 = \emptyset \), that the matrix \( A \) satisfies (1.7) and that \( b > 0 \). Furthermore, assume that there exists a \( C^1 \) vector field \( H \) verifying (1.19), (1.20), (1.21).

(a) If the memory kernel \( k \) satisfies (1.9) and \( g \) satisfies (1.12) and (1.15). Then there exist two positive constants \( C_1, C_2 \) such that for any regular solution of problem (1.1)–(1.4),

\[
E(t) \leq C_1 E(0) e^{-C_2 t}, \quad \forall t > 0.
\]  

(1.25)

(b) If the memory kernel \( k \) satisfies (1.10). Let \( r \) be a real number such that

\[
\frac{1}{p+1} < r < 1
\]

and let \( g \) satisfy (1.12), (1.13) and (1.16) with \( q = 1 + \frac{2}{(1-r)(p+1)} \). Then, for any regular solution \( u \) of problem (1.1)–(1.4),

\[
E(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}}, \quad \forall t > 0
\]  

(1.26)

for a suitable positive constant \( C \) depending on \( E(0) \) and on \( r \).

Note that in the case of \( k \) satisfying (1.10) we do not have that the energy decays at the same rate as \( k \) (i.e., \( 1/(1+t)^p \)) but, for every \( 0 < s < p \), we obtain

\[
E(t) \leq \frac{C}{(1+t)^s},
\]

for a suitable positive constant \( C \) depending on \( E(0) \) and on \( s \).

It is well-known that the term \( bu_t \) in the boundary condition (1.3) is dissipative, this means that the condition

\[
\frac{\partial u}{\partial \nu_A} + bu_t = 0 \quad \text{on } \Gamma_1,
\]
allows to stabilize the wave equation (1.1) with Dirichlet boundary condition (1.2) on \( \Gamma_0 \). On the contrary, the memory term alone seems to be not sufficient to obtain a stability result. Indeed the term \( bu_t \) gives a term \( -b \int_{\Gamma_1} u^2_t(s) \, ds \) in the derivative of the energy (see Lemma 2.3) that allows to balance an analogous term with an opposite sign coming from the multiplier method. We therefore do not know if it is possible to obtain stabilization results in the case \( b = 0 \). Analogous remarks hold for the term \( bg(u_t) \) in condition (1.11) under appropriate assumptions on \( g \). Note further that from the above theorem, we see some interactions between the two damping terms since the worse term always governs the decay of the energy.

Finally in the case \( \Gamma_0 \cap \Gamma_1 \neq \emptyset \), using the techniques from [7,3] (mainly based on an adequate multiplier method) and an adapted feedback law, we can also obtain some stabilization results for the wave equation. Similarly in the case \( \Gamma_0 = \emptyset \), by modifying adequately the boundary condition as in [25], we prove the exponential decay of the energy when \( k \) satisfies (1.9).

Let us now compare our paper with existing stabilization results. In [8] the author studies the wave equation (i.e., \( u_{tt} - \Delta u = 0 \)) with boundary condition of memory type when the kernel \( k \) satisfies (compare with (1.9))

\[
\begin{align*}
  k \geq 0, \quad k' \leq 0, \quad k'' \geq -\alpha k',
\end{align*}
\]

for some \( \alpha > 0 \) and the additional assumption

\[
\alpha \inf_{x \in \Gamma_1} k(x,0) > -2 \inf_{x \in \Gamma_1} k'(x,0).
\]

If \( u_0 \) is zero on \( \Gamma_1 \), Guesmia shows the exponential stability of the system using the multiplier method and integral inequalities. The extension to a nonlinear boundary condition of memory type was made in [1] with the same assumptions on \( k \) but without the above assumption on \( u_0 \). Using the same techniques than the ones in [8], the authors show the exponential stability if \( g \) is linear and a polynomial decay if \( g \) is linear at infinity and satisfies

\[
C_1 |x|^p \leq |g(x)| \leq C_2 |x|^{1/p}, \quad \forall |x| \leq 1,
\]

for some \( C_1, C_2 > 0 \). In [10], the wave equation was considered with a boundary condition of memory type when \( b = 0 \) and therefore the lack of dissipativeness is compensated by an internal damping term. For \( k(x,t) = p(x)e^{-t} \), with \( p(x) \geq 0 \), the exponential stability is proved by introducing an appropriate Lyapounov functional. In [2], the authors consider the one-dimensional nonlinear wave equation

\[
u_{tt} - [\sigma(u_x)]_x = f \quad \text{in } (0,1) \times \mathbb{R}^+,
\]

with Dirichlet boundary condition \( u(0,t) = 0 \) and a nonlinear boundary condition of memory type at 1, in the sense that \( u_x \) is replaced by \( \sigma(u_x) \), for some smooth enough function \( \sigma \). If the kernel \( k \) satisfies

\[
k' < 0, \quad -\alpha k' \leq k'' \leq -\beta k',
\]
for some positive real number $\alpha, \beta$, the authors obtain the exponential stability using an appropriate Lyapunov functional. Similarly if $k$ satisfies

$$k' < 0, \quad \alpha[k']^{1+1/p} \leq k'' \leq \beta[k']^{1+1/p},$$

$$\int_0^\infty [k'(t)]^{1-1/p} dt < \infty,$$

the polynomial decay is proved. In [21], the author considers the one-dimensional linear wave equation

$$u_{tt} - \mu(t)u_{xx} = 0 \quad \text{in} \ (0, 1) \times \mathbb{R}^+,$$

with Dirichlet boundary condition $u(0, t) = 0$ and a linear boundary condition of memory type at 1, with $u_x$ replaced by $\mu(t)u_x$. Under the assumptions

$$0 < k(t) \leq b_0 e^{-\gamma_0 t},$$

$$-b_1 k \leq k' \leq -b_2 k, \quad -b_3 k' \leq k'' \leq -b_4 k',$$

for some positive real numbers $\gamma_0$ and $b_i$ and using a Lyapunov functional technique, the author shows the exponential stability of the system. If $k$ decays polynomially (with quite strong restrictions), the author further shows the polynomial decay. Finally the multidimensional nonlinear wave equation

$$u_{tt} - \Delta u + F(x, t, u, \nabla u) = 0$$

with Dirichlet boundary condition on a part of the boundary and a boundary condition of memory type on the remainder part is studied in [4]. If $k$ satisfies

$$k \geq 0, \quad k' \leq 0, \quad k'' \geq \alpha[k']^{1+q},$$

for some $\alpha > 0$ and $q \geq 0$ and the additional assumption that $\inf_{t \geq t_0} k(t)$ is small enough for some $t_0 > 0$, then using a Lyapunov functional technique, the authors obtain the exponential decay if $q = 0$ and $u_0$ is zero on $\Gamma_1$ (if $u_0 \neq 0$ on $\Gamma_1$, a weaker decay is nevertheless obtained) and a polynomial decay if $0 < q < 1/2$ and $u_0$ is zero on $\Gamma_1$.

From the above papers, we can say that mainly the case of the wave equation with constant coefficients and linear boundary condition of memory type is treated in the literature (with the exception of [1]). Our main contribution is the systematic treatment of the wave equation with variable coefficients (in space) with a Dirichlet boundary condition on a part of the boundary and a linear or nonlinear boundary condition of memory type on the remainder part. Moreover our assumptions on $k$ and on $g$ are coherent with the ones imposed in the above mentioned papers and are sometimes weaker.

The paper is organized as follows: Well-posedness of the problem (1.1)–(1.4) is analyzed in Section 2 under appropriate conditions on $k$ using semigroup theory. We further show that the same conditions on $k$ guarantee the decay of the energy. In Section 3 we recall some results from differential geometry. We show in Section 4 the exponential stability of our system under some geometric assumptions. In Section 5
we prove the polynomial decay of the energy under polynomial growth conditions on $k$ and $k'$. Section 6 is devoted to the study of problem (1.1), (1.2), (1.3) with the nonlinear boundary feedback (1.11). We prove a well-posedness result using nonlinear semigroup theory and exponential or polynomial stability results under appropriate assumptions on $g$. Section 7 analyzes briefly the case of $\overline{T}_0 \cap \overline{T}_1 \neq \emptyset$ and the case $I_0 = \emptyset$. Finally in Section 8 we end up with some examples, where the geometric assumptions are illustrated.

2. Well-posedness of the linear problem

In this section, we consider the existence and regularity of solutions to our system (1.1)–(1.4). We further show that our system is dissipative by showing that its energy decays.

With our assumptions (1.9) or (1.10), problem (1.1)–(1.4) can be formulated as an evolutionary integral equation of variational type [18,19]. Therefore the results from [18,19] allow to state the following results, where we recall that

$$H^1_{I_0}(\Omega) := \{ u \in H^1(\Omega): u = 0 \text{ on } I_0 \}.$$

Theorem 2.1. Let the above assumptions on $k$ be satisfied. Then for all initial data $(u_0, u_1) \in H^1_{I_0}(\Omega) \times L^2(\Omega)$, there exists a unique weak solution $u \in C^1(\mathbb{R}^+; L^2(\Omega)) \cap C(\mathbb{R}^+; H^1_{I_0}(\Omega))$ of (1.1)–(1.4).

If furthermore $(u_0, u_1) \in (H^2(\Omega) \cap H^1_{I_0}(\Omega)) \times H^1_{I_0}(\Omega)$ satisfies the compatibility condition (1.23), then the weak solution $u$ of (1.1)–(1.4) has the regularity $u \in C^2(\mathbb{R}^+; L^2(\Omega)) \cap C^1(\mathbb{R}^+; H^1_{I_0}(\Omega)) \cap C(\mathbb{R}^+; H^2(\Omega))$. Such a solution is called a regular solution.

Remark 2.2. In [4,21,22], instead of (1.3) the authors consider the boundary condition

$$u(t) + \int_0^t g(t - s) \frac{\partial u}{\partial \nu}(s) \, ds = 0 \quad \text{on } I_1 \times (0, +\infty),$$

(2.1)

for some function $g$. But in these papers it was shown, applying Volterra’s inverse operator, that (2.1) can be rewritten as

$$\frac{\partial u}{\partial \nu}(t) = -\frac{1}{g(0)} \left( u(t) + k(0)u(t) - k(t)u(0) + \int_0^t k'(t - s)u(s) \, ds \right),$$

(2.2)

where the resolvent kernel $k$ satisfies

$$k(t) + \frac{1}{g(0)} \int_0^t g'(t - s)k(s) \, ds = -\frac{1}{g(0)} g'(t).$$

Moreover, under the assumption $u_0 = u(0) = 0$ on $I_1$, condition (2.2) implies (2.1). Therefore our analysis below also holds for such a boundary condition. Note further that any strong solution of problem (1.1), (1.2), (2.1) and (1.4) satisfies $u(0) = 0$ on $I_1$ and therefore the assumption $u_0 = 0$ on $I_1$ is necessary to have strong solutions to that system.
Lemma 2.3. Any regular solution of the problem (1.1)–(1.4) verifies

\[ E'(t) = \frac{1}{2} \int_{\Gamma_1} k'(t)[u(t) - u(0)]^2 \, d\Gamma - b \int_{\Gamma_1} u_t^2 \, d\Gamma \]

\[ - \frac{1}{2} \int_{\Gamma_1} \int_0^t k''(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma \leq 0. \]  

(2.3)

Proof. Differentiating (1.22), we have

\[ E'(t) = \int_{\Omega} \{ u_t u_{tt} + \langle A\nabla u, \nabla u_t \rangle \} \, dx + \frac{1}{2} \int_{\Gamma_1} k'(t)[u(t) - u(0)]^2 \, d\Gamma \]

\[ - \frac{1}{2} \int_{\Gamma_1} \int_0^t k''(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma + \int_{\Gamma_1} k(t)[u(t) - u(0)]u_t(t) \, d\Gamma \]

\[ - \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]u_t(t) \, ds \, d\Gamma. \]  

(2.4)

From (2.4) integrating by parts we obtain

\[ E'(t) = \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu_A} \, d\Gamma - \frac{1}{2} \int_{\Gamma_1} \int_0^t k''(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma - \int_{\Gamma_1} k(t)u(0)u_t(t) \, d\Gamma \]

\[ + \int_{\Gamma_1} k(t)u(t)u_t(t) \, d\Gamma + \int_{\Gamma_1} u_t(t) \int_0^t k'(t-s)u(s) \, ds \, d\Gamma + \frac{1}{2} \int_{\Gamma_1} k'(t)[u(t) - u(0)]^2 \, d\Gamma \]

\[ - \int_{\Gamma_1} u(t)u_t(t) \int_0^t k'(t-s) \, ds \, d\Gamma \]

and then

\[ E'(t) = \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu_A} \, d\Gamma - \frac{1}{2} \int_{\Gamma_1} \int_0^t k''(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma - \int_{\Gamma_1} k(t)u(0)u_t(t) \, d\Gamma \]

\[ + \int_{\Gamma_1} k(0)u(t)u_t(t) \, d\Gamma + \int_{\Gamma_1} u_t(t) \int_0^t k'(t-s)u(s) \, ds \, d\Gamma \]

\[ + \frac{1}{2} \int_{\Gamma_1} k'(t)[u(t) - u(0)]^2 \, d\Gamma. \]  

(2.5)

Now, observe that the boundary condition (1.3) can be rewritten as

\[ \frac{\partial u}{\partial \nu_A} + bu_t(t) + \int_0^t k'(t-s)u(s) \, ds - k(t)u(0) + k(0)u(t) = 0, \]  

on \( \Gamma_1 \).  

(2.6)

Using (2.6) we can rewrite

\[ \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu_A} \, d\Gamma = -b \int_{\Gamma_1} u_t^2(t) \, d\Gamma - \int_{\Gamma_1} u_t(t) \int_0^t k'(t-s)u(s) \, ds \, d\Gamma \]

\[ + \int_{\Gamma_1} k(t)u(0)u_t(t) \, d\Gamma - \int_{\Gamma_1} k(0)u(t)u_t(t) \, d\Gamma, \]  

(2.7)
which substituted in (2.5) gives the identity (2.3). The fact that the energy is decreasing follows by the assumptions on \(k, k', k''\). □

Our next purpose is to find sufficient conditions on \(\Omega\) and on the operator \(A\) in order to guarantee the exponential decay of the energy under the assumption (1.9) and the polynomial decay under assumption (1.10).

3. Riemannian metric generated by the spatial operator

Here we describe in more details the Riemannian metric defined by (1.17) and we give some calculation rules.

For \(f \in C^1(\Omega)\) we define the gradient of \(f\) in the Riemannian metric \(g\) by

\[
\nabla_g f(x) = A(x) \nabla f(x),
\]

where \(\nabla f\) is the usual Euclidean gradient and the co-normal derivative of \(f\) is defined in (1.8).

Denote the Levi-Civita connection in the Riemannian metric \(g\) by \(\mathcal{D}\). Let

\[
H = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i}, \quad X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}
\]

be vector fields on \((\mathbb{R}^n, g)\). The covariant differential \(\mathcal{D}H\) of \(H\) determines a bilinear form on \(\mathbb{R}^n \times \mathbb{R}^n\), for any \(x \in \mathbb{R}^n\), defined by

\[
\mathcal{D}H(Y, X) = \langle \mathcal{D}_X H, Y \rangle_g \quad \forall X, Y \in \mathbb{R}^n,
\]

where \(\mathcal{D}_X H\) is the covariant derivative of \(H\) with respect to \(X\)

\[
\mathcal{D}_X H = \sum_{k=1}^{n} \mathcal{D}_X \left( h_k \frac{\partial}{\partial x_k} \right) = \sum_{k=1}^{n} X \cdot \nabla h_k \frac{\partial}{\partial x_k} + \sum_{k,j=1}^{n} h_k \xi_j \mathcal{D}_{\partial/\partial x_j} \left( \frac{\partial}{\partial x_k} \right),
\]

where

\[
\mathcal{D}_{\partial/\partial x_j} \left( \frac{\partial}{\partial x_k} \right) = \sum_{l=1}^{n} \Gamma_{jk}^{l} \frac{\partial}{\partial x_l},
\]

\(\Gamma_{jk}^{l}\) being the Christoffel symbols of the connection \(\mathcal{D}\),

\[
\Gamma_{jk}^{l} = \frac{1}{2} \sum_{p=1}^{n} \alpha_{kp} \left( \frac{\partial g_{jp}}{\partial x_l} + \frac{\partial g_{jp}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_p} \right),
\]

Then, by (3.3) and (3.4)

\[
\mathcal{D}_X H = \sum_{l=1}^{n} \left( X \cdot \nabla h_l + \sum_{k,i=1}^{n} h_k \xi_i \Gamma_{jk}^{l} \right) \frac{\partial}{\partial x_l}.
\]
Therefore, by (3.5)

\[ DH(X, X) = \langle DXH, X \rangle_g = \sum_{i,j=1}^{n} \left( \sum_{l=1}^{n} \frac{\partial h_l}{\partial x_i} g_{ij} + \sum_{k,l=1}^{n} h_k g_{ij} F_{ik}^d \right) \xi_i \xi_j. \]  

(3.6)

See [24] for more details.

**Remark 3.1.** Observe that assumption (1.19) is verified if there exists a function \( v \) of class \( C^2 \) strictly convex with respect to the metric \( g \), that is a function \( v \) such that

\[ D^2v(X, X) = \langle DX(\nabla g v), X \rangle_g \geq a_0 |X|^2_g, \quad \forall x \in \Omega, \forall X \in \mathbb{R}^n. \]

In that case (1.19) holds with \( H = \nabla g v \), see [14].

We will need the following lemma. See [24, Lemma 2.1] for a proof.

**Lemma 3.2.** Let \( f, h \) scalar functions in \( C^1(\overline{\Omega}) \) and let \( H \) be a \( C^1 \) vector field. Then

\[ \langle \nabla g f, \nabla g h \rangle_g = \nabla g f \cdot \nabla h = \nabla f \cdot A \nabla h, \]  

(3.7)

\[ \langle \nabla g f, \nabla g (H \cdot \nabla f) \rangle_g = DH(\nabla g f, \nabla g f) - \frac{1}{2} \nabla g f^2_g \text{div} \, H + \frac{1}{2} \text{div}(|\nabla g f|_g^2 H), \]  

(3.8)

where \( A \) is given in (1.6).

### 4. The exponential stability result

Consider the standard energy

\[ E(t) := \int_{\Omega} \left\{ u_t^2 + \langle A \nabla u, \nabla u \rangle \right\} \, dx, \]  

(4.1)

and define

\[ Mu = 2(H \cdot \nabla u) + \theta u, \]  

(4.2)

where \( H \) is defined in assumption (1.19) and \( \theta \) is a constant such that

\[ \sup_{\Omega} \text{div} \, H - 2a_0 < \theta < \inf_{\Omega} \text{div} \, H, \]  

(4.3)

By the assumption (1.20) such a constant \( \theta \) exists.

**Lemma 4.1.** For any regular solution of the problem (1.1)–(1.4)

\[ \frac{d}{dt} \left\{ \int_{\Omega} u_t Mu \, dx \right\} \leq \int_{\Omega} (\theta - \text{div} \, H) u_t^2 \, dx + \int_{\Omega} (\text{div} \, H - \theta - 2a_0) \langle A \nabla u, \nabla u \rangle \, dx \]

\[ + \int_{\Gamma_1} H \cdot \nu (u_t^2 - \langle A \nabla u, \nabla u \rangle) \, d\Gamma + \int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2H \cdot \nabla u + \theta u) \, d\Gamma. \]  

(4.4)
Proof. Differentiating \( \int_{\Omega} u_t M u \, dx \) we have
\[
\frac{d}{dt} \left\{ \int_{\Omega} u_t M u \, dx \right\} = 2 \int_{\Omega} u_t H \cdot \nabla u_t \, dx + \theta \int u_t^2 \, dx - 2 \int_{\Omega} A u (H \cdot \nabla u) \, dx - \theta \int_{\Omega} u A u \, dx.
\] (4.5)

Applying Green’s formula and the identities (3.7), (3.8), we get
\[
\int_{\Omega} A u (H \cdot \nabla u) \, dx = \int_{\Omega} A \nabla u \cdot \nabla (H \cdot \nabla u) \, dx - \int_{\Gamma} \frac{\partial u}{\partial \nu} A H \cdot \nabla u \, d\Gamma
\]
\[
+ \frac{1}{2} \int_{\Omega} \text{div} (\langle A \nabla u, \nabla u \rangle H) \, dx - \frac{1}{2} \int_{\Omega} \langle A \nabla u, \nabla u \rangle \text{div} H \, dx.
\] (4.6)

Now, recalling (1.19), (4.6) gives
\[
\int_{\Omega} A u (H \cdot \nabla u) \, dx \geq \int_{\Omega} a_0 \langle A \nabla u, \nabla u \rangle \, dx - \frac{1}{2} \int_{\Omega} \langle A \nabla u, \nabla u \rangle \text{div} H \, dx + \frac{1}{2} \int_{\Gamma} (\langle A \nabla u, \nabla u \rangle H \cdot \nu) \, d\Gamma. \tag{4.7}
\]

Moreover, integrating by parts, we may write
\[
\int_{\Omega} u_t A u \, dx = \int_{\Omega} \langle A \nabla u, \nabla u \rangle \, dx - \int_{\Gamma} \frac{\partial u}{\partial \nu} A u \, d\Gamma,
\] (4.8)

and
\[
\int_{\Omega} 2 u_t H \cdot \nabla u_t \, dx = \int_{\Omega} H \cdot \nabla u_t^2 \, dx = - \int_{\Omega} u_t^2 \, \text{div} H \, dx + \int_{\Gamma} H \cdot \nu u_t^2 \, d\Gamma. \tag{4.9}
\]

Using (4.7), (4.8) and (4.9) in (4.5) we have
\[
\frac{d}{dt} \left\{ \int_{\Omega} u_t M u \, dx \right\} \leq \int_{\Omega} (\theta - \text{div} H) u_t^2 \, dx + \int_{\Omega} (\text{div} H - \theta - 2a_0) \langle A \nabla u, \nabla u \rangle \, dx + \int_{\Gamma} H \cdot \nu (u_t^2 - \langle A \nabla u, \nabla u \rangle) \, d\Gamma + \int_{\Gamma} \frac{\partial u}{\partial \nu} (2 H \cdot \nabla u + \theta u) \, d\Gamma. \tag{4.10}
\]

Now, to obtain (4.4) it suffices to observe that by (1.2),
\[
\nabla u = \langle \nabla u, \nu \rangle \nu = \frac{\partial u}{\partial \nu} \nu, \quad \text{on } \Gamma_0.
\]
So, by the boundary condition (1.2), the ellipticity assumption (1.7) and the assumption (1.21) on $H$, we obtain

$$\int_{\Gamma_0} \frac{\partial u}{\partial \nu} (2H \cdot \nabla u + \theta u) \, d\Gamma - \int_{\Gamma_0} H \cdot \nu \langle A \nabla u, \nabla u \rangle \, d\Gamma$$

$$= 2 \int_{\Gamma_0} \frac{\partial u}{\partial \nu} H \cdot \nu \langle A \nabla u, \nu \rangle \, d\Gamma - \int_{\Gamma_0} \left( \frac{\partial u}{\partial \nu} \right)^2 H \cdot \nu \langle A \nu, \nu \rangle \, d\Gamma$$

$$= \int_{\Gamma_0} \left( \frac{\partial u}{\partial \nu} \right)^2 H \cdot \nu \langle A \nu, \nu \rangle \, d\Gamma \leq 0.$$ 

This ends the proof. \(\Box\)

**Proposition 4.2.** Assume that $k$ satisfies (1.9). Then, for any regular solution of the problem (1.1)–(1.4)

$$\frac{d}{dt} \left\{ \int_{\Omega} u_t M u \, dx \right\} \leq -c_0 \mathcal{E}(t) + C \left\{ \int_{\Gamma_1} u_t^2 \, d\Gamma + \int_{\Gamma_1} k(t)[u(t) - u(0)]^2 \, d\Gamma \right.$$ 

$$\left. - \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma \right\},$$

(4.11)

for suitable positive constants $c_0, C$.

**Proof.** Using Young’s inequality, (4.4) implies

$$\frac{d}{dt} \left\{ \int_{\Omega} u_t M u \, dx \right\} \leq \int_{\Omega} \left( \sup_{\Omega} \text{div} \, H - \theta - 2a_0 \right) \langle A \nabla u, \nabla u \rangle \, dx$$

$$+ \int_{\Omega} \left( \theta - \inf_{\Omega} \text{div} \, H \right) u_t^2 \, dx + \int_{\Gamma_1} H \cdot \nu u_t^2 \, d\Gamma - \delta \int_{\Gamma_1} \langle A \nabla u, \nabla u \rangle \, d\Gamma$$

$$+ \frac{C}{\varepsilon} \int_{\Gamma_1} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\Gamma + \varepsilon \int_{\Gamma_1} (|\nabla u|^2 + u^2) \, d\Gamma,$$

(4.12)

where $C$ is a suitable positive constant, $\delta$ is as in assumption (1.21) and $\varepsilon$ is an arbitrary small constant.

Using Poincaré’s theorem, the trace inequality and recalling (1.7), from (4.12), by choosing $\varepsilon$ sufficiently small, we obtain

$$\frac{d}{dt} \left\{ \int_{\Omega} u_t M u \, dx \right\} \leq -c_0 \mathcal{E}(t) + C \int_{\Gamma_1} u_t^2 \, d\Gamma + C \int_{\Gamma_1} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\Gamma,$$

(4.13)

for suitable positive constants $c_0, C$.

Note that the boundary condition (1.3) may be rewritten as

$$\frac{\partial u}{\partial \nu} = bu_t(t) - \int_0^t k'(t-s)[u(t) - u(s)] \, ds + k(t)[u(t) - u(0)] = 0,$$  

(4.14)
and from Cauchy–Schwarz’s inequality
\[
\left( \int_0^t k'(t-s)[u(t)-u(s)] \, ds \right)^2 \leq \left[ k(0) - k(t) \right] \int_0^t (-k'(t-s)) \left[ u(t) - u(s) \right]^2 \, ds.
\] (4.15)

Then, using (4.14) and (4.15) we can estimate
\[
\int_{\Gamma_1} \left( \frac{\partial u}{\partial v_A} \right)^2 \, d\Gamma \leq C \left\{ \int_{\Gamma_1} u_t^2 \, d\Gamma + \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma 
+ \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma \right\},
\] (4.16)

that substituted in (4.13) gives (4.11). □

**Proof of Theorem 1.2(a), case \( g \) linear.** By the assumptions on \( k, k', k'' \), we can obtain from the identity (2.3) the estimate
\[
E'(t) \leq -b \int_{\Gamma_1} u_t^2(t) \, d\Gamma - \gamma_0 \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma
+ \frac{\gamma_1}{2} \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma.
\] (4.17)

Define the Lyapunov functional

\[
\tilde{E}(t) := E(t) + \gamma \int_{\Omega} u(t) M u(t) \, dx,
\] (4.18)

where \( \gamma \) is a positive constant chosen sufficiently small later on.

Then, by the inequalities (4.11) and (4.16), for \( \gamma \) sufficiently small we have
\[
\tilde{E}'(t) \leq -c_0 \gamma \tilde{E}(t) - \frac{b}{2} \int_{\Gamma_1} u_t^2(t) \, d\Gamma - \frac{\gamma_0}{4} \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma
+ \frac{\gamma_1}{4} \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma,
\] (4.19)

from which follows,
\[
\tilde{E}'(t) \leq -c_1 \tilde{E}(t), \quad \forall t > 0,
\] (4.20)

for a suitable constant \( c_1 \).

Observing that \( \tilde{E}(t) \leq CE(t) \), for a suitable positive constant \( C \), inequality (4.19) implies
\[
\tilde{E}'(t) \leq -c_1 \tilde{E}(t).
\]

This means that \( \tilde{E} \) is exponentially decreasing. Therefore, since \( E(t) \leq c\tilde{E}(t) \) for a suitable constant \( c > 0 \) (for \( \gamma \) sufficiently small), the estimate (1.25) immediately follows. □
Remark 4.3. Note that if the initial datum $u_0$ belongs to $H^1_0(\Omega)$, in order to have exponential decay of the energy it is sufficient to assume that there exists $T > 0$ for which

$$\sup_{\{T,+\infty\}} k(t)$$

is sufficiently small, instead to assume that $k$ is exponentially decreasing. Indeed, in this case, the term

$$-\frac{1}{2} \int_{\Gamma_1} k(t)[u(t) - u(0)]^2 \, d\Gamma$$

in (4.11) can be estimated using the trace inequality and Poincaré’s theorem.

Remark 4.4. Note that everything still holds if the function $k$ depends also on the variable space, that is,

$$k : \Gamma_1 \times [0, \infty) \rightarrow \mathbb{R}, \quad k \in C^2(\Gamma_1 \times [0, \infty)).$$

The same remark applies to the following section. We have considered $k := k(t)$ only for simplicity of notation.

Remark 4.5. The above exponential stability result holds also if in condition (1.3) $b = b(x)$ with $b : \Gamma_1 \rightarrow \mathbb{R}$ continuous and satisfying

$$b(x) > 0, \quad \forall x \in \Gamma_1.$$

In this case (2.3) becomes

$$E'(t) = \frac{1}{2} \int_{\Gamma_1} k'(t)[u(t) - u(0)]^2 \, d\Gamma - \int_{\Gamma_1} b(x)u_t^2 \, d\Gamma$$

$$-\frac{1}{2} \int_{\Gamma_1} \int_0^t k''(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma \leq 0.$$

Lemma 4.1 and Proposition 4.2 remain unchanged, while (4.18) becomes

$$\tilde{E}'(t) \leq -c_0 \tilde{\gamma} \tilde{E}(t) - \frac{1}{2} \min_{\Gamma_1} b(x) \int_{\Gamma_1} u_t^2(t) \, d\Gamma - \frac{\gamma_0}{4} \int_{\Gamma_1} k(t)[u(t) - u(0)]^2 \, d\Gamma$$

$$+ \frac{\gamma_1}{4} \int_{\Gamma_1} \int_0^t k'(t-s)[u(t) - u(s)]^2 \, ds \, d\Gamma,$$

from which follows (4.19) and therefore (1.25). Analogous remark applies to the polynomial stability result proved in the next section. See also Remark 4.6 for the case $b(x) := (x - x_0) \cdot \nu(x)$.

Remark 4.6. If $H(x) \cdot \nu(x) = 0$ for some $x \in \Gamma_1$, then a term like

$$\int_0^T \int_{\Gamma_1} |\nabla_r u|^2 \, d\Gamma \, dt,$$
where $\nabla_\tau$ denotes the tangential gradient, remains in the estimate of boundary terms. For the standard boundary condition

$$\frac{\partial u}{\partial \nu} + u_t = 0,$$

this term is usually eliminated (see, e.g., [14]) using micro-local analysis and compactness arguments. This compactness argument cannot be invoked here since we need an estimate independent of $T$.

Note that a feedback like

$$\frac{\partial u}{\partial \nu} + H \cdot \nu \left( \int_0^t k(t - s) u'(s) \, ds + b u(t) \right) = 0$$

avoids this problem and allows to consider $H \cdot \nu \geq 0$ on $\Gamma_1$ (cf. [13]).

5. Polynomial decay of the energy

In this section we assume that the function $k$ in the boundary condition (1.3) satisfies the assumptions (1.10). Even in this case Lemma 4.1 holds. Moreover we obtain, instead of Proposition 4.2, the following result.

**Proposition 5.1.** Assume that $k$ satisfies (1.10). Then, for any regular solution of the problem (1.1)–(1.4)

$$\frac{d}{dt} \left\{ \int_\Omega u_t M u \, dx \right\} \leq -c_0 E(t) + C \int_{\Gamma_1} [k(t)]^{1+\frac{1}{p}} \left[ u(t) - u(0) \right]^2 \, d\Gamma$$

$$+ \int_{\Gamma_1} \int_0^t \left[ -k'(t - s) \right]^{1+\frac{1}{p+1}} [u(t) - u(s)]^2 \, ds \, d\Gamma,$$

(5.20)

for suitable positive constants $c_0, C$.

**Proof.** Note that (4.13) still holds. From Cauchy–Schwarz’s inequality

$$\left\{ \int_0^t \left[ -k'(t - s) \right] [u(t) - u(s)] \, ds \right\}^2 \leq \int_0^t \left[ -k'(s) \right]^{\frac{p}{p+1}} \, ds \cdot \int_0^t \left[ -k'(t - s) \right]^{\frac{p+2}{p+1}} [u(t) - u(s)]^2 \, ds.$$

(5.21)

By (1.10) the function $-k'$ is decreasing as $1/(t + 1)^{p+1}$, then

$$\int_0^t \left[ -k'(s) \right]^{\frac{p}{p+1}} \, ds, \quad t > 0,$$
is bounded. So, (5.21) and the boundedness of \( k \) allow to obtain by (4.14)
\[
\int_{\Gamma_1} \left( \frac{\partial u}{\partial u_A} \right)^2 \, d\Gamma \leq C \left\{ \int_{\Gamma_1} u^2 \, d\Gamma + \int_{\Gamma_1} [k(t)]^{1+\gamma} \, d\Gamma \right\}
\]
\[
+ \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right]^{1+\gamma} \, ds \, d\Gamma \right\}, \tag{5.22}
\]
Using (5.22) in (4.13), inequality (5.20) immediately follows. \( \square \)

We are ready to prove the stability result.

**Proof of Theorem 1.2(b), case \( g \) linear.** By the assumption (1.10) we can obtain from the identity (2.3) the estimate
\[
E'(t) \leq -b \int_{\Gamma_1} \frac{1}{4} \Gamma_1 [k]^{1+\gamma} \, [u(t) - u(0)]^2 \, d\Gamma
\]
\[
- \gamma_1 \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right]^{1+\gamma} \, ds \, d\Gamma. \tag{5.23}
\]
Let \( \tilde{E}(\cdot) \) be the functional defined by (4.17). By (5.20) and (5.23), for \( \gamma \) sufficiently small, we have
\[
\tilde{E}'(t) \leq -c_0 \gamma \tilde{E}(t) - \frac{b}{2} \int_{\Gamma_1} \frac{1}{4} \Gamma_1 [k]^{1+\gamma} \, [u(t) - u(0)]^2 \, d\Gamma
\]
\[
- \frac{\gamma_1}{4} \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right]^{1+\gamma} \, ds \, d\Gamma. \tag{5.24}
\]
From Hölder’s inequality (see [21, Lemma 4.1]), we may write
\[
\int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right] \, [u(t) - u(s)]^2 \, ds \, d\Gamma
\]
\[
\leq \left\{ \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right]^{1+\gamma} \, ds \, d\Gamma \right\}^{\frac{1}{1+\gamma}} \left\{ \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right] \, [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{\frac{1}{1+\gamma}}.
\]
We then get
\[
\left\{ \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right] \, [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{\frac{1}{1+\gamma}}
\]
\[
\leq \int_{\Gamma_1} \int_0^t \left[ k'(t-s) \right]^{1+\gamma} \, ds \, d\Gamma
\]
\begin{eqnarray}
\times \left\{ \int_{\Gamma_t} \int_0^t [-k'(t-s)]^r [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{\frac{1}{1 - r(p+1)}} \\
\leq 2 \int_{\Gamma_t} \int_0^t [-k'(t-s)]^{1+\frac{1}{p+1}} [u(t) - u(s)]^2 \, ds \, d\Gamma \\
\times \left\{ \|u(s) - u(0)\|_{L^\infty(0,t;L^2(\Gamma_1))} \int_0^t [-k'(s)]^r \, ds \right\}^{\frac{1}{(1-r)(p+1)}}.
\end{eqnarray} \tag{5.25}

Now, observe that, by the trace inequality and Poincaré's theorem
\begin{eqnarray}
\int_{\Gamma_1} [u(t) - u(0)]^2 \, d\Gamma \\
\leq 2 \left( \int_{\Gamma_1} u^2(t) \, d\Gamma + \int_{\Gamma_1} u^2(0) \, d\Gamma \right) \\
\leq c(E(t) + 1) \leq 2c(E(0) + 1), \quad \forall t > 0. \tag{5.26}
\end{eqnarray}

Moreover, by (1.10) recalling that $\frac{1}{p+1} < r < 1$,
\begin{eqnarray}
\int_0^\infty [-k'(s)]^r \, ds < \infty. \tag{5.27}
\end{eqnarray}

So, using (5.26) and (5.27) in (5.25), we obtain
\begin{eqnarray}
\left\{ \int_{\Gamma_t} \int_0^t [-k'(t-s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{\frac{1}{1 - r(p+1)}} \\
\leq c \int_{\Gamma_t} \int_0^t [-k'(t-s)]^{1+\frac{1}{p+1}} [u(t) - u(s)]^2 \, ds \, d\Gamma, \quad \forall t > 0, \tag{5.28}
\end{eqnarray}

for a suitable positive constant $c$. By (5.28) and (5.24) it follows that
\begin{eqnarray}
\tilde{E}'(t) \leq -c \left\{ E(t) + \int_{\Gamma_t} [k(t)]^{1+\frac{1}{p}} [u(t) - u(0)]^2 \, d\Gamma \\
+ \left( \int_{\Gamma_t} \int_0^t [-k(t-s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \right)^{\frac{1}{1 - (r)(p+1)}} \right\}, \quad \forall t > 0, \tag{5.29}
\end{eqnarray}

for a suitable constant $c > 0$.

Now, observing that
\begin{eqnarray}
\frac{1}{(1-r)(p+1)} > \frac{1}{p}.
\end{eqnarray}
using the fact that the energy $E(t)$ is bounded, Hölder’s inequality and (5.26), we can estimate
\[
\{ \int_{1} k(t) [u(t) - u(0)]^2 \, d\Gamma \}^{1 + \frac{1}{\beta}} \leq c \{ \int_{1} k(t) [u(t) - u(0)]^2 \, d\Gamma \}^{1 + \frac{1}{\beta}}
\]
\[
\leq c \int_{1} [k(t)]^{1 + \frac{1}{\beta}} [u(t) - u(0)]^2 \, d\Gamma \cdot \{ \int_{1} [u(t) - u(0)]^2 \, d\Gamma \}^{\frac{1}{\beta}}
\]
\[
\leq c \int_{1} [k(t)]^{1 + \frac{1}{\beta}} [u(t) - u(0)]^2 \, d\Gamma.
\]
This estimate and the fact that $\mathcal{E}(t)$ is bounded ($\mathcal{E}(t) \leq 2E(t) \leq 2E(0)$) imply
\[
\mathcal{E}(t) + \int_{1} [k(t)]^{1 + \frac{1}{\beta}} [u(t) - u(0)]^2 \, d\Gamma + \{ \int_{1} \int_{0} [-k' (t-s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \}^{1 + \frac{1}{\beta}}
\]
\[
\geq c \{ \mathcal{E}(t) + \int_{1} [k(t)]^{1 + \frac{1}{\beta}} [u(t) - u(0)]^2 \, d\Gamma + \int_{1} \int_{0} [-k' (t-s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \}^{1 + \frac{1}{\beta}}
\]
\[
\geq c [\mathcal{E}(t)]^{1 + \frac{1}{\beta}}.
\]
(5.30)
Therefore, by (5.29) and (5.30),
\[
\mathcal{E}(t) \leq -c [\mathcal{E}(t)]^{1 + \frac{1}{\beta}},
\]
which gives, for a suitable positive constant $c$,
\[
\mathcal{E}(t) \leq -c \mathcal{E}(t),
\]
This implies (1.26) because $E(t) \leq cE(t)$. □

6. Nonlinear feedbacks

In this section, we consider a nonlinear feedback law on $I_1$, namely we study the system (1.1) with Dirichlet boundary condition (1.2) on $I_0$, initial conditions (1.4) and the nonlinear boundary condition (1.11) on $I_1$, where the function $g \in C(\mathbb{R})$ is a nondecreasing function satisfying (1.12) and (1.13).

For the well-posedness of this problem, we use nonlinear semigroup theory and an idea from [17] by introducing the new unknown
\[
\tilde{v}(s) = \int_{0}^{s} \tilde{v}(t - \tau) \, d\tau,
\]
where $\tilde{v}(t) = u(t)$, and
\[
\tilde{v}(t) = \begin{cases} \nu(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}
\]
Note that $\bar{v}^t$ is constant for $s > t$, since

$$\bar{v}^t(s) = \int_0^t v(t-\tau) \, d\tau, \quad \forall s > t.$$  

We further remark that

$$\partial_s \bar{v}^t(s) = \tilde{v}(t-s), \quad \forall t, s > 0,$$  

(6.31)

$$\partial_t \bar{v}^t(s) = v(t) - \partial_s \bar{v}^t(s), \quad \forall t, s > 0.$$  

(6.32)

With these notations and properties, we see that (1.11) is equivalent to

$$\frac{\partial u}{\partial t}(t) - \int_0^\infty k'(s) \bar{v}^t(s) \, ds + bg(v(t)) = 0 \text{ on } \Gamma_1 \times (0, +\infty).$$  

(6.33)

At this stage, we set

$$U = \begin{pmatrix} u \\ u_t \\ \bar{v}^t \end{pmatrix}.$$  

Then we see that (1.1) and (6.32) imply that

$$\partial_t U + \hat{A} U = 0,$$  

(6.34)

where the operator $\hat{A}$ is defined by

$$\hat{A} \begin{pmatrix} u \\ v \\ \bar{v} \end{pmatrix} = \begin{pmatrix} -v \\ \hat{A} u \\ -v + \partial_s \bar{v} \end{pmatrix}.$$  

The (nonlinear) operator $\hat{A}$ is actually defined on the Hilbert space

$$\mathcal{H} := H^1_0(\Omega) \times L^2(\Omega) \times L^2_{-k'}(\Gamma_1 \times (0, \infty)),$$  

(6.35)

where $L^2_{-k'}(\Gamma_1 \times (0, \infty))$ means the space of square integrable functions with respect to the measure $-k'(s) \, ds \, d\Gamma$, i.e.,

$$L^2_{-k'}(\Gamma_1 \times (0, \infty)) := \left\{ \bar{v} : -\int_{\Gamma_1 \times (0, \infty)} k'(s) |\bar{v}(x, s)|^2 \, d\Gamma \, ds < \infty \right\}.$$  

(6.36)

This space is a Hilbert space with the natural inner product. Note that the assumptions made on $k$ guarantee that the constant functions belong to $L^2_{-k'}(\Gamma_1 \times (0, \infty))$. 

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As a reader, I found that the expressions became complex due to the use of subscripts and superscripts. However, the text is structured logically, and understanding it requires careful reading. The notation and symbols used are consistent, which made following the proof easier.
The space \( \mathcal{H} \) is equipped with the inner product
\[
\left( \begin{pmatrix} u \\ v \\ \bar{v} \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \\ \bar{v}_1 \end{pmatrix} \right)_{\mathcal{H}} = \int_{\Omega} (A \nabla u \cdot \nabla u_1 + v v_1) \, dx - \int_{\Gamma_1 \times (0, \infty)} k'(s) \bar{v}(x, s) \bar{v}_1(x, s) \, d\Gamma \, ds.
\]

Finally the domain of \( \tilde{A} \) is defined by
\[
D(\tilde{A}) := \left\{ \begin{pmatrix} u \\ v \\ \bar{v} \end{pmatrix} \in \mathcal{H} : \ A u \in L^2(\Omega); v \in H^1_0(\Omega); \right\}
\]
\[
\partial_s \bar{v} \in L^2_{-k'}(\Gamma_1 \times (0, \infty)) \text{ and satisfying } (6.35) \}.
\]

\[
\frac{\partial u}{\partial \nu_A} - \int_0^\infty k'(s) \bar{v}(s) \, ds + b g(v) = 0 \quad \text{on } \Gamma_1.
\]

Note that, in this definition, the boundary condition (6.35) is meaningful (as an equality in \( L^2(\Gamma_1) \)) due to the assumption (1.13).

By nonlinear semigroup theory [23], problem (6.34) has a unique solution for any initial datum in \( \mathcal{H} \) if \( \tilde{A} \) is a maximal monotone operator with a dense domain. We first start with the density result:

**Lemma 6.1.** The domain \( D(\tilde{A}) \) of \( \tilde{A} \) is dense in \( \mathcal{H} \).

**Proof.** Let \( \begin{pmatrix} u_1 \\ v_1 \\ \bar{v}_1 \end{pmatrix} \) in \( \mathcal{H} \) be orthogonal to \( D(\tilde{A}) \), i.e.,
\[
\int_{\Omega} (A \nabla u \cdot \nabla u_1 + v v_1) \, dx - \int_{\Gamma_1 \times (0, \infty)} k'(s) \bar{v}(x, s) \bar{v}_1(x, s) \, d\Gamma \, ds, \quad \forall \begin{pmatrix} u \\ v \\ \bar{v} \end{pmatrix} \in D(\tilde{A}).
\]

We first take \( v = 0 \) and \( \bar{v} = 0 \). Then (6.36) reduces to (recall that \( g(0) = 0 \))
\[
\int_{\Omega} A \nabla u \cdot \nabla u_1 \, dx = 0, \quad \forall u \in H^2(\Omega) \cap H^1_0(\Omega) \text{ such that } \frac{\partial u}{\partial \nu_A} = 0.
\]

As the space \{ \( u \in H^2(\Omega) \cap H^1_0(\Omega) ; \frac{\partial u}{\partial \nu_A} = 0 \) \} is dense in \( H^1_0(\Omega) \), we deduce that \( u_1 = 0 \). Now in (6.36) we take \( u = 0, \bar{v} = 0 \) and \( v \in D(\Omega) \), this yields
\[
\int_{\Omega} v v_1 \, dx = 0, \quad \forall v \in D(\Omega),
\]
and consequently \( v_1 = 0 \).

Finally we take \( v = 0, \bar{v} = \bar{v}_1 \) and \( u \in H^1_0(\Omega) \) such that \( \tilde{A} u = 0 \) and
\[
\frac{\partial u}{\partial \nu_A} = \int_0^\infty k'(s) \bar{v}_1(s) \, ds.
\]
This is meaningful since the above right-hand side belongs to $L^2(I_1)$ and therefore $u \in H^1_{10}(\Omega)$ is the unique solution of
\[
\int_{\Omega} A \nabla u \cdot \nabla u_2 \, dx = \int_{I_1} \int_0^\infty k'(s) \bar{v}_1(x, s) \, ds \, u_2(x) \, d\Gamma, \quad \forall u_2 \in H^1_{10}(\Omega).
\]
This construction guarantees that the triple $\left( \frac{u}{v_1} \right)$ belongs to $D(\hat{A})$ and by (6.36) we obtain
\[
- \int_{\Gamma_1 \times (0, \infty)} k'(s) |\bar{v}_1(x, s)|^2 \, d\Gamma \, ds = 0.
\]
This proves that $\bar{v}_1 = 0$ and the requested density result. \(\square\)

**Lemma 6.2.** Under the above assumptions on $k$ and $g$, the operator $\hat{A}$ is a maximal monotone operator in $\mathcal{H}$.

**Proof.** We start with the monotonicity: $\hat{A}$ is monotone if and only if
\[
(\hat{A}U - \hat{A}U_1, U - U_1)_{\mathcal{H}} \geq 0, \quad \forall U, U_1 \in D(\hat{A}).
\]
From the definition of $\hat{A}$, this is equivalent to
\[
\int_{\Omega} (A \nabla (v - v_1) \cdot \nabla (u - u_1) - \mathcal{A}(u - u_1)(v - v_1)) \, dx
\]
\[
- \int_{\Gamma_1 \times (0, \infty)} k'(s)(v - v_1 - \partial_s(\bar{v} - \bar{v}_1))(\bar{v} - \bar{v}_1) \, d\Gamma \, ds \leq 0,
\]
for all $\left( \frac{u}{v}, \frac{u_1}{v_1} \right) \in D(\hat{A})$. Now applying Green's formula in the first term, we get equivalently
\[
\int_{\Gamma_1} \frac{\partial}{\partial \nu_A} (u - u_1)(v - v_1) \, d\Gamma - \int_{\Gamma_1 \times (0, \infty)} k'(s)(v - v_1 - \partial_s(\bar{v} - \bar{v}_1))(\bar{v} - \bar{v}_1) \, d\Gamma \, ds \leq 0.
\]
Now using the boundary condition (6.35) satisfied by $U$ and $U_1$, we obtain
\[
-b \int_{\Gamma_1} (g(v) - g(v_1))(v - v_1) \, d\Gamma + \int_{\Gamma_1 \times (0, \infty)} k'(s) \partial_s(\bar{v} - \bar{v}_1)(\bar{v} - \bar{v}_1) \, d\Gamma \, ds \leq 0.
\]
The first term of this left-hand side is clearly nonpositive because $g$ is nondecreasing and $b > 0$. For the second term, an integration by parts yields
\[
\int_{\Gamma_1 \times (0, \infty)} k'(s) \partial_s(\bar{v} - \bar{v}_1)(\bar{v} - \bar{v}_1) \, d\Gamma \, ds = -\frac{1}{2} \int_{\Gamma_1 \times (0, \infty)} k''(s)(\bar{v} - \bar{v}_1)^2 \, d\Gamma \, ds \leq 0,
\]
since we recall that $k'' \geq 0$. 

\[\text{22}\]

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\[\text{ASY ios2a v.2006/03/16 Prn:2/08/2006; 13:27 F:asy784.tex; VTEX/DL p. 22}\]
Let us go on with the maximality of $A$: This means that for all $F = \left( \begin{array}{c} f \\ g \\ h \end{array} \right) \in \mathcal{H}$, there exists $U = \left( \begin{array}{c} u \\ v \\ \bar{v} \end{array} \right) \in D(\hat{A})$ solution of
\begin{equation}
(I + \hat{A})U = F,
\end{equation}
or equivalently
\begin{align}
&u - v = f, \\
v + Au = g, \\
&\bar{v} - (v - \partial_s \bar{v}) = h.
\end{align}
This is still equivalent to
\begin{align}
v &= u - f, \\
\bar{v} + \partial_s \bar{v} &= h - f + u, \\
u + Au &= g + f.
\end{align}
This system seems to be triangular (and then solvable from the bottom) but the three equations are coupled via the boundary condition (6.35). Nevertheless assume that $u$ exists, then we directly get $v$ by (6.38) and obtain $\bar{v}$ by simply solving the differential equation (6.39), namely
\begin{equation}
\bar{v}(x, s) = (u(x) - f(x))(1 - e^{-s}) + H(x, s),
\end{equation}
where
\begin{equation}
H(x, s) = \int_0^s h(x, \sigma) e^{-(s-\sigma)} \, d\sigma.
\end{equation}
Using the expression (6.41) and (6.38) in the boundary condition (6.35), we see that $u$ must satisfy (at least formally)
\begin{equation}
\frac{\partial u}{\partial \nu} + bg(u - f) + \delta u = \delta f + \int_0^\infty k'(s)H(s) \, ds \quad \text{on } \Gamma_1,
\end{equation}
where we set $\delta = - \int_0^\infty k'(s)(1 - e^{-s}) \, ds$ (which is $\geq 0$).

In summary we are looking for $u \in H^1_{\Gamma_0}(\Omega)$ solution of (6.40) and satisfying (6.42). For that purpose, we introduce the following form $\beta$ (linear on $u_1$ but not on $u$)
\begin{equation}
\beta(u, u_1) = \int_\Omega (A \nabla u \cdot \nabla u_1 + uu_1) \, dx + \int_{\Gamma_1} (\delta u + bg(u - f))u_1 \, d\Gamma, \quad \forall u, u_1 \in H^1_{\Gamma_0}(\Omega).
\end{equation}
This form depends on $f$, which is now assumed to be fixed in $H^1_{\Gamma_0}(\Omega)$. 
The variational formulation of problem (6.40) and (6.42) is then
\[ \beta(u, u_1) = F(u_1), \quad \forall u_1 \in H^{1,0}_0(\Omega), \]  
(6.43)

where
\[ F(u_1) = \int_\Omega [(1 + \delta)f + g]u_1 \, dx + \int_{\Gamma_1 \times (0, \infty)} k'(s)H(x, s)u_1(x) \, d\Gamma \, ds. \]

Introducing the (nonlinear) mapping
\[ B : H^{1,0}_0(\Omega) \rightarrow H^{1,0}_0(\Omega)', \quad u \rightarrow Bu, \]
where \( Bu(u_1) = \beta(u, u_1) \), we see that (6.43) is equivalent to
\[ Bu = F, \]
since \( F \) clearly belongs to \( H^{1,0}_0(\Omega)' \). This means that the solvability of (6.43) is equivalent to the surjectivity of \( B \). This surjectivity is obtained using Corollary II.2.2 of [23], which states that \( B \) is surjective if \( B \) is monotone, hemicontinuous, bounded and coercive. Let us then check these properties: \( B \) is monotone if and only if
\[ [Bu - Bv](u - v) \geq 0, \quad \forall u, v \in H^{1,0}_0(\Omega). \]
In view of the definition of \( B \), this is equivalent to
\[ \int_\Omega (A\nabla(u - v) \cdot \nabla(u - v) + (u - v)^2) \, dx \]
\[ + \int_{\Gamma_1} (\delta(u - v)^2 + b(g(u - f) - g(v - f))(u - v)) \, d\Gamma \geq 0. \]
This clearly holds because
\[ \int_{\Gamma_1} b(g(u - f) - g(v - f))(u - v) \, d\Gamma = \int_{\Gamma_1} b(g(u - f) - g(v - f))(u - f - (v - f)) \, d\Gamma \geq 0, \]
as \( g \) is nondecreasing.

The boundedness of \( B \) follows from the property (1.13) satisfied by \( g \), Cauchy–Schwarz’s inequality and a trace theorem (reminding that \( f \) is fixed).

The hemicontinuity of \( B \) means that the function
\[ t \rightarrow B(u + tu_1)(u_1) \]
is continuous for each \( u, u_1 \in H^{1,0}_0(\Omega) \). In our case, this follows from the continuity of \( g \).
Corollary 6.3. Under the above assumptions on $k$ and $g$, for all initial data $(u_0, u_1) \in H_{Γ_0}^1(Ω) \times L^2(Ω)$, there exists a unique weak solution $u \in H_{Γ_0}^1(Ω) \cap C([0, T]; H_{Γ_0}^1(Ω))$ of (1.1), (1.2), (1.4) and (1.11). If furthermore $(u_0, u_1) \in (H^2(Ω) \cap H_{Γ_0}^1(Ω)) \times H_{Γ_0}^1(Ω)$ satisfies the compatibility condition (1.24), then the weak solution $u$ of (1.1), (1.2), (1.4) and (1.11) has the regularity $u \in W^{2,∞}(Ω) \cap L^∞(Ω)$ and $D(A)$ (also called a regular solution), where

$$D(A) = \left\{ u \in H_{Γ_0}^1(Ω) : Au \in L^2(Ω), \frac{∂u}{∂ν_A} \in L^2(Γ) \right\}.$$  

The energy of the solution $u$ of (1.1), (1.2), (1.4) and (1.11) is still defined by (1.22) and is nonincreasing because

$$E'(t) = \frac{1}{2} \int_{Γ_1} k'(t)[u(t) - u(0)]^2 \mathrm{d}Γ - b \int_{Γ_1} u_t g(u_t) \mathrm{d}Γ$$

$$- \frac{1}{2} \int_{Γ_1} \int_0^t k''(t - s)[u(t) - u(s)]^2 \mathrm{d}s \mathrm{d}Γ \leq 0. \quad (6.45)$$
Lemma 4.1 remains the same, while the identity (4.11) from Proposition 4.2 becomes
\[
\frac{d}{dt} \left\{ \int_\Omega u_t M u \, dx \right\} \leq -c_0 \mathcal{E}(t) + C \left\{ \int_{\Gamma_1} (u_t^2 + g(u_t)^2) \, d\Gamma + \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma \right. \\
- \left. \int_{\Gamma_1} \int_0^t k'(t-s) [u(t) - u(s)]^2 \, ds \, d\Gamma \right\},
\]
(6.46)
for suitable positive constants \(c_0, C\).

Now we can prove the exponential stability result.

**Proof of Theorem 1.2(a), case \(g\) nonlinear.** The proof is similar to the one of Theorem 1.2. Indeed we define the Lyapunov functional \(\mathcal{E}\) by (4.17), with the parameter \(\tilde{\gamma}\) fixed later on. By the assumptions on \(k\), we have
\[
\mathcal{E}'(t) \leq -b \int_{\Gamma_1} u_t g(u_t) \, d\Gamma - \frac{\gamma_0}{2} \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma \\
+ \frac{\gamma_1}{2} \int_{\Gamma_1} \int_0^t k'(t-s) [u(t) - u(s)]^2 \, ds \, d\Gamma.
\]
(6.47)
By the inequalities (6.46), and the property (1.15) for \(\tilde{\gamma}\) sufficiently small it holds
\[
-b \int_{\Gamma_1} u_t g(u_t) \, d\Gamma + C \tilde{\gamma} \int_{\Gamma_1} (u_t)^2 + (g(u_t))^2 \, d\Gamma \leq 0.
\]
The remainder of the proof is similar to the one of Theorem 1.2. \(\Box\)

Now we can continue with the polynomial decay.

Combining the arguments from Theorem 4(b) (linear case) and the one from [26], we can prove the polynomial stability result.

**Proof of Theorem 1.2(b), case \(g\) nonlinear.** By the assumption (1.10) and the identity (6.45) we have
\[
\mathcal{E}'(t) \leq -b \int_{\Gamma_1} u_t g(u_t) \, d\Gamma - \frac{\gamma_0}{2} \int_{\Gamma_1} [k(t)]^{1+\frac{1}{p-1}} [u(t) - u(0)]^2 \, d\Gamma \\
- \frac{\gamma_1}{2} \int_{\Gamma_1} \int_0^t [-k'(t-s)]^{1+\frac{1}{p-1}} [u(t) - u(s)]^2 \, ds \, d\Gamma.
\]
(6.48)
The other hand, using the estimate (6.46) and the assumptions on \(k\) and \(g\), the estimate (5.20) still holds.

Now we define a Lyapunov functional adapted to the nonlinearity \(g\) (compare with [26]), namely we take
\[
\tilde{\mathcal{E}}(t) := \mathcal{E}(t) + \tilde{\gamma} [\mathcal{E}(t)]^{\frac{p-1}{2}} \int_{\Omega} u_t(t) M u(t) \, dx,
\]
(6.49)
where \(\tilde{\gamma}\) is a positive constant chosen sufficiently small later on.
Using (5.20) and the fact that \( \int_\Omega u_t(t) M u(t) \, dx \leq C E(t) \), for some \( C > 0 \), we deduce that

\[
\frac{d}{dt} \left( [E(t)]^{\frac{q+1}{2}} \int_\Omega u_t(t) M u(t) \, dx \right)
\]

\[
= \frac{(q-1)}{2} [E(t)]^{\frac{q+1}{2}} E'(t) \int_\Omega u_t(t) M u(t) \, dx + [E(t)]^{\frac{q+1}{2}} \frac{d}{dt} \left( \int_\Omega u_t(t) M u(t) \, dx \right)
\]

\[
\leq -CE'(t) + [E(t)]^{\frac{q+1}{2}} \left( -c_0 E(t) + C \left\{ \int_{\Gamma_1} u_t^2 \, d\Gamma + \int_{\Gamma_1} [k(t)]^{1+\frac{1}{p}} [u(t) - u(0)]^2 \, d\Gamma \right. \right.
\]

\[
+ \int_{\Gamma_1} \int_0^t [k'(t-s)]^{1+\frac{1}{p-1}} [u(t) - u(s)]^2 \, ds \, d\Gamma \left. \right) \). \tag{6.50}
\]

This estimate and (6.48) lead to

\[
\tilde{E}'(t) \leq -c_0 \gamma [E(t)]^{\frac{q+1}{2}} E(t) - \frac{b}{2} \int_{\Gamma_1} u_t g(u_t) \, d\Gamma + \gamma C [E(t)]^{\frac{q+1}{2}} \int_{\Gamma_1} |u_t|^2 \, d\Gamma
\]

\[
- \frac{\gamma_0}{4} \int_{\Gamma_1} [k(t)]^{1+\frac{1}{p}} [u(t) - u(0)]^2 \, d\Gamma
\]

\[
- \frac{\gamma_1}{4} \int_{\Gamma_1} \int_0^t [-k'(t-s)]^{1+\frac{1}{p-1}} [u(t) - u(s)]^2 \, ds \, d\Gamma.
\]

Now we showed in Theorem 1.2(b), case linear, that

\[
\int_{\Gamma_1} [k(t)]^{1+\frac{1}{p}} [u(t) - u(0)]^2 \, d\Gamma + \int_{\Gamma_1} \int_0^t [-k'(t-s)]^{1+\frac{1}{p-1}} [u(t) - u(s)]^2 \, ds \, d\Gamma
\]

\[
\geq c \left\{ \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma \right\}^{1+\frac{1}{n-p+1}}
\]

\[
+ \left\{ \int_{\Gamma_1} \int_0^t [-k'(t-s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{1+\frac{1}{n-p+1}}
\]

This estimate and the estimate \( E(t) \geq \mathcal{E}(t) \) yield

\[
[E(t)]^{\frac{q+1}{2}} \mathcal{E}'(t) + \int_{\Gamma_1} [k(t)]^{1+\frac{1}{p}} [u(t) - u(0)]^2 \, d\Gamma
\]

\[
+ \int_{\Gamma_1} \int_0^t [-k'(t-s)]^{1+\frac{1}{p-1}} [u(t) - u(s)]^2 \, ds \, d\Gamma
\]

\[
\geq [\mathcal{E}(t)]^{\frac{q+1}{2}} + c \left\{ \int_{\Gamma_1} k(t) [u(t) - u(0)]^2 \, d\Gamma \right\}^{1+\frac{1}{n-p+1}}
\]

\[
+ \left\{ \int_{\Gamma_1} \int_0^t [-k'(t-s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{1+\frac{1}{n-p+1}}
\]
\[
\geq c \left\{ \mathcal{E}(t) + \int_{\Gamma_1} k(t)[u(t) - u(0)]^2 \, d\Gamma + \int_{\Gamma_1} \int_0^t [-k'(t - s)] [u(t) - u(s)]^2 \, ds \, d\Gamma \right\}^{\frac{q+1}{2}}
\]
\[
\geq c [E(t)]^{\frac{q+1}{2}},
\]
since \(\frac{q+1}{2} = 1 + \frac{1}{(1-r)(p+1)}\). As \(E(t) \leq cE(t)\) for some \(c > 0\), this last estimate in (6.50) allows to write
\[
\tilde{E}'(t) \leq -c [\tilde{E}(t)]^{\frac{q+1}{2}} - \frac{b}{2} \int_{\Gamma_1} u_t g(u_t) \, d\Gamma + \hat{\gamma} C [E(t)]^{\frac{q+1}{2}} \int_{\Gamma_1} |u_t|^2 \, d\Gamma.
\]
Using the assumptions on \(g\), we show as in Theorem 2.1 of [26] that
\[
-\frac{b}{2} \int_{\Gamma_1} u_t g(u_t) \, d\Gamma + \hat{\gamma} C [E(t)]^{\frac{q+1}{2}} \int_{\Gamma_1} |u_t|^2 \, d\Gamma \leq \gamma C_1 [E(t)]^{\frac{q+1}{2}},
\]
for some \(C_1 > 0\). Therefore choosing \(\hat{\gamma}\) small enough we deduce that (since \(E(t) \leq c\tilde{E}(t)\) for \(\hat{\gamma}\) small enough)
\[
\tilde{E}'(t) \leq -c [\tilde{E}(t)]^{\frac{q+1}{2}}.
\]
This yields the conclusion. □

**Remark 6.4.** Clearly the energy will decay polynomially, if \(g\) satisfies (1.15) and \(k\) the hypotheses (1.10), or if \(g\) satisfies (1.13) and (1.16) for some \(q > 1\) and \(k\) satisfies (1.9).

**7. Other geometric configurations**

7.1. The case \(\Gamma_0 \cap \Gamma_1\) nonempty

In the case \(\Gamma_0 \cap \Gamma_1 \neq \emptyset\), our above analysis fails since the solution of (1.1)–(1.4) (or of (1.1), (1.2), (1.4) and (1.11)) has a singular behavior along \(\Gamma_0 \cap \Gamma_1\) (see for instance [6,7]). Nevertheless in the case of the Laplace operator, i.e., \(A = -\Delta\), then the singular behavior can be described in an appropriate way [7,3] so that the estimate (4.7) remains valid if \(H(x) = m(x) = x - x_0\), for some \(x_0 \in \mathbb{R}^n\) and if \(\Gamma_0\) and \(\Gamma_1\) are given by
\[
\begin{align*}
\Gamma_0 &= \{ x \in \Gamma : m(x) \cdot \nu(x) < 0 \}, \\
\Gamma_1 &= \{ x \in \Gamma : m(x) \cdot \nu(x) \geq 0 \},
\end{align*}
\]
and are such that \(\overline{\Gamma}_0 \cap \Gamma_1\) is a submanifold of class \(C^3\) of dimension \(n-2\) and
\[
m(x) \cdot \tau(x) \leq 0, \quad \forall x \in \overline{\Gamma}_0 \cap \Gamma_1,
\]
when \( \tau(x) \) is the unit vector in the tangent plane of \( \Gamma \) at \( x \) orientated from \( \Gamma_1 \) to \( \Gamma_0 \). In this setting, we further remark that

\[
u \in H^1_{\Gamma_0}(\Omega) \quad \text{such that} \quad \Delta \nu \in L^2(\Omega) \quad \text{and} \quad \frac{\partial \nu}{\partial \nu} \in H^{1/2}(\Gamma_1),
\]

only satisfies

\[
m \cdot \nu |\nabla \nu|^2 \in L^1(\Gamma).
\]

Therefore the previous feedback law (1.11) has to be replaced by

\[
\frac{\partial u}{\partial \nu}(t) + m \cdot \nu \left( \int_0^t k(t-s)u(s)\,ds + bg(u(t)) \right) = 0
\]

in order to guarantee that some boundary terms are well defined.

With these geometrical hypotheses and this new feedback law, the same stabilization results hold under the same hypotheses on \( k \) and \( g \) and with the additional assumption on \( g \) to be globally Lipschitz.

### 7.2. The case \( \Gamma_0 \) empty

If \( \Gamma_0 = \emptyset \), every constant function is a solution of energy zero. In the case \( k \equiv 0 \), that is without memory effects, this difficulty to obtain a stabilization result has been overcome in Zuazua [25] by considering a feedback in the form \(- (m \cdot \nu)(y_t + ay)\).

In our case, instead of (1.3), we then consider as boundary condition

\[
\frac{\partial u}{\partial \nu}(A(t)) + \int_0^t k(t-s)u(s)\,ds + bu(t) + au(t) = 0 \quad \text{on} \quad \Gamma \times (0, +\infty),
\]

where \( a \) is a positive constant.

Moreover, we consider a new energy

\[
E_{\text{new}}(t) = E(t) + \frac{a}{2} \int_\Gamma u^2(x,t)\,d\Gamma.
\]

For a solution of problem (1.1), (1.4), (7.52), differentiating and integrating by parts we obtain

\[
E_{\text{new}}'(t) = \frac{1}{2} \int_\Gamma k'(t)[u(t) - u(0)]^2\,d\Gamma - b \int_\Gamma u^2_t\,d\Gamma
\]

\[
- \frac{1}{2} \int_\Gamma \int_0^t k''(t-s)[u(t) - u(s)]^2\,ds\,d\Gamma \leq 0.
\]

Note that (7.54) implies

\[
E_{\text{new}}'(t) \leq -b \int_\Gamma u^2_t\,d\Gamma,
\]
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and then

$$\int_{\Gamma} u_t^2 \, d\Gamma \leq - \frac{1}{b} E_{new}'(t).$$  \hspace{1cm} (7.55)$$

It is easy to see that Lemma 4.1 remains unchanged, while Proposition 4.2 becomes

Proposition 7.1. Assume that $k$ satisfies (1.9). Then, for any regular solution of the problem (1.1), (1.4), (7.52),

$$\frac{d}{dt} \left\{ \int_{\Omega} u_t M u \, dx \right\} \leq - c_0 E(t) + C \int_{\Gamma} k(t) \left[ u(t) - u(0) \right]^2 \, d\Gamma$$
$$- \int_{\Gamma} \int_{0}^{t} k'(t-s) \left[ u(t) - u(s) \right] \, ds \, d\Gamma,$$  \hspace{1cm} (7.56)

for suitable positive constants $c_0, C$.

Applying an argument of Conrad and Rao [5], we can prove the following lemma.

Lemma 7.2. Let $u$ be a regular solution of the problem (1.1), (1.4), (7.52). Then, there exists $C > 0$

such that for all $\eta \in (0, 1)$ and all pairs $S, T$ with $0 \leq S < T < +\infty$,

$$\int_{S}^{T} \int_{\Gamma} u_t^2 \, d\Gamma \, dt \leq \frac{C}{\eta} E_{new}(S) + \eta \int_{S}^{T} E_{new}(t) \, dt.$$  \hspace{1cm} (7.57)

Proof. For any $t \geq 0$, we define $z = z(t)$ by

Then, $z$ verifies

$$\int_{\Omega} z_t^2 \, dx \leq c \int_{\Gamma} u_t^2 \, dx \leq c E_{new}(t),$$  \hspace{1cm} (7.58)

and, from (7.55),

$$\int_{\Omega} z_t^2 \, dx \leq c \int_{\Gamma} u_t^2 \, dx \leq - c_0 E_{new}'(t),$$  \hspace{1cm} (7.59)

for suitable positive constants $c, c_0$. From the method of Conrad–Rao we have

$$\int_{S}^{T} \int_{\Gamma} a u_t^2 \, d\Gamma \, dt \leq - \left[ \int_{\Omega} z u_t \, dx \right]_S^T + \int_{S}^{T} \int_{\Omega} z_t u_t \, dx \, dt - \int_{S}^{T} \int_{\Gamma} b u u_t \, d\Gamma \, dt,$$
and therefore, from Cauchy–Schwarz’s inequality,

\[
\int_S \left( \int_G au^2 \, d\Gamma \right) dt \leq \left\| z \right\|_{L^2(G)} \left\| u_t \right\|_{L^2(G)} \right]_T^T + \frac{1}{2} \int_S \int_G au^2 \, d\Gamma \, dt + \frac{1}{2} \int_T T \int_G \frac{b^2}{a} u_t^2 \, d\Gamma \, dt. \tag{7.60}
\]

From (7.60), applying (7.59), we obtain

\[
\int_S \left( \int_G au^2 \, d\Gamma \right) dt \leq c E_{\text{new}}(S) + c \theta c E_{\text{new}}(S) + \frac{1}{2} \int_S \int_G au^2 \, d\Gamma \, dt + \frac{1}{2} \int_T T \int_G \frac{b^2}{a} u_t^2 \, d\Gamma \, dt \tag{7.61}
\]

for a suitable positive constant \( c \) and all \( \theta > 0 \).

Now, from (7.55) we have

\[
\int_S \int_G \frac{b^2}{a} u_t^2 \, d\Gamma \, dt \leq -c \int_T T \int_G E'_{\text{new}}(t) \, dt \leq c E_{\text{new}}(S),
\]

which used in (7.61) gives (7.57). \( \square \)

Using the above results we can give an exponential stability result when \( k \) satisfies (1.9).

Indeed, from (7.54) and (1.9), we obtain

\[
E'_{\text{new}}(t) \leq -b \int_G u_t^2(t) \, d\Gamma - \frac{70}{2} \int_G k(t) \left[ u(t) - u(0) \right]^2 \, d\Gamma \]

\[+ \frac{\gamma_1}{4} \int_S \int_G k'(t - s) \left[ u(t) - u(s) \right]^2 \, ds \, d\Gamma. \tag{7.62}
\]

Define the Lyapounov functional

\[
\tilde{E}(t) := E_{\text{new}}(t) + \tilde{\gamma} \int_S u_t(t) M u(t) \, dx, \tag{7.63}
\]

where \( \tilde{\gamma} \) is a positive constant.

From (7.56) and (7.62), for \( \tilde{\gamma} \) sufficiently small, we have

\[
\tilde{E}'(t) \leq -c_0 \tilde{E}(t) - b \int_G u_t^2(t) \, d\Gamma - \frac{70}{4} \int_G k(t) \left[ u(t) - u(0) \right]^2 \, d\Gamma \]

\[+ \frac{\gamma_1}{4} \int_S \int_G k'(t - s) \left[ u(t) - u(s) \right]^2 \, ds \, d\Gamma + c \int_G u_t^2 \, d\Gamma. \tag{7.64}
\]
from which follows

\[ \tilde{E}'(t) \leq -c_1 E_{\text{new}}(t) + c' \int_{\Gamma} u^2 \, d\Gamma, \quad (7.65) \]

for suitable positive constants \( c_1, c' \). Integrating (7.65) in time and using Lemma 7.2, we obtain

\[ \int_{S}^{T} \tilde{E}'(t) \, dt \leq -c_1 \int_{S}^{T} E_{\text{new}}(t) \, dt + c \eta \int_{S}^{T} E_{\text{new}}(t) \, dt, \quad (7.66) \]

for a suitable positive constant \( c \) and any \( \eta \in (0, 1) \). So, from (7.66), for \( \eta \) sufficiently small,

\[ \int_{S}^{T} \tilde{E}'(t) \, dt \leq -c_1 \int_{S}^{T} E_{\text{new}}(t) \, dt + c_2 E_{\text{new}}(S), \]

or, equivalently,

\[ \tilde{E}(T) - \tilde{E}(S) \leq -c_1 \int_{S}^{T} E_{\text{new}}(t) \, dt + c_2 E_{\text{new}}(S), \quad (7.67) \]

for some constants \( c_1, c_2 > 0 \).

Now, observe that for \( \hat{\gamma} \) sufficiently small

\[ C_1 E_{\text{new}}(t) \leq \tilde{E}(t) \leq C_2 E_{\text{new}}(t), \]

for suitable positive constants \( C_1, C_2 \) independent of \( t \). Then, (7.67) allows to obtain

\[ \int_{S}^{T} E_{\text{new}}(t) \, dt \leq C E_{\text{new}}(S), \]

and, taking the limit for \( T \to +\infty \),

\[ \int_{S}^{+\infty} E_{\text{new}}(t) \, dt \leq C E_{\text{new}}(S), \quad \forall S > 0. \]

As well-known, this implies

\[ E_{\text{new}}(t) \leq E_{\text{new}}(0) e^{1-t/C}, \]

that is the exponential stability result.

Unfortunately when \( I_0 = \emptyset \), we are not able to prove the polynomial decay of the energy \( E_{\text{new}} \) when \( k \) satisfies (1.10), because the analogue of the estimate (7.57) of Lemma 7.2 with the term \( \eta \int_{S}^{T} E_{\text{new}}(t) \, dt \) replaced by \( \eta \int_{S}^{T} [E_{\text{new}}(t)]^{1+\beta} \, dt \), with \( \beta > 0 \) seems to be difficult to prove.
8. Examples

We end up with some examples that illustrate the geometric assumption (1.19) and the assumption (1.20).

Example 1. We take $A$ such that

$$a_{ij}(x) = a(x)\delta_{ij},$$

for some smooth function $a$ satisfying $a \geq \alpha > 0$ in $\Omega$. Then clearly

$$g_{ij}(x) = a(x)^{-1}\delta_{ij},$$

and direct calculations yield

$$I_{jk}^i = -\frac{1}{2a}\left(\frac{\partial a}{\partial x_j}\delta_{kl} + \frac{\partial a}{\partial x_k}\delta_{jl} - \frac{\partial a}{\partial x_l}\delta_{jk}\right).$$

We now choose $H(x) = a(x)(x-x_0)$ for some $x_0 \in \mathbb{R}^n$ and again standard calculations give

$$DH(X, X) = a^{-1}\sum_{i,j=1}^{n} \left( a\delta_{ij} + \frac{\partial a}{\partial x_i}(x_j - x_{0j}) - \frac{1}{2}(x-x_0) \cdot \nabla a \delta_{ij}\right)\xi_i \xi_j.$$

Therefore the assumption (1.19) is equivalent to

$$a|X|^2 + X \cdot \nabla a X \cdot (x-x_0) - \frac{1}{2}|X|^2(x-x_0) \cdot \nabla a \geq a_0|X|^2, \quad \forall x \in \Omega, X \in \mathbb{R}^n,$$

(8.68)

where $|X|$ means the Euclidean norm of $X$. Since Cauchy–Schwarz’s inequality leads to

$$|X \cdot \nabla a X \cdot (x-x_0) - \frac{1}{2}|X|^2(x-x_0) \cdot \nabla a| \leq \frac{3}{2}\|x-x_0\|_{\infty}\|\nabla a\|_{\infty}|X|^2,$$

(8.69)

where $\|w\|_{\infty} = \sup_{x \in \Omega} |w(x)|$, the estimate (8.68) will be true if

$$a - \frac{3}{2}\|x-x_0\|_{\infty}\|\nabla a\|_{\infty} \geq a_0 \quad \text{in} \; \Omega,$$

or equivalently if

$$\frac{3}{2}\|x-x_0\|_{\infty}\|\nabla a\|_{\infty} < \inf_{x \in \Omega} a(x),$$

(8.69)

and in that case we may take

$$a_0 = \inf_{x \in \Omega} a(x) - \frac{3}{2}\|x-x_0\|_{\infty}\|\nabla a\|_{\infty}. $$
Since
\[
\text{div} \, H(x) = na(x) + \nabla a(x) \cdot (x - x_0),
\]
the condition (1.20) is equivalent to
\[
\sup_{x \in \Omega} (na(x) + \nabla a(x) \cdot (x - x_0)) < \inf_{x \in \Omega} (na(x) + \nabla a(x) \cdot (x - x_0)) + 2a_0. \quad (8.70)
\]
Again using Cauchy–Schwarz’s inequality, this estimate is satisfied if
\[
n\|a\|_\infty + 2\|x - x_0\|_\infty \|\nabla a\|_\infty < n \inf_{x \in \Omega} a(x) + 2a_0.
\]
Taking the above value of \(a_0\), we deduce that (1.20) holds if
\[
n\|a\|_\infty + 5\|x - x_0\|_\infty \|\nabla a\|_\infty < (n + 2) \inf_{x \in \Omega} a(x). \quad (8.71)
\]
Roughly speaking the conditions (8.69) and (8.71) hold together if \(a\) does not vary too much in \(\Omega\).

**Example 2.** We keep the setting of Example 1 with \(a(x) = 1 + |x|^2\) and \(H(x) = a(x)x\). Then from the above considerations (1.19) reduces to (see (8.68))
\[
|X|^2(1 + |x|^2) + 2(X \cdot x)^2 - |X|^2|x|^2 \geq a_0|X|^2, \quad \forall x \in \Omega, X \in \mathbb{R}^n,
\]
that is,
\[
a_0|X|^2 \leq |X|^2 + 2(X \cdot x)^2,
\]
which always holds by choosing \(a_0 = 1\).
Concerning (1.20), it is equivalent to (with the choice \(a_0 = 1\))
\[
(n + 2)r_{\max}^2 < (n + 2)r_{\min}^2 + 2,
\]
where for shortness we write
\[
r_{\min} = \inf_{x \in \Omega} |x|, \quad r_{\max} = \sup_{x \in \Omega} |x|.
\]
In other words, (1.20) is equivalent to
\[
r_{\max}^2 - r_{\min}^2 < \frac{2}{n + 2}.
\]
For instance it holds if \(\Omega = B(0, r_{\max}) \setminus B(0, r_{\min})\) with the above constraint between \(r_{\min}\) and \(r_{\max}\). In that case \(\Gamma_0 = S(0, r_{\min})\) and \(\Gamma_1 = S(0, r_{\max})\).
Example 3. As in Example 4.1 of [24] we take

\[ a_{ij}(x) = (1 + |x|^2)^2 \delta_{ij}. \]

Here contrary to [24] we simply take \( H(x) = (1 + |x|^2)^2 x \). As before (1.19) is equivalent to

\[ |X^2(1 + |x|^2) + 4(X \cdot x)^2 - 2|X|^2x|^2| \geq \frac{a_0}{1 + |x|^2} |X|^2, \quad x \in \Omega, X \in \mathbb{R}^n. \tag{8.72} \]

This inequality will be satisfied if

\[ |X^2(1 + |x|^2) - 2|X|^2x|^2| \geq \frac{a_0}{1 + |x|^2} |X|^2, \quad x \in \Omega, X \in \mathbb{R}^n, \]

or equivalently

\[ 1 - |x|^2 \geq \frac{a_0}{1 + |x|^2}, \quad x \in \Omega. \tag{8.73} \]

If \( \mathbb{T} \subset B(0, 1) = \{ x \in \mathbb{R}^n : |x| < 1 \} \), then the above condition holds with \( a_0 = 1 - r_{\text{max}}^4 \).

Since \( \text{div} H(x) = (1 + |x|^2)(n + (4 + n)|x|^2) \), the condition (1.20) is equivalent to

\[ (1 + r_{\text{max}}^2)(n + (4 + n)r_{\text{max}}^2) < (1 + r_{\text{min}}^2)(n + (4 + n)r_{\text{min}}^2) + 2 - 2r_{\text{max}}^4, \]

which is satisfied if \( r_{\text{max}} \) is not too far from \( r_{\text{min}} \).

Example 4. We take the matrix \( A(x) \) as a perturbation of a symmetric positive definite matrix \( A \). More precisely we take

\[ A(x) = A + R(x), \]

and assume that the perturbation \( R \) is small in the sense that there exists \( r \in (0, 1) \) such that

\[ \sup_{x \in \Omega} \| A^{-1} \|_2 \| R(x) \|_2 \leq r, \quad \sup_{k=1,\ldots,n} \sup_{x \in \Omega} \| \frac{\partial R}{\partial x_k} \|_2 \leq r, \tag{8.73} \]

where \( \| \cdot \|_2 \) means the Euclidean matrix norm. This condition implies that

\[ \| A^{-1} R(x) \|_2 \leq \| A^{-1} \|_2 \| R(x) \|_2 < 1, \]

and using a Neumann series, the matrix \( A(x) \) is invertible and its inverse matrix is given by

\[ G(x) = A(x)^{-1} = \sum_{k=0}^{\infty} (A^{-1} R(x))^k A^{-1}. \tag{8.74} \]
From this expression, we see that
\[
\frac{\partial G}{\partial x_k} = \sum_{k=1}^{\infty} k(A^{-1} R(x))^k A^{-1} \frac{\partial R}{\partial x_k} A^{-1},
\]
and therefore by the assumptions (8.73), we deduce that
\[
\left\| \frac{\partial G}{\partial x_k} \right\|_2 \leq \left( r(1 - r) \right) 2^{\frac{1}{2}}. \tag{8.75}
\]
Now we choose
\[
H(x) = A(x)(x - x_0),
\]
as and find
\[
\frac{\partial h}{\partial x_k} = a_{ik}(x) + \sum_{j=1}^{n} \frac{\partial r_{ij}}{\partial x_k} (x_j - x_{0j}).
\]
From this identity and the estimate (8.75), we deduce that
\[
D H(X, X) = |X|^2 + (R_1(x) X, X), \quad \forall x \in \Omega, \quad X \in \mathbb{R}^n,
\]
where the matrix function \(R_1(x)\) satisfies
\[
\left\| R_1(x) \right\|_2 \leq C r, \quad \forall x \in \Omega,
\]
for some \(C > 0\). Therefore for \(r\) small enough, the assumption (1.19) will be satisfied.

Similarly as
\[
\text{div } H(x) = \text{tr } A + r_2(x),
\]
with
\[
|r_2(x)| \leq C r, \quad \forall x \in \Omega,
\]
for some \(C > 0\), the condition (1.20) holds if \(r\) is small enough.

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