Universal Regular Autonomous Asynchronous Systems: Fixed Points, Equivalencies and Dynamic Bifurcations

Serban E. Vlad
str. Zimbrului, nr. 3, bl. PB68, ap. 11, 410430, Oradea, Romania, E-mail: serban_e_vlad@yahoo.com

Abstract

The asynchronous systems are the non-deterministic models of the asynchronous circuits from the digital electrical engineering. In the autonomous version, such a system is a set of functions \( x : R \rightarrow \{0, 1\}^n \) called states (\( R \) is the time set). If an asynchronous system is defined by making use of a so called generator function \( \Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n \), then it is called regular. The property of universality means the greatest in the sense of the inclusion.

The purpose of the paper is that of defining and of characterizing the fixed points, the equivalencies and the dynamical bifurcations of the universal regular autonomous asynchronous systems. We use analogies with the dynamical systems theory.

1 Preliminaries

Definition 1 We denote by \( B = \{0, 1\} \) the binary Boole algebra, endowed with the discrete topology and with the usual laws.

Definition 2 Let be the Boolean function \( \Phi : B^n \rightarrow B^n, \Phi = (\Phi_1, ..., \Phi_n) \) and \( \nu \in B^n, \nu = (\nu_1, ..., \nu_n) \). We define \( \Phi' : B^n \rightarrow B^n \) by \( \forall \mu \in B^n, \)

\[ \Phi'(\mu) = (\bar{\nu}_1 \cdot \mu_1 \oplus \nu_1 \cdot \Phi_1(\mu), ..., \bar{\nu}_n \cdot \mu_n \oplus \nu_n \cdot \Phi_n(\mu)). \]

Remark 3 \( \Phi' \) represents the function resulting from \( \Phi \) when this one is not computed, in general, on all the coordinates \( \Phi_i, i = 1, n : \) if \( \nu_i = 0 \), then \( \Phi_i \) is not computed, \( \Phi'_i(\mu) = \mu_i \) and if \( \nu_i = 1 \), then \( \Phi_i \) is computed, \( \Phi'_i(\mu) = \Phi_i(\mu) \).

Definition 4 Let be the sequence \( \alpha^0, \alpha^1, ..., \alpha^k, ... \in B^n \). The functions \( \Phi^{\alpha_0 \alpha^1 ... \alpha^k} : B^n \rightarrow B^n \) are defined iteratively by \( \forall k \in N, \forall \mu \in B^n, \)

\[ \Phi^{\alpha_0 \alpha^1 ... \alpha^k}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha_0 \alpha^1 ... \alpha^k}(\mu)). \]
Definition 5 The sequence $\alpha^0, \alpha^1, ..., \alpha^k, ... \in \mathbb{B}^n$ is called progressive if
\[ \forall i \in \{1, ..., n\}, \text{ the set } \{ k | k \in \mathbb{N}, \alpha^k_i = 1 \} \text{ is infinite.} \]
The set of the progressive sequences is denoted by $\Pi_n$.

Remark 6 Let be $\mu \in \mathbb{B}^n$. When $\alpha = \alpha^0, \alpha^1, ..., \alpha^k, ...$ is progressive, each coordinate $\Phi_i, i = 1, n$ is computed infinitely many times in the sequence $\Phi^{\alpha^0, \alpha^1, ..., \alpha^k} (\mu), k \in \mathbb{N}$. This is the meaning of the progress property, giving the so called 'unbounded delay model' of computation of the Boolean functions.

Definition 7 The initial value, denoted by $x(-\infty + 0) \text{ or } \lim_{t \to -\infty} x(t) \in \mathbb{B}^n$ and the final value, denoted by $x(\infty - 0) \text{ or } \lim_{t \to \infty} x(t) \in \mathbb{B}^n$ of the function $x : \mathbb{R} \to \mathbb{B}^n$ are defined by
\[ \exists t' \in \mathbb{R}, \forall t < t', x(t) = x(-\infty + 0), \]
\[ \exists t' \in \mathbb{R}, \forall t > t', x(t) = x(\infty - 0). \]

Definition 8 The function $x : \mathbb{R} \to \mathbb{B}^n$ is called (pseudo)periodical with the period $T_0 > 0$ if
a) $\lim_{t \to \infty} x(t)$ does not exist and
b) $\exists t' \in \mathbb{R}, \forall t \geq t', x(t) = x(t + T_0)$.

Definition 9 The characteristic function $\chi_A : \mathbb{R} \to \mathbb{B}$ of the set $A \subset \mathbb{R}$ is defined in the following way:
\[ \chi_A(t) = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{otherwise} \end{cases} . \]

Notation 10 We denote by $Seq$ the set of the real sequences $t_0 < t_1 < ... < t_k < ...$ which are unbounded from above.

Remark 11 The sequences $(t_k) \in Seq$ act as time sets. At this level of generality of the exposure, a double uncertainty exists in the real time iterative computations of the function $\Phi : \mathbb{B}^n \to \mathbb{B}^n :$ we do not know precisely neither the coordinates $\Phi_i$ of $\Phi$ that are computed, nor when the computation happens. This uncertainty implies the non determinism of the model and its origin consists in structural fluctuations in the fabrication process, the variations in ambiental temperature and the power supply etc.
Figure 1: Circuit with the logical gate NOT

**Definition 12** A *signal* (or *n-signal*) is a function \( x : \mathbb{R} \rightarrow \mathbb{B}^n \) of the form

\[
x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \tag{1}
\]

... \( \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... \\
with \( (t_k) \in \text{Seq} \). The set of the signals is denoted by \( S^{(n)} \).

**Remark 13** The signals \( x \in S^{(n)} \) model the electrical signals from the digital electrical engineering. They have by definition initial values and they avoid ‘Dirichlet type’ properties (called Zeno properties by the engineers) such as

\[
\exists t \in \mathbb{R}, \forall \varepsilon > 0, \exists t' \in (t - \varepsilon, t), \exists t'' \in (t - \varepsilon, t), x(t') \neq x(t''),
\]

\[
\exists t \in \mathbb{R}, \forall \varepsilon > 0, \exists t' \in (t, t + \varepsilon), \exists t'' \in (t, t + \varepsilon), x(t') \neq x(t'')
\]

because these properties cannot characterize the inertial devices.

**Notation 14** We denote by \( P^* \) the set of the non-empty subsets of a set.

**Definition 15** The *autonomous asynchronous systems* are the non-empty sets \( X \in P^*(S^{(n)}) \).

**Example 16** We give the following simple example that shows how the autonomous asynchronous systems model the asynchronous circuits. In Figure 1 we have drawn the (logical) gate NOT with the input \( u \in S^{(1)} \) and the state (the output) \( x \in S^{(1)} \). For \( \lambda \in \mathbb{B} \) and

\[
u(t) = \lambda,
\]

the state \( x \) represents the computation of the negation of \( u \) and it is of the form

\[
x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \chi_{[t_0,t_1)}(t) \oplus \chi_{[t_1,t_2)}(t) \oplus ... \oplus \chi_{[t_k,t_{k+1})}(t) \oplus ...
\]
Figure 2: Circuit with feedback with the logical gate NOT

\[
= \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\lambda} \cdot \chi_{[t_0,\infty)}(t),
\]

where \( \mu \in \mathbb{B} \) is the initial value of \( x \) and \((t_k) \in \text{Seq}\) is arbitrary. As we can see, \( x \) depends on \( t_0, \mu, \lambda \) only and it is independent on \( t_1, t_2, \ldots \)

In Figure 2, we have

\[
x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus \ldots
\]

\[
\oplus \overline{\mu} \cdot \chi_{[t_{2k},t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus \ldots
\]

thus this circuit is modeled by the autonomous asynchronous system

\[
X = \{ \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \overline{\mu} \cdot \chi_{[t_0,t_1)}(t) \oplus \mu \cdot \chi_{[t_1,t_2)}(t) \oplus \ldots
\]

\[
\oplus \overline{\mu} \cdot \chi_{[t_{2k},t_{2k+1})}(t) \oplus \mu \cdot \chi_{[t_{2k+1},t_{2k+2})}(t) \oplus \ldots | \mu \in \mathbb{B}, (t_k) \in \text{Seq} \} \in P^*(S^{(1)}).
\]

**Definition 17** The **progressive functions** \( \rho : \mathbb{R} \to \mathbb{B}^n \) are by definition the functions

\[
\rho(t) = \alpha^0 \cdot \chi_{(t_0)}(t) \oplus \alpha^1 \cdot \chi_{(t_1)}(t) \oplus \ldots \oplus \alpha^k \cdot \chi_{(t_k)}(t) \oplus \ldots
\]  \hspace{1cm} (2)

where \((t_k) \in \text{Seq}\) and \( \alpha^0, \alpha^1, \ldots, \alpha^k, \ldots \in \Pi_n \). The set of the progressive functions is denoted by \( P_n \).

**Definition 18** For \( \Phi : \mathbb{B}^n \to \mathbb{B}^n \) and \( \rho \in P_n \) like at (2), we define \( \Phi^\rho : \mathbb{R} \times \mathbb{B}^n \to \mathbb{B}^n \) by \( \forall t \in \mathbb{R}, \forall \mu \in \mathbb{B}^n \),

\[
\Phi^\rho(t, \mu) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu) \cdot \chi_{(t_0,t_1)}(t) \oplus \ldots \oplus \Phi^{\alpha^k}(\mu) \cdot \chi_{(t_k,t_{k+1})}(t) \oplus \ldots
\]

**Remark 19** The previous equation reminds the iterations of a discrete time real dynamical system. The time is not exactly discrete in it, but some sort of intermediate situation occurs between the discrete and the real time; on the other hand the iterations of \( \Phi \) do not happen on all the coordinates (synchronicity), but on some coordinates only, such that any coordinate \( \Phi_i \) is computed infinitely many times, \( i = 1, n \) (asynchronicity) when \( t \in \mathbb{R} \).
2 Discrete time

Notation 20 We denote by

\[ N_\mathcal{=} N \cup \{-1\} \]

the discrete time set.

Definition 21 Let be \( \Phi : \mathcal{B}^n \to \mathcal{B}^n \) and \( \alpha \in \Pi_n, \alpha = \alpha^0, ..., \alpha^k, ... \). We define the function \( \hat{\Phi}^\alpha : N_\times \mathcal{B}^n \to \mathcal{B}^n \) by \( \forall (k, \mu) \in N_\times \mathcal{B}^n \),

\[ \hat{\Phi}^\alpha(k, \mu) = \begin{cases} \mu, k = -1, \\ \Phi^{\alpha^0...\alpha^k}(\mu), k \geq 0. \end{cases} \]

Notation 22 Let us denote

\[ \hat{\Pi}_n = \{ \alpha | \alpha \in \Pi_n, \forall k \in N, \alpha^k \neq (0, ..., 0) \}. \]

Definition 23 The equivalence of \( \rho, \rho' \in P_n \) is defined by: \( \exists (t_k) \in \text{Seq}, \exists (t'_k) \in \text{Seq}, \exists \alpha \in \hat{\Pi}_n \) such that (2) and

\[ \rho'(t) = \alpha^0 \cdot \chi_{[t_0]}(t) \oplus \alpha^1 \cdot \chi_{[t_1]}(t) \oplus ... \oplus \alpha^k \cdot \chi_{[t_k]}(t) \oplus ... \]

are true.

Definition 24 The 'canonical surjection' \( s : P_n \to \hat{\Pi}_n \) is by definition the function \( \forall \rho \in P_n \),

\[ s(\rho) = \alpha \]

where \( \alpha \in \hat{\Pi}_n \) is the only sequence such that \( (t_k) \in \text{Seq} \) exists, making the equation (2) true.

Remark 25 The relation between the continuous and the discrete time is the following: for any \( \mu \in \mathcal{B}^n \) and any \( \rho \in P_n, \alpha \in \hat{\Pi}_n \) and \( (t_k) \in \text{Seq} \) exist making the equation (2) true and we have

\[ \Phi^\rho(t, \mu) = \hat{\Phi}^\alpha(-1, \mu) \cdot \chi_{(-\infty, t_0]}(t) \oplus \hat{\Phi}^\alpha(0, \mu) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \]

\[ ... \oplus \hat{\Phi}^\alpha(k, \mu) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... \]

Equivalent progressive functions \( \rho, \rho' \in P_n \) (i.e. \( s(\rho) = s(\rho') \)) give 'equivalent' functions \( \Phi^\rho(t, \mu), \Phi^{\rho'}(t, \mu) \) in the sense that the computations of \( \Phi \) are the same, but the time flow is piecewise faster or slower in the two situations.
3 Regular autonomous asynchronous systems

Definition 26 The universal regular autonomous asynchronous system $\Xi_\Phi \in P^*(S^{(n)})$ that is generated by the function $\Phi : B^n \rightarrow B^n$ is defined by
$$\Xi_\Phi = \{\Phi^\rho(\cdot, \mu) | \mu \in B^n, \rho \in P_n\}.$$ 

Definition 27 An autonomous asynchronous system $X \in P^*(S^{(n)})$ is called regular, if $\Phi$ exists such that $X \subseteq \Xi_\Phi$. In this case $\Phi$ is called the generator function\(^1\) of $X$.

Remark 28 In the last two definitions, the attribute 'regular' refers to the existence of a generator function $\Phi$ and the attribute 'universal' means maximal relative to the inclusion. For a regular system, $\Phi$ is not unique in general.

Example 29 For any $\mu^0 \in B^n$ and $\rho^* \in P_n$, the autonomous systems $\{\Phi^\rho(\cdot, \mu^0)\}_{\rho \in P_n}$, $\{\Phi^\rho(\cdot, \mu)|\mu \in B^n\}$ and $\Xi_\Phi$ are regular.

For $\Phi = 1_{B^n}$, the system $\Xi_{1_{B^n}} = \{\mu | \mu \in B^n\} = B^n$ is regular.

Another example of universal regular autonomous asynchronous system is given by $\Phi = \mu^0$, the constant function, for which $\Xi_{\mu^0} = \{x | x_i = \mu_i \cdot \chi_{(-\infty, t_i]} + \mu_i^0 \cdot \chi_{[t_i, \infty)}, \mu_i \in B, t_i \in R, i = 1, n\}$.

Remark 30 These examples suggest several possibilities of defining the systems $X \subseteq \Xi_\Phi$, which are not universal. For example by putting appropriate supplementary requests on the functions $\rho$, one could rediscover the 'bounded delay model' of computation of the Boolean functions.

4 Orbits and state portraits

Definition 31 Let be $\rho \in P_n$. Two things are understood by orbit, or (state, or phase) trajectory ([1], page 19; [2], page 3; [4], page 8; [5], page 24; [6], page 2) of $\Xi_\Phi$ starting at $\mu \in B^n$:

a) the function $\Phi^\rho(\cdot, \mu) : R \rightarrow B^n$;

b) the set $\text{Or}_\rho(\mu) = \{\Phi^\rho(t, \mu)| t \in R\}$ representing the values of the previous function.

Sometimes ([2], page 4; [3], page 91; [5], page 24; [6], page 2) the function from a) is called the motion (or the dynamic) of $\mu$ through $\Phi^\rho$.

\(^1\)The terminology of 'generator function' is also used in [1], page 18 meaning the vector field of a discrete time dynamical system. In [3] the terminology of 'generator' (function) of a dynamical system is mentioned too. Moisil called $\Phi$ 'network function' in a non-autonomous, discrete time context; for Moisil, 'network' means 'system' or 'circuit'.
Definition 32 The equivalent properties
\[ \exists t \in \mathbb{R}, \Phi^\rho(t, \mu) = \mu' \]
and
\[ \mu' \in \text{Or}_\rho(\mu) \]
are called of accessibility; the points \( \mu' \in \text{Or}_\rho(\mu) \) are said to be accessible.

Remark 33 The orbits are the curves in \( \mathbb{B}^n \), parametrized by \( \rho \) and \( t \). On the other hand \( \rho \in P_n, t' \in \mathbb{R} \) imply \( \rho \cdot \chi_{(t', \infty)} \in P_n \) and we see the truth of the implication
\[ \mu' = \Phi^\rho(t', \mu) \implies \forall t \geq t', \Phi^\rho(t, \mu) = \Phi^\rho_{\chi_{(t', \infty)}}(t, \mu'). \]

Definition 34 The state (or the phase) portrait of \( \Xi_\Phi \) is the set of its orbits ([2], page 4; [3], page 92; [4], page 10; [6], page 2).

Example 35 The function \( \Phi : \mathbb{B}^2 \to \mathbb{B}^2 \) is defined by the following table
\[
\begin{array}{|c|c|}
\hline
(\mu_1, \mu_2) & \Phi(\mu_1, \mu_2) \\
\hline
(0, 0) & (0, 0) \\
(0, 1) & (1, 0) \\
(1, 0) & (1, 1) \\
(1, 1) & (1, 1) \\
\hline
\end{array}
\]
The state portrait of \( \Xi_\Phi \) is:
\[
\{ (0, 1) \cdot \chi_{(-\infty, t_0)} \oplus (0, 0) \cdot \chi_{[t_0, \infty)} | t_0 \in \mathbb{R} \} \cup \\
\{ (0, 1) \cdot \chi_{(-\infty, t_0)} \oplus (1, 0) \cdot \chi_{[t_0, t_1)} \oplus (1, 1) \cdot \chi_{[t_1, \infty)} | t_0, t_1 \in \mathbb{R}, t_0 < t_1 \} \cup \\
\{ (0, 1) \cdot \chi_{(-\infty, t_0)} \oplus (1, 1) \cdot \chi_{[t_0, \infty)} | t_0 \in \mathbb{R} \} \cup \\
\{ (0, 0) \} \cup \{(1, 1)\}.
\]
This set is drawn in Figure 3, where the arrows show the increase of time. One might want to put arrows from \((0, 0)\) to itself and from \((1, 1)\) to itself.

5 Nullclins
Definition 36 Let be \( \Phi : \mathbb{B}^n \to \mathbb{B}^n \). For any \( i \in \{1, \ldots, n\} \), the nullclins of \( \Phi \) are the sets
\[ NC_i = \{ \mu | \mu \in \mathbb{B}^n, \Phi_i(\mu) = \mu_i \}. \]
If \( \mu \in NC_i \), then the coordinate \( i \) is said to be not excited, or not enabled, or stable and if \( \mu \in \mathbb{B}^n \setminus NC_i \) then it is called excited, or enabled, or unstable.
Remark 37 Sometimes, instead of indicating \( \Phi \) by a table like previously, we can replace Figure 3 by Figure 4, where we have underlined the unstable coordinates. For example in Figure 4, \((0, 1)\) means that \(\Phi(0, 1) = (1, 0)\), \((1, 0)\) means that \(\Phi(1, 0) = (1, 1)\) etc.

In fact Figure 4 results uniquely from Figure 3, one could know by looking at Figure 3 which coordinates should be underlined and which should be not.

6 Fixed points

Definition 38 A point \( \mu \in B^n \) that fulfills \(\Phi(\mu) = \mu\) is called a fixed point (an equilibrium point, a critical point, a singular point) ([1], page 43; [2], page 4; [3], page 92; [4], page 9; [5], page 24; [6], page 2), shortly an equilibrium of \( \Phi \). A point that is not fixed is called ordinary.

Theorem 39 The following statements are equivalent for \( \mu \in B^n \):

\[
\Phi(\mu) = \mu, \tag{3}
\]
\[
\exists \rho \in P_n, \forall t \in \mathbb{R}, \Phi^\rho(t, \mu) = \mu, \tag{4}
\]
\[
\forall \rho \in P_n, \forall t \in \mathbb{R}, \Phi^\rho(t, \mu) = \mu, \tag{5}
\]
\[
\exists \rho \in P_n, Or^\rho(\mu) = \{\mu\}, \tag{6}
\]
\[ \forall \rho \in P_n, Or_{\rho}(\mu) = \{\mu\}, \quad (7) \]
\[ \mu \in NC_1 \cap ... \cap NC_n. \quad (8) \]

**Proof.** (3)\(\Rightarrow\)(4) We take \(\rho \in P_n\) in the following way
\[ \rho(t) = (1, ..., 1) \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus (1, ..., 1) \cdot \chi_{\{t_k\}}(t) \oplus ... \]
with \((t_k) \in Seq\). For the sequence
\[ \forall k \in \mathbb{N}, \alpha^k = (1, ..., 1) \]
from \(\Pi_n\) we can prove by induction on \(k\) that
\[ \forall k \in \mathbb{N}, \Phi^{\alpha^1...\alpha^k}(\mu) = \mu \quad (9) \]
wherefrom
\[ \Phi^\rho(t, \mu) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus \mu \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \mu \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ... = \mu \quad (10) \]

(4)\(\Rightarrow\)(3) From (4) we have the existence of \(\alpha \in \Pi_n\) and \((t_k) \in Seq\) with the property that (10) is true, thus (9) is true. We denote
\[ I_0 = \{i|i \in \{1, ..., n\}, \alpha^0_i = 1\}, \]
\[ I_1 = \{i|i \in \{1, ..., n\}, \alpha^1_i = 1\}, \]
... \[ I_k = \{i|i \in \{1, ..., n\}, \alpha^k_i = 1\}, \]
... and we have from (9):
\[ \forall i \in \{1, ..., n\}, \]
\[ \Phi^\rho_i(\mu) = \begin{cases} \Phi_i(\mu), & i \in I_0 \setminus I_0 = \mu_i; \\ \mu_i, & i \in \{1, ..., n\} \setminus I_0 \end{cases} \]
\[ \forall i \in \{1, ..., n\}, \Phi^{\alpha^0\alpha^1}(\mu) = \Phi^{\alpha^1}(\Phi^{\alpha^0}(\mu)) = \]
\[ = \Phi^{\alpha^1}(\mu) = \begin{cases} \Phi_i(\mu), & i \in I_1 \setminus I_1 = \mu_i; \\ \mu_i, & i \in \{1, ..., n\} \setminus I_1 \end{cases} \]
... \[ \forall i \in \{1, ..., n\}, \Phi^{\alpha^1...\alpha^k}(\mu) = \Phi^{\alpha^k}(\Phi^{\alpha^1...\alpha^{k-1}}(\mu)) = \]
\[ = \Phi^{\alpha^k}(\mu) = \begin{cases} \Phi_i(\mu), & i \in I_k \setminus I_k = \mu_i; \\ \mu_i, & i \in \{1, ..., n\} \setminus I_k \end{cases} \]
\[ \forall k \in \mathbb{N}, \forall i \in I_0 \cup I_1 \cup \ldots \cup I_k, \Phi_i(\mu) = \mu_i. \]

Some \( k' \in \mathbb{N} \) exists with the property that
\[ I_0 \cup I_1 \cup \ldots \cup I_{k'} = \{1, \ldots, n\}, \]
thus (3) is true.

(3) \( \implies \) (5) Let be
\[ \rho(t) = \alpha^0(t_0)(t) \oplus \ldots \oplus \alpha^k(t_k)(t) \oplus \ldots \quad (11) \]
with \( \alpha^0, \ldots, \alpha^k, \ldots \in \Pi_n \) and \( (t_k) \in \text{Seq} \) arbitrary. It is proved by induction on \( k \) the validity of (9) and this implies the truth of (10).

(5) \( \implies \) (3) This is true because (5) \( \implies \) (4) and (4) \( \implies \) (3) are true.

(4) \( \iff \) (6) and (5) \( \iff \) (7) are obvious.

(3) \( \iff \) (8) \( \Phi(\mu) = \mu \iff \Phi_1(\mu) = \mu_1 \) and...and \( \Phi_n(\mu) = \mu_n \iff \mu \in NC_1 \) and...and \( \mu \in NC_n \iff \mu \in NC_1 \cap \ldots \cap NC_n \).

**Definition 40** If \( \Phi(\mu) = \mu \), then \( \forall \rho \in P_n \), the orbit \( \Phi^\rho(t, \mu) = \mu \) is called rest position.

### 7 Fixed points vs. final values of the orbits

**Theorem 41** ([7], Theorem 49) The following fixed point property is true
\[ \forall \mu \in \mathbb{B}^n, \forall \mu' \in \mathbb{B}^n, \forall \rho \in P_n, \lim_{t \to \infty} \Phi^\rho(t, \mu) = \mu' \implies \Phi(\mu') = \mu'. \]

**Proof.** Let \( \mu \in \mathbb{B}^n, \mu' \in \mathbb{B}^n, \rho \in P_n \) be arbitrary and fixed. Some \( t' \in \mathbb{R} \) exists such that \( \forall t \geq t' \),
\[ \mu' = \Phi^\rho(t, \mu) \quad \text{Remark 33} \quad \Phi^\rho \chi_{[t', \infty)}(t, \mu') \]
and from Theorem 39, (4) \( \implies \) (3) we have \( \Phi(\mu') = \mu' \).

**Remark 42** Theorem 41 shows that the final values of the states of a system are fixed points of \( \Phi \).

**Theorem 43** ([7], Theorem 50) We have \( \forall \mu \in \mathbb{B}^n, \forall \mu' \in \mathbb{B}^n, \forall \rho \in P_n, \)
\[ (\Phi(\mu') = \mu' \text{ and } \exists t' \in \mathbb{R}, \Phi^\rho(t', \mu) = \mu') \implies \forall t \geq t', \Phi^\rho(t, \mu) = \mu'. \]
**Proof.** For arbitrary $\mu \in B^n, \mu' \in B^n, \rho \in P_n$ we suppose that $\Phi(\mu') = \mu'$ and $\Phi^\rho(t', \mu) = \mu'$. We have $\forall t \geq t'$,

\[
\Phi^\rho(t, \mu) = \Phi^\rho(t', \mu') = \Phi^\rho(\chi(t', \infty))(t, \mu') = (5)
\]

\[
= \mu'.
\]

\[
\square
\]

**Remark 44** As resulting from Theorem 43, the accessible fixed points are final values of the states of the systems.

The properties of the fixed points that are expressed by Theorems 39, 41, 43 give a better understanding of Example 35.

### 8 Transitivity

**Definition 45** The system $\Xi_\Phi$ (or $\Phi$) is **transitive** ([1], page 22; [2], page 3), or **minimal** ([1], page 23) if one of the following non-equivalent properties holds true:

\[
\forall \mu \in B^n, \forall \mu' \in B^n, \exists \rho \in P_n, \exists t \in R, \Phi^\rho(t, \mu) = \mu', \tag{12}
\]

\[
\forall \mu \in B^n, \forall \mu' \in B^n, \forall \rho \in P_n, \exists t \in R, \Phi^\rho(t, \mu) = \mu'. \tag{13}
\]

**Remark 46** The property of transitivity may be considered one of surjectivity or one of accessibility.

If $\Phi$ is transitive, then it has no fixed points.

**Example 47** The property (12) of transitivity is exemplified in Figure 5 and the property (13) of transitivity is exemplified in Figure 6.

### 9 The equivalence of the dynamical systems

**Notation 48** Let $h : B^n \to B^n$ and $x : R \to B^n$ be some functions. We denote by $h(x) : R \to B^n$ the function

\[
\forall t \in R, h(x)(t) = h(x(t)).
\]
Figure 6: Transitivity

Remark 49 If \( h : \mathbb{B}^n \to \mathbb{B}^n \) and \( x \in S^{(n)} \) is expressed by
\[
x(t) = x(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...
\]
then
\[
h(x)(t) = h(x(-\infty + 0)) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(x(t_0)) \cdot \chi_{[t_0, t_1)}(t) \oplus ...
\]
\[
... \oplus h(x(t_k)) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ...
\]

Notation 50 For \( h : \mathbb{B}^n \to \mathbb{B}^n \) and \( \alpha = \alpha^0, ..., \alpha^k, ... \in \mathbb{B}^n \), we denote by \( \hat{h}(\alpha) \) the sequence \( h(\alpha^0), ..., h(\alpha^k), ... \in \mathbb{B}^n \).

Notation 51 Let be \( k \geq 2 \) arbitrary and we denote for \( \mu^1, ..., \mu^k \in \mathbb{B}^n \),
\[
\mu^1 \cup ... \cup \mu^k = (\mu^1 \cup ... \cup \mu^1, ..., \mu^1 \cup ... \cup \mu^k).
\]

Notation 52 We denote by \( \Omega_n \) the set of the functions \( h : \mathbb{B}^n \to \mathbb{B}^n \) that fulfill

i) \( h \) is bijective;

ii) \( h(0, ..., 0) = (0, ..., 0) \), \( h(1, ..., 1) = (1, ..., 1) \);

iii) \( \forall k \geq 2, \forall \mu^1 \in \mathbb{B}^n, ..., \forall \mu^k \in \mathbb{B}^n \),
\[
\mu^1 \cup ... \cup \mu^k = (1, ..., 1) \iff h(\mu^1) \cup ... \cup h(\mu^k) = (1, ..., 1).
\]

Theorem 53 a) \( \Omega_n \) is group relative to the composition ‘\( \circ' \) of the functions;

b) \( \forall h \in \Omega_n, \forall \alpha \in \Pi_n, \hat{h}(\alpha) \in \Pi_n \);

c) \( \forall h \in \Omega_n, \forall \rho \in P_n, h(\rho) \in P_n \).

Proof. a) The fact that \( 1_{\mathbb{B}^n} \in \Omega_n \), \( \forall h \in \Omega_n, \forall h' \in \Omega_n, h \circ h' \in \Omega_n \) and \( \forall h \in \Omega_n, h^{-1} \in \Omega_n \) is obvious.

b) Let \( h \in \Omega_n \) and \( \alpha = \alpha^0, ..., \alpha^k, ... \in \mathbb{B}^n \) be arbitrary. We denote for \( p \geq 1 \)
\[
\{\mu^1, ..., \mu^p\} = \{\mu | \mu \in \mathbb{B}^n, \{k | k \in \mathbb{N}, \alpha^k = \mu\} \text{ is infinite}\}.
\]
and we remark that
\[
\alpha \in \Pi_n \iff \mu^1, \ldots, \mu^p, \mu^1, \ldots, \mu^p, \mu^1, \ldots \in \Pi_n \iff
\]
\[
\left\{
\begin{array}{l}
\mu^1 = (1, \ldots, 1), p = 1 \\
\mu^1 \cup \ldots \cup \mu^p = (1, \ldots, 1), p \geq 2
\end{array}
\right.
\]
\[
\tilde{h}(\alpha) \in \Pi_n \iff h(\mu^1), \ldots, h(\mu^p), h(\mu^1), \ldots, h(\mu^p), h(\mu^1), \ldots \in \Pi_n \iff
\]
\[
\left\{
\begin{array}{l}
h(\mu^1) = (1, \ldots, 1), p = 1 \\
h(\mu^1) \cup \ldots \cup h(\mu^p) = (1, \ldots, 1), p \geq 2
\end{array}
\right.
\]

Case \( p = 1, \)
\[
\alpha \in \Pi_n \implies \mu^1 = (1, \ldots, 1) \implies h(\mu^1) = (1, \ldots, 1) \implies \tilde{h}(\alpha) \in \Pi_n.
\]

Case \( p \geq 2, \)
\[
\alpha \in \Pi_n \implies \mu^1 \cup \ldots \cup \mu^p = (1, \ldots, 1) \implies h(\mu^1) \cup \ldots \cup h(\mu^p) = (1, \ldots, 1) \implies
\]
\[
\implies \tilde{h}(\alpha) \in \Pi_n.
\]

c) Let us take arbitrarily some \( h \in \Omega_n \) and a function \( \rho \in P_n, \)
\[
\rho(t) = \alpha^0 \cdot \chi_{[t_0]}(t) \oplus \ldots \oplus \alpha^k \cdot \chi_{[t_k]}(t) \oplus \ldots
\]
where \( \alpha \in \Pi_n \) and \( (t_k) \in \text{Seq}. \) We have
\[
h(\rho)(t) = h(\rho(t)) =
\]
\[
h((0, \ldots, 0) \cdot \chi_{(\infty, t_0]}(t) \oplus \alpha^0 \cdot \chi_{[t_0]}(t) \oplus (0, \ldots, 0) \cdot \chi_{(t_0, t_1]}(t) \oplus \ldots
\]
\[
\ldots \oplus \alpha^k \cdot \chi_{(t_k]}(t) \oplus (0, \ldots, 0) \cdot \chi_{(t_k, t_{k+1}]}(t) \oplus \ldots
\]
\[
= h(0, \ldots, 0) \cdot \chi_{(\infty, t_0]}(t) \oplus h(\alpha^0) \cdot \chi_{[t_0]}(t) \oplus h(0, \ldots, 0) \cdot \chi_{(t_0, t_1]}(t) \oplus \ldots
\]
\[
\ldots \oplus h(\alpha^k) \cdot \chi_{(t_k]}(t) \oplus h(0, \ldots, 0) \cdot \chi_{(t_k, t_{k+1}]}(t) \oplus \ldots
\]
\[
= h(\alpha^0) \cdot \chi_{(t_0]}(t) \oplus \ldots \oplus h(\alpha^k) \cdot \chi_{(t_k]}(t) \oplus \ldots
\]
Because \( \tilde{h}(\alpha) \in \Pi_n, \) taking into account b), we conclude that \( h(\rho) \in P_n. \)

\[\text{Theorem 54}\]
Let be the generator functions \( \Phi, \Psi : B^n \rightarrow B^n \) of the systems \( \Xi_\Phi, \Xi_\Psi \) and the bijections \( h : B^n \rightarrow B^n, h' \in \Omega_n. \) The following statements are equivalent:

a) \( \forall \nu \in B^n, \) the diagrams
\[
\begin{array}{ccc}
B^n & \xrightarrow{\Phi^\nu} & B^n \\
\downarrow h & & \downarrow h \\
B^n & \xrightarrow{\Psi h'(\nu)} & B^n
\end{array}
\]

13
are commutative;
b) \( \forall \mu \in B^n, \forall \alpha \in \Pi_n, \forall k \in \mathbb{N}_- \),
\[ h(\Phi^\alpha(k, \mu)) = \tilde{\Psi}^{\beta(\alpha)}(k, h(\mu)) \];
c) \( \forall \mu \in B^n, \forall \rho \in P_n, \forall t \in \mathbb{R} \),
\[ h(\Phi^\rho(t, \mu)) = \Psi^{h'(\rho)}(t, h(\mu)) \]. (14)

**Proof.**
a)\( \implies \)b) It is sufficient to prove that \( \forall \mu \in B^n, \forall \alpha \in \Pi_n, \forall k \in \mathbb{N} \),
\[ h(\Phi^{\alpha\ldots\alpha^k}(\mu)) = \tilde{\Psi}^{h'(\alpha\ldots\alpha^k)}(h(\mu)) \] (15)
since this is equivalent with b).

We fix arbitrarily some \( \mu \) and some \( \alpha \) and we use the induction on \( k \). For \( k = 0 \) the statement is proved, thus we suppose that it is true for \( k \) and we prove it for \( k + 1 \):
\[ h(\Phi^{\alpha\ldots\alpha^k\alpha^{k+1}}(\mu)) = h(\Phi^{\alpha^{k+1}}(\Phi^{\alpha\ldots\alpha^k}(\mu))) = \tilde{\Psi}^{h'(\alpha^{k+1})}(h(\Phi^{\alpha\ldots\alpha^k}(\mu))) = \]
\[ = \tilde{\Psi}^{h'(\alpha^{k+1})}(\tilde{\Psi}^{h'(\alpha\ldots\alpha^k)}(h(\mu))) = \tilde{\Psi}^{h'(\alpha^{k})h'(\alpha^{k+1})}(h(\mu)). \]

b)\( \implies \)c) For arbitrary \( \mu \in B^n \) and \( \rho \in P_n \),
\[ \rho(t) = \rho(t_0) \cdot \chi_{\{t_0\}}(t) + \ldots + \rho(t_k) \cdot \chi_{\{t_k\}}(t) + \ldots \]
\((t_k) \in \text{Seq}, \rho(t_0), \ldots, \rho(t_k), \ldots \in \Pi_n \) we have that
\[ h'(\rho)(t) = h'(\rho(t)) = h'(\rho(t_0)) \cdot \chi_{\{t_0\}}(t) + \ldots + h'(\rho(t_k)) \cdot \chi_{\{t_k\}}(t) + \ldots \] (16)
is an element of \( P_n \) (see Theorem 53 c)) and
\[ h(\Phi^{\rho}(t, \mu)) = h(\mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi^{\rho(t_0)}(\mu) \cdot \chi_{[t_0, t_1]}(t) \oplus \ldots \]
\[ = h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(\Phi^{\rho(t_0)}(\mu)) \cdot \chi_{[t_0, t_1]}(t) \oplus \ldots \]
\[ = h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus \Psi^{h'(\rho(t_0))}(h(\mu)) \cdot \chi_{[t_0, t_1]}(t) \oplus \ldots \]
\[ = h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus \Psi^{h'(\rho(t_0))}(h(\mu)) \cdot \chi_{[t_0, t_{k+1}]}(t) \oplus \ldots \]
\[ = \tilde{\Psi}^{h'(\rho)(t)}(t, h(\mu)). \] (16)
c)\( \implies \)a) Let \( \nu, \mu \in B^n \) be arbitrary and fixed and we consider \( \rho \in P_n \),
\[ \rho(t) = \nu \cdot \chi_{\{t_0\}}(t) + \rho(t_1) \cdot \chi_{\{t_1\}}(t) + \ldots + \rho(t_k) \cdot \chi_{\{t_k\}}(t) + \ldots \]
with \((t_k) \in \text{Seq}\) fixed too. We have

\[
h(\Phi'(t, \mu)) = h(\mu \cdot \chi_{(-\infty, t_0)}(t) \oplus \Phi'(\mu) \cdot \chi_{[t_0, t_1)}(t) \oplus \Phi'(\nu(1))(\mu) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots) =
\]

\[
= h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus h(\Phi'(\mu)) \cdot \chi_{[t_0, t_1)}(t) \oplus h(\Phi'(\nu))(\mu) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots
\]

But

\[
h'(\rho)(t) = h'(\rho(t)) = h'(\nu) \cdot \chi_{[t_0, t_1)}(t) \oplus h'(\rho(t_1)) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots
\]

\[
= h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus \Psi^{h(\rho)}(h(\mu)) =
\]

\[
= h(\mu) \cdot \chi_{(-\infty, t_0)}(t) \oplus \Psi^{h(\nu)} \cdot \chi_{[t_0, t_1)}(t) \oplus \Psi^{h(\nu)} h'(\rho(t_1)) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots
\]

and from (14), for \(t \in [t_0, t_1)\), we obtain

\[
h(\Phi'(\mu)) = \Psi^{h(\nu)}(h(\mu)).
\]

\[\square\]

**Definition 55** We consider the generator functions \(\Phi, \Psi : \mathbb{B}^n \to \mathbb{B}^n\) and the universal asynchronous systems \(\Xi_\Phi, \Xi_\Psi\). If two bijections \(h : \mathbb{B}^n \to \mathbb{B}^n, h' \in \Omega_n\) exist such that one of the equivalent properties a), b), c) from Theorem 54 is satisfied, then \(\Xi_\Phi, \Xi_\Psi\) are called **equivalent** ([1], page 35; [3], page 102; [4], page 40; [5], page 32; [6], page 6) and \(\Phi, \Psi\) are called **conjugated**. In this case we denote \(\Phi \overset{(h,h')}\rightarrow \Psi\).

**Definition 56** We fix \(\Phi\). The fact that \(\Psi \neq \Phi\) exists such that the previous property holds, makes us say that \(\Phi\) is **structurally stable** (Peixoto [3], page 121). \(\Psi\) is called an **admissible** (or **allowable**) **perturbation** of \(\Phi\).

**Remark 57** The equivalence of the universal regular autonomous asynchronous systems is indeed an equivalence and it should be understood as a change of coordinates. Thus \(\Phi\) and \(\Psi\) are indistinguishable.

**Example 58** \(\Phi, \Psi : \mathbb{B}^2 \to \mathbb{B}^2\) are given by, see Figure 7

\[
\forall (\mu_1, \mu_2) \in \mathbb{B}^2, \Phi(\mu_1, \mu_2) = (\mu_1 \oplus \mu_2, \mu_2),
\]

\[
\forall (\mu_1, \mu_2) \in \mathbb{B}^2, \Psi(\mu_1, \mu_2) = (\mu_1 \mu_2 \cup \mu_1 \mu_2)
\]

and the bijection \(h : \mathbb{B}^2 \to \mathbb{B}^2\) is

\[
\forall (\mu_1, \mu_2) \in \mathbb{B}^2, h(\mu_1, \mu_2) = (\mu_2, \mu_1).
\]

15
Figure 7: Equivalent systems

The diagram

\[ \Phi : B \overset{\Phi}{\rightarrow} B \]
\[ h \downarrow \quad \downarrow h \]
\[ B \overset{\Psi}{\rightarrow} B \]

commutes for \( \nu = \nu' = (0,0) \) and for \( \nu = \nu' = (1,1) \) we have the assignments

\[
(0,0) \xrightarrow{\Phi} (0,1) \quad (0,1) \xrightarrow{\Phi} (1,0) \quad (1,0) \xrightarrow{\Phi} (1,1) \quad (1,1) \xrightarrow{\Phi} (0,0) \\
(1,1) \xrightarrow{\Psi} (0,1) \quad (0,1) \xrightarrow{\Psi} (1,0) \quad (1,0) \xrightarrow{\Psi} (0,0) \quad (0,0) \xrightarrow{\Psi} (1,1)
\]

We denote \( \pi_i : B \rightarrow B, \forall (\mu_1, \mu_2) \in B^2 \),

\[
\pi_i(\mu_1, \mu_2) = \mu_i, i = 1, 2.
\]

For \( \nu = (0,1), \nu' = (1,0) \) we have

\[
(0,0) \xrightarrow{(\pi_1, \Phi_2)} (0,1) \quad (0,1) \xrightarrow{(\pi_1, \Phi_2)} (0,0) \quad (1,0) \xrightarrow{(\pi_1, \Phi_2)} (1,1) \quad (1,1) \xrightarrow{(\pi_1, \Phi_2)} (1,0) \\
(1,1) \xrightarrow{(\Psi_1, \pi_3)} (0,1) \quad (0,1) \xrightarrow{(\Psi_1, \pi_3)} (1,1) \quad (1,0) \xrightarrow{(\Psi_1, \pi_3)} (0,0) \quad (0,0) \xrightarrow{(\Psi_1, \pi_3)} (1,0)
\]

and for \( \nu = (1,0), \nu' = (0,1) \) the assignments are

\[
(0,0) \xrightarrow{(\Phi_1, \pi_2)} (0,0) \quad (0,1) \xrightarrow{(\Phi_1, \pi_2)} (1,1) \quad (1,0) \xrightarrow{(\Phi_1, \pi_2)} (1,0) \quad (1,1) \xrightarrow{(\Phi_1, \pi_2)} (0,1) \\
(1,1) \xrightarrow{(\pi_1, \Psi_2)} (1,1) \quad (0,1) \xrightarrow{(\pi_1, \Psi_2)} (0,0) \quad (1,0) \xrightarrow{(\pi_1, \Psi_2)} (1,0) \quad (0,0) \xrightarrow{(\pi_1, \Psi_2)} (0,1)
\]

respectively. \( \Phi \) and \( \Psi \) are conjugated.
Example 59 The functions $h, h' : B^2 \rightarrow B^2$ are given in the following table

<table>
<thead>
<tr>
<th>$(\mu_1, \mu_2)$</th>
<th>$h(\mu_1, \mu_2)$</th>
<th>$h'(\mu_1, \mu_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(1, 1)</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>(0, 0)</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(1, 0)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

and the state portraits of the two systems are given in Figure 8. $\Xi_\Phi$ and $\Xi_\Psi$ are equivalent.

Theorem 60 If $\Phi$ and $\Psi$ are conjugated, then the following possibilities exist:

a) $\Phi = \Psi = 1_{B^n}$;

b) $\Phi \neq 1_{B^n}$ and $\Psi \neq 1_{B^n}$.

Proof. We presume that $\Phi \xrightarrow{h,h'} \Psi$. In the equation

$$\forall \nu \in B^n, \forall \mu \in B^n, h(\Phi^\nu(\mu)) = \Psi^{h(\nu)}(h(\mu))$$

we put $\Psi = 1_{B^n}$ and we have

$$\forall \nu \in B^n, \forall \mu \in B^n, h(\Phi^\nu(\mu)) = h(\mu)$$

thus $\forall \nu \in B^n, \Phi^\nu = 1_{B^n}$ and finally $\Phi = 1_{B^n}$.  

Theorem 61 We suppose that $\Xi_\Phi$ and $\Xi_\Psi$ are equivalent and let be $h, h'$ such that $\Phi \xrightarrow{h,h'} \Psi$.

a) If $\mu$ is a fixed point of $\Phi$, then $h(\mu)$ is a fixed point of $\Psi$.

b) For any $\mu \in B^n$ and any $\rho \in P_n$, if $\Phi^\rho(t, \mu)$ is periodical with the period $T_0$, then $\Psi^{h(\rho)}(t, h(\mu))$ is periodical with the period $T_0$.

c) If $\Xi_\Phi$ is transitive, then $\Xi_\Psi$ is transitive.
Proof. a) The commutativity of the diagram

\[
\begin{array}{ccc}
B^n & \xrightarrow{\Phi^\nu} & B^n \\
\downarrow h & & \downarrow h \\
B^n & \xrightarrow{\Psi h^{(\nu)}} & B^n 
\end{array}
\]

for \( \nu = (1, \ldots, 1) \) gives

\[
h(\mu) = h(\Phi(\mu)) = h(\Phi^{(1,\ldots,1)}(\mu)) = \Psi^{h^{(1,\ldots,1)}}(h(\mu)) = \\
= \Psi^{(1,\ldots,1)}(h(\mu)) = \Psi(h(\mu)).
\]

b) The hypothesis states that \( \exists t' \in \mathbb{R}, \forall t \geq t', \)

\[
\Phi^\rho(t, \mu) = \Phi^\rho(t + T_0, \mu)
\]

and in this situation

\[
\Psi^{h^{(\rho)}}(t, h(\mu)) = h(\Phi^\rho(t, \mu)) = h(\Phi^\rho(t + T_0, \mu)) = \Psi^{h^{(\rho)}}(t + T_0, h(\mu)).
\]

c) Let \( \mu, \mu' \in B^n \) be arbitrary and fixed. The hypothesis (12) states that

\[
\exists \rho \in P_n, \exists t \in \mathbb{R}, \Phi^\rho(t, h^{-1}(\mu)) = h^{-1}(\mu'),
\]

wherefrom

\[
\Psi^{h^{(\rho)}}(t, \mu) = \Psi^{h^{(\rho)}}(t, h(h^{-1}(\mu))) = h(\Phi^\rho(t, h^{-1}(\mu)) = h(h^{-1}(\mu')) = \mu'.
\]

The situation with (13) is similar. \( \blacksquare \)

10 Dynamic bifurcations

Remark 62 Let be the generator function \( \Phi : B^n \times B^m \to B^n, B^n \times B^m \ni (\mu, \lambda) \to \Phi(\mu, \lambda) \in B^n \) that depends on the parameter \( \lambda \in B^m \). Intuitively speaking (Ott, [2], page 137) a dynamic bifurcation is a qualitative change in the dynamic of the system \( \Xi_{\Phi(\cdot, \lambda)} \) that occurs at the variation of the parameter \( \lambda \).

Definition 63 If for any parameters \( \lambda, \lambda' \in B^m \) the systems \( \Xi_{\Phi(\cdot, \lambda)} \) and \( \Xi_{\Phi(\cdot, \lambda')} \) are equivalent, then \( \Phi \) is called **structurally stable** ([3], page 117; [5], page 43; [6], page 9); the existence of \( \lambda, \lambda' \) such that \( \Xi_{\Phi(\cdot, \lambda)} \) and \( \Xi_{\Phi(\cdot, \lambda')} \) are not equivalent is called a **dynamic bifurcation** ([4], page 57; [6], page 9).

Equivalently, let us fix an arbitrary \( \lambda \in B^m \). If \( \forall \lambda' \in B^m, \Phi(\cdot, \lambda') \) is an admissible perturbation of \( \Phi(\cdot, \lambda) \) (Definition 56), then \( \Phi \) is said to be **structurally stable**, otherwise we say that \( \Phi \) has a **dynamic bifurcation**.
Remark 64  If $\forall \lambda \in B^m, \forall \lambda' \in B^m$ the bijections $h : B^n \rightarrow B^n, h' \in \Omega_n$ exist such that $\forall \nu \in B^n$, the diagram

\[
\begin{array}{ccc}
B^n & \xrightarrow{\Phi^\nu(\cdot, \lambda)} & B^n \\
h \downarrow & & \downarrow h \\
B^n & \xrightarrow{\Phi^{h'(\cdot, \lambda')}(\cdot, \lambda')} & B^n
\end{array}
\]

commutes, then $\Phi$ is structurally stable, otherwise we have a dynamic bifurcation.

Example 65  In Figure 9 ($n = 2, m = 1$), $\Phi$ is structurally stable and the bijections $h, h'$ are defined accordingly to the following table:

\[
\begin{array}{ccc}
(\mu_1, \mu_2) & h(\mu_1, \mu_2) & h'(\mu_1, \mu_2) \\
(0, 0) & (0, 1) & (0, 0) \\
(0, 1) & (1, 1) & (1, 0) \\
(1, 0) & (0, 0) & (0, 1) \\
(1, 1) & (1, 0) & (1, 1)
\end{array}
\]

Example 66  In Figure 10 ($n = 2, m = 1$), $\Phi$ has a dynamic bifurcation.

Definition 67  The bifurcation diagram ([4], page 61) is a partition of the set of systems $\{\Xi_{\Phi(\cdot, \lambda)} | \lambda \in B^n\}$ in classes of equivalence given by the equivalence of the systems, together with representative state portraits for each class of equivalence.
Example 68 Figure 10 is a bifurcation diagram.

Definition 69 The bifurcation diagram ([2], page 5) is the graph that gives the position of the fixed points depending on a parameter, such that a bifurcation exists.

Remark 70 Such a(n informal) definition works for calling Figure 10 a bifurcation diagram, since there fixed points exist. However for Figure 11 this definition does not work, because a bifurcation exists there, but no fixed points.

Definition 71 Let be $\Phi, \Psi : B^n \times B^n \rightarrow B^n$. The families of systems $(\Xi_{\Phi(\cdot, \lambda)})_{\lambda \in B^m}$ and $(\Xi_{\Psi(\cdot, \lambda)})_{\lambda \in B^m}$ are called equivalent ([6], pages 7, 17) if there exists a bijection $h^n : B^m \rightarrow B^m$ such that $\forall \lambda \in B^m, \Xi_{\Phi(\cdot, \lambda)}$ and $\Xi_{\Psi(\cdot, h^n(\lambda))}$ are equivalent in the sense of Definition 55.

References