Asymptotically $H^2$-Optimal Tuning of Low Gain Robust Controllers for DPS

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Abstract

It is well known that a low-gain controller of the form $C_\varepsilon(s) = \sum_{k=-n}^{n} \varepsilon K_k/(s - i\omega_k)$ is able to track and reject constant and linear combinations of sinusoidal reference and disturbance signals, asymptotically for stable plants $P$ in the Callier-Desoer algebra, and in the $L^2$ sense for exponentially stable well-posed systems.

In this paper we investigate the optimal tuning of the matrix gains $K_k$ of the controller $C_\varepsilon(s)$ as the positive scalar gain $\varepsilon \to 0$. The cost function is taken to be the maximum energy of the error between the reference signal and the measured output signal over all suitably bounded reference and disturbance signal amplitudes.

It is shown that as $\varepsilon \to 0$ the cost function decomposes into a sum of simpler cost functions, each depending only on $K_k$ and $P(i\omega_k)$. Hence the optimization problem can be decomposed into $2n+1$ simpler subproblems, each depending on only one matrix gain $K_k$. Using the decomposition closed form solutions for the subproblems are found in certain special cases, and upper and lower bounds for the cost function are given in the general case.

No plant model is necessary since the only information we need to know of the plant are the values of the plant transfer matrix at the reference and disturbance signal frequencies $\omega_k$, and these can be measured from the open loop plant with input-output experiments.

1 Introduction

In a previous paper [1] the authors solved the following robust regulation problem: Given a stable plant $P$ in the Callier-Desoer algebra (CD-algebra) and reference and disturbance signals of the form

$$a_0 + \sum_{k=1}^{n} a_k \sin(\omega_k t + \phi_k), \quad a_k \in \mathbb{R},$$

find a low order finite-dimensional controller so that the outputs asymptotically track the reference signals, asymptotically reject the disturbance signals, and the closed-loop system is stable and robust with respect to a class of perturbations in the plant, see Figure 1.

In [1] it is shown that a low-gain controller $C_\varepsilon$ given by

$$C_\varepsilon(s) = \sum_{k=-n}^{n} \frac{\varepsilon K_k}{s - i\omega_k},$$

solves the robust regulation problem provided that the positive scalar gain $\varepsilon$ is small enough and the matrix gains $K_k$ satisfy the stability conditions

$$\sigma(P(i\omega_k)K_k) \subset \mathbb{C}^+, \quad k = -n, \ldots, n,$$
where $\omega_0 = 0$ and $\omega_k = -\omega_k$ for $k = 1, \ldots, n$. Recently Rebarber and Weiss have extended the result to exponentially stable well-posed systems [2].

Conditions (3) give an easily verifiable condition for the stabilizing matrix gains $K_k$. Unfortunately there are no analogous conditions for the scalar gain $\varepsilon$. In general we can only say that the closed-loop system will remain stable if $\varepsilon$ is below some bound $\varepsilon^*$. In this paper we concentrate on tuning the matrix gains $K_k$ and leave the tuning of $\varepsilon$ to a future paper.

Conditions (3) show that there is a great deal of freedom in the choice of the gains $K_k$. The important feature of conditions (3) is that the only information needed from the plant is the value $P(i\omega_k)$ of its transfer function at the reference and disturbance signal frequencies. Thus a model for the plant is not needed, because $P(i\omega_k)$ can be determined by input-output measurements from the open loop plant. Hence the aim of this paper is to find values for the matrix gains $K_k$ that are optimal in some sense and also depend only on the values $P(i\omega_k)$.

Because the structure of the controller (2) is fixed and the reference and disturbance signals have the known form (1) (instead of being general $L^2$-functions for example), the usual $H^\infty$ and $H^2$ optimization methods, as described for example in [3], [4] and [5] (and in [6] for finite-dimensional systems), are not applicable. Hence the approach taken in this paper is to minimize the error signal resulting from inputs of the form (1). A natural choice for the cost function would be the $H^2$-norm of the error, $J(\varepsilon, K) = \sup_{r} \int e(i\omega)\,d\omega$, where the supremum is taken over all suitably bounded reference and disturbance signal amplitudes $e_k$ and the argument $K = (K_k)_{k=-n}^{n}$ is the $(2n+1)$-tuple of stabilizing matrix gains of the controller (2). Since we are interested in small values of $\varepsilon$, investigating what happens as $\varepsilon \rightarrow 0$ seems like a sensible strategy. It is proved that for small positive $\varepsilon$ we have $J(\varepsilon, K) \approx J(K)/\varepsilon$ for some $J(K)$ that does not depend on $\varepsilon$. Since we are minimizing with respect to $K$, it makes sense to define the cost function as $J(K) = \lim_{\varepsilon \rightarrow 0} \varepsilon J(\varepsilon, K)$.

The main result of the paper shows that $J(K) = \sum_{k=-n}^{n} J_k(K_k)$ where each $J_k$ depends only on $K_k$ and $P(i\omega_k)$. Hence the optimization problem is decomposed into $2n+1$ simpler subproblems, each depending on only one matrix gain $K_k$ corresponding to the frequency $\omega_k$ of the reference signal. The subproblems can be solved independently of one another. Also, a model for the plant is not needed, the values $P(i\omega_k)$ are sufficient.

Closed form solution for the minima of the $J_k$ are found in the special case of single output (but possibly multiple inputs). The optimal matrix gains are given for each $k$ by (up to a scaling constant) $K_{ka} = P(i\omega_k)^* (P(i\omega_k)P(i\omega_k)^*)^{-1}$ or $K_{kb} = P(i\omega_k)^* (P(i\omega_k)P(i\omega_k)^*)^{-1/2}$. Upper and lower bounds for $J_k$ are given in the general case of multiple outputs. The gains $K_{ka}$ and $K_{kb}$ are shown to be good values also in the case of multiple outputs.

Interestingly, $K_{kb}$ was shown to be the asymptotically optimal matrix gain in a previous paper by the authors where $H^\infty$-norm of the error was used instead of the $H^2$-norm [7].

2 Notation

$\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, and $\mathbb{C}_\alpha^+ = \{ z \in \mathbb{C} \mid \text{Re} z > \alpha \}$. The complex conjugate of a complex number $z$ is denoted by $\overline{z}$. $\mathbb{R}(s)$ is the field of rational functions with real coefficients in the indeterminate $s$. $\mathbb{F}_{p \times n}$ is the class of $p \times m$ matrices with elements in the set $\mathbb{F}$, $\mathbb{F}$ is the same as $\mathbb{Z}$, $I_p$ is the $p \times p$ identity matrix, and $A^*$ is the conjugate transpose of the matrix $A$. The spectrum of a linear operator $A$ is denoted by $\sigma(A)$. The
norms of \( x \in \mathbb{C}^n \) and \( A \in \mathbb{C}^{m \times n} \) are defined by \( \|x\|_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2} \) and \( \|A\|_2 = \sigma_{\text{max}}(A) = \) the largest singular value of \( A \). The smallest singular value of \( A \) is denoted by \( \sigma_{\text{min}}(A) \). In a normed space we denote the open ball with center \( x \) and radius \( r \) by \( B(x; r) \). The space of bounded analytic functions in \( \mathbb{C}^+ \) is denoted by \( H_\infty^\alpha \) and \( H_\infty = H_0^\infty \). The norm of \( P \in (H_\infty^\alpha)^{p \times m} \) is given by \( \|P\|_\infty = \sup_{\omega \in \mathbb{R}} \|P(i\omega)\|_2 \). For \( \beta \in \mathbb{R} \) we define \( L^2_{\text{loc}}(0, \infty) = \{ f \in L^2_{\text{loc}}(0, \infty) \mid \int_0^\infty e^{-2\beta t}|f(t)|^2 \, dt < \infty \} \). A transfer function is exponentially stable if it is in \( H_\infty^\alpha \) for some \( \alpha < 0 \). The feedback system in Figure 1 is exponentially stable if the transfer functions from inputs, to the controller can be chosen such that for every \( \varepsilon \in (0, \varepsilon^*) \) the closed loop system in Figure 1 is exponentially stable provided that the matrix gains \( K_k \) satisfy the stability conditions

\[
\sigma(P(i\omega_k)K_k) \subset \mathbb{C}^+, \quad k = -n, \ldots, n.
\]  

To simplify notation we have defined \( \omega_0 = 0 \) and \( \omega_{-k} = -\omega_k \) for \( k = 1, \ldots, n \).

The following theorem was proved in [1] for \( P \) in CD-Algebra and later extended to the case where \( P \) is in \( H_\infty^\alpha \) in [2].

**Theorem 1.** Let the controller \( C_\varepsilon \) be defined by

\[
C_\varepsilon(s) = \sum_{k=-n}^{n} \frac{\varepsilon K_k}{s - i\omega_k},
\]

where \( K_k \in \mathbb{C}^{m \times p} \) for \( k = -n, \ldots, n \). Then there exists an \( \varepsilon^* > 0 \) such that for every \( \varepsilon \in (0, \varepsilon^*) \) the closed loop system in Figure 1 is exponentially stable provided that the matrix gains \( K_k \) satisfy the stability conditions

\[
\sigma(P(i\omega_k)K_k) \subset \mathbb{C}^+, \quad k = -n, \ldots, n.
\]

\[
\sigma(P(i\omega_k)K_k) \subset \mathbb{C}^+, \quad k = -n, \ldots, n.
\]

If the plant is in CD-Algebra the outputs asymptotically track the reference signals and asymptotically reject the disturbance signals, i.e., \( \lim_{t \to \infty} e(t) = 0 \). If the plant is in \( H_\infty^\alpha \) the best we can say is that \( e \in L^2_{\text{loc}}(0, \infty) \) for some \( \beta < 0 \). A consequence of condition (5) is that \( P(i\omega_k) \) must satisfy the rank condition

\[
\text{rank } P(i\omega_k) = p, \quad k = -n, \ldots, n.
\]

Let us define

\[
Q_k(s) = sI_p + P(i\omega_k)K_k, \quad k = -n, \ldots, n.
\]

It is clear that conditions (5) are equivalent to conditions

\[
\det Q_k(s) \text{ is stable for } k = -n, \ldots, n.
\]

If the plant satisfies the condition \( P(i\omega) = P(-i\omega) \) for \( \omega \in \mathbb{R} \) the controller can be chosen to be real, i.e., \( C_\varepsilon(s) \in \mathbb{R}(s)^{m \times p} \), by choosing \( K_0 \in \mathbb{R}^{m \times p} \) and \( K_k = \overline{K_k} \) for \( k = 1, \ldots, n \). In this case only \( K_0, K_1, \ldots, K_n \) need be assigned.
3.2 The Optimization Problem

In this subsection the optimization problem for the matrix gains \( K_k \) will be formulated. The transfer matrix from the reference signal \( r \) to the error signal \( e \) is \( H(s) = (I_p + P(s)C_\varepsilon(s))^{-1} \).

The error signal \( e \) can be written as \( e(s) = H(s)(r(s) - P(s)d(s)) = \sum_{k=-n}^{n} e_k(s), \) where

\[
e_k(s) = \frac{H(s)(r_k - P(s)d_k)}{s - i\omega_k}. \quad (9)
\]

To simplify notation we define the \((2n+1)\)-tuple of matrix gains \( K = (K_k)_{k=-n}^{n} \in (\mathbb{C}^{m \times p})^{2n+1} \), the \((2n+1)\)-tuple of reference signal amplitudes \( r_0 = (r_k)_{k=-n}^{n} \in (\mathbb{C}^p)^{2n+1} \), and the \((2n+1)\)-tuple of disturbance signal amplitudes \( d_0 = (d_k)_{k=-n}^{n} \in (\mathbb{C}^m)^{2n+1} \).

For \( k = -n, \ldots, n \) let \( b_{r,k}, b_{d,k} \) be nonnegative parameters, and define \( S = S_{-n} \times \cdots \times S_{n} \) where the constraint sets \( S_k \) for the matrix gains \( K_k \) are given by \( S_k = \{ K_k \in \mathbb{C}^{m \times p} : \sigma(P(i\omega_k)K_k) \subset \mathbb{C}^+ \text{ and } \|K_k\|_2 \leq b_{r,k} \} \). The constraint sets \( D_r \) and \( D_d \) for the reference and disturbance signal amplitudes \( r_k \) and \( d_k \), respectively, are defined by \( D_r = B(0; b_{r,-n}) \times \cdots \times B(0; b_{r,n}) \subset (\mathbb{C}^p)^{2n+1} \) and \( D_d = B(0; b_{d,-n}) \times \cdots \times B(0; b_{d,n}) \subset (\mathbb{C}^m)^{2n+1} \).

A natural choice for the cost function would be

\[
\tilde{J}(\varepsilon, K) = \sup_{r_0 \in D_r} \sup_{d_0 \in D_d} \int_{\mathbb{R}} \|e(i\omega)\|_2^2 \, d\omega = \sup_{r_0 \in D_r} \sup_{d_0 \in D_d} \|e\|_2^2, \quad (10)
\]

which measures the maximum energy of the error signal over all bounded reference and disturbance signal amplitudes. However, as seen later in the paper, for small positive \( \varepsilon \) we have \( \tilde{J}(\varepsilon, K) \approx J(K)/\varepsilon \) for some \( J(K) \) that does not depend on \( \varepsilon \). Hence we define the cost function to be

\[
J(K) = \lim_{\varepsilon \downarrow 0} \tilde{J}(\varepsilon, K). \quad (11)
\]

Now the optimization problem can be formulated as

**Problem 1.** Given the cost function (11), find

\[
J_0 = \inf_{K \in S} J(K). \quad (12)
\]

4 Solution of the Optimization Problem

Lemma 3 shows that the main contribution to the cost function comes from a neighborhood of each of the the frequencies \( \omega_k \) as \( \varepsilon \downarrow 0 \). To prove Lemma 3 we need Lemma 2, proved in [7]. For the proofs we need the following definitions: \( U_k(s) = Q_k(s)/(s + 1) \) and

\[
U_{\varepsilon,k}(s) = U_k\left(\frac{s - i\omega_k}{\varepsilon}\right) = \frac{(s - i\omega_k)I_p + \varepsilon P(i\omega_k)K_k}{s - i\omega_k + \varepsilon}, \quad (13)
\]

\[
Q_{\varepsilon,k}(s) = Q_k\left(\frac{s - i\omega_k}{\varepsilon}\right) = \frac{1}{\varepsilon}((s - i\omega_k)I_p + \varepsilon P(i\omega_k)K_k). \quad (14)
\]

In [1] it is shown that \( U_k \) and \( U_{\varepsilon,k} \) are unimodular and \( \|U_{\varepsilon,k}(i\omega)^{-1}\|_2 \leq \|U_k^{-1}\|_\infty \) for \( \varepsilon \rightarrow 0 \) and \( \omega \in \mathbb{R} \). In the sequel \( B_k \) is the interval \( B_k = (\omega_k - \sqrt{\varepsilon}, \omega_k + \sqrt{\varepsilon}) \).

**Lemma 2.** For \( s = i\omega \) and \( \omega \in B_k \) the transfer matrix \( H(s) = (I_p + P(s)C_\varepsilon(s))^{-1} \) can be written in the form

\[
H(s) = (I_p + F_{\varepsilon,k}(s))U_{\varepsilon,k}(s)^{-1} \frac{s - i\omega_k}{s - i\omega_k + \varepsilon}. \quad (15)
\]

For sufficiently small \( \varepsilon > 0 \) and some constant \( M_k > 0 \) we have the bound

\[
\|F_{\varepsilon,k}(i\omega)\|_2 \leq \frac{M_k\sqrt{\varepsilon}}{1 - M_k\sqrt{\varepsilon}} \quad \text{for all } \omega \in B_k. \quad (16)
\]

Now we prove Lemma 3.
Lemma 3. Let \( \varepsilon > 0 \) be such that the intervals \( B_k \) are nonoverlapping for \( k = -n, \ldots, n \) and let \( B_c \) be the complement of \( \bigcup_{k=-n}^{n} B_k \) in \( \mathbb{R} \). Then

(a) The error signal \( e \) satisfies

\[
\lim_{\varepsilon \to 0} \sup_{r_0 \in D_r} \sup_{d_0 \in D_d} \int_{B_c} \varepsilon \|e(i\omega)\|^2 \, d\omega = 0. \tag{17}
\]

(b) For any \( k = -n, \ldots, n \) and any \( j \neq k, l \neq k \) we have

\[
\lim_{\varepsilon \to 0} \sup_{r_0 \in D_r} \sup_{d_0 \in D_d} \int_{B_k} \varepsilon |(e_j(i\omega), e_l(i\omega))| \, d\omega = 0. \tag{18}
\]

(c) For any \( k = -n, \ldots, n \) and any \( j \neq k \) we have

\[
\lim_{\varepsilon \to 0} \sup_{r_0 \in D_r} \sup_{d_0 \in D_d} \int_{B_k} \varepsilon |(e_j(i\omega), e_k(i\omega))| \, d\omega = 0. \tag{19}
\]

(d) For any \( k = -n, \ldots, n \) we have

\[
\lim_{\varepsilon \to 0} \sup_{r_0 \in D} \sup_{d_0 \in D} \int_{B_k} \varepsilon \|e_k(i\omega)\|^2 \, d\omega = J_k(K_k), \tag{20}
\]

where

\[
J_k(K_k) = \sup_{\|r_k\| \leq b_{r,k}} \sup_{\|d_k\| \leq b_{d,k}} \int_{\mathbb{R}} \|Q_k(i\omega)^{-1}(r_k - P(i\omega)dk)\|^2 \, d\omega, \tag{21}
\]

and \( e_k \) and \( Q_k \) were defined in Eqs. (9) and (7), respectively.

Proof. Throughout the proof we let \( M = \max_{-n \leq k \leq n} (b_{r,k} + \|P\|_{\infty}b_{d,k}) \).

(a) Let \( k \) be arbitrary. An arbitrary \( \omega \in B_c \) satisfies \(|\omega - \omega_k| \geq \sqrt{\varepsilon} \) for \( k = -n, \ldots, n \). Hence for any \( \omega \in B_c \) we have \( \|P(i\omega)C_\varepsilon(i\omega)\|_2 \leq Mc_\varepsilon \sqrt{\varepsilon} \), where \( M_c = \|P\|_{\infty} \sum_{k=-n}^{n} b_{r,k,k} \). Clearly \( \|P(i\omega)C_\varepsilon(i\omega)\|_2 < 1 \) when \( \varepsilon \) is sufficiently small. Hence \( H(i\omega) \) can be expanded into a Neumann series giving the bound \( \|H(i\omega)\|_2 \leq 1/(1 - M_c \sqrt{\varepsilon}) \). Thus

\[
\int_{B_c} \varepsilon \|e_k(i\omega)\|^2 \, d\omega \leq \frac{\varepsilon M^2}{(1 - M_c \sqrt{\varepsilon})^2} \int_{B_c} \frac{d\omega}{(\omega - \omega_k)^2}.
\]

We have \( B_c = (-\infty, \omega_n - \sqrt{\varepsilon}) \cup B' \cup (\omega_n + \sqrt{\varepsilon}, \infty) \), where \( B' = \bigcup_{j=-n}^{n-1} (\omega_j + \sqrt{\varepsilon}, \omega_{j+1} - \sqrt{\varepsilon}) \). Since \( \omega_n \leq \omega_k \leq \omega_n \) we have

\[
\int_{\omega_n + \sqrt{\varepsilon}}^{\infty} \frac{d\omega}{(\omega - \omega_k)^2} = \frac{1}{\omega_n - \omega_k + \sqrt{\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}},
\]

and since \( \omega_j \leq \omega_k \) if \( j \leq k \) we have

\[
\int_{\omega_j + \sqrt{\varepsilon}}^{\omega_{j+1} - \sqrt{\varepsilon}} \frac{d\omega}{(\omega - \omega_k)^2} = -\frac{1}{\omega_j + \sqrt{\varepsilon} - \omega_k - \sqrt{\varepsilon}} + \frac{1}{\omega_j + \sqrt{\varepsilon} - \omega_k + \sqrt{\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}}.
\]

Hence

\[
\int_{B_c} \varepsilon \|e_k(i\omega)\|^2 \, d\omega \leq \frac{2(n+1)M^2 \sqrt{\varepsilon}}{(1 - M_c \sqrt{\varepsilon})^2} \to 0 \text{ as } \varepsilon \downarrow 0
\]

independently of \( r_k \) and \( d_k \). Now the result follows from the inequality

\[
\int_{B_c} \|e(i\omega)\|^2 \, d\omega \leq (2n + 1) \sum_{k=-n}^{n} \int_{B_c} \|e_k(i\omega)\|^2 \, d\omega.
\]
(b) Since \(|\omega - \omega_k|/|i\omega - i\omega_k + \varepsilon| \leq 1\) for \(\omega \in B_k\) we have

\[
\|H(i\omega)\|_2 \leq \frac{\|U_{k}^{-1}\|_\infty}{1 - M_k \sqrt{\varepsilon}}.
\]

Because \(\omega_j, \omega_l \notin B_k\) we have \(|\omega - \omega_j| \geq \sqrt{\varepsilon}, |\omega - \omega_l| \geq \sqrt{\varepsilon}\) and hence

\[
\int_{B_k} \varepsilon |\langle e_j(i\omega), e_l(i\omega)\rangle| \, d\omega \leq \int_{B_k} \varepsilon \|e_j(i\omega)\| \|e_l(i\omega)\| \, d\omega
\]

\[
\leq \varepsilon M^2 \|U^{-1}_{k}\|_\infty^2 \int_{B_k} \frac{d\omega}{|\omega - \omega_j| |\omega - \omega_l|}
\]

\[
\leq \frac{2 \sqrt{\varepsilon} M^2 \|U^{-1}_{k}\|_\infty^2}{(1 - M_k \sqrt{\varepsilon})^2} \to 0 \text{ as } \varepsilon \downarrow 0
\]

independently of \(r_0\) and \(d_0\).

(c) For \(\omega \in B_k\) we have

\[
\|H(i\omega)\|_2 \leq \frac{\|U_{k}^{-1}\|_\infty}{1 - M_k \sqrt{\varepsilon}} \frac{|\omega - \omega_k|}{|i(\omega - \omega_k) + \varepsilon|}.
\]

Since \(|\omega - \omega_j| \geq \sqrt{\varepsilon}\) we get

\[
\int_{B_k} \varepsilon |\langle e_j(i\omega), e_k(i\omega)\rangle| \, d\omega \leq \int_{B_k} \varepsilon \|e_j(i\omega)\| \|e_k(i\omega)\| \, d\omega
\]

\[
\leq \varepsilon M^2 \|U^{-1}_{k}\|_\infty^2 \int_{B_k} \frac{|\omega - \omega_k|}{|i(\omega - \omega_k) + \varepsilon|^2 |\omega - \omega_j|} \, d\omega
\]

\[
\leq \frac{\sqrt{\varepsilon} M^2 \|U^{-1}_{k}\|_\infty^2}{(1 - M_k \sqrt{\varepsilon})^2} \int_{B_k} \frac{|\omega - \omega_k|}{(\omega - \omega_k)^2 + \varepsilon^2} \, d\omega.
\]

Because

\[
\int_{B_k} \frac{|\omega - \omega_k|}{(\omega - \omega_k)^2 + \varepsilon^2} \, d\omega = \int_{\omega_k - \sqrt{\varepsilon}}^{\omega_k} \frac{-(\omega - \omega_k)}{(\omega - \omega_k)^2 + \varepsilon^2} \, d\omega + \int_{\omega_k}^{\omega_k + \sqrt{\varepsilon}} \frac{\omega - \omega_k}{(\omega - \omega_k)^2 + \varepsilon^2} \, d\omega
\]

\[
= \left[ \frac{\omega_k}{\omega_k - \sqrt{\varepsilon}} - \frac{1}{2} \ln((\omega - \omega_k)^2 + \varepsilon^2) \right]_{\omega_k}^{\omega_k + \sqrt{\varepsilon}} + \frac{1}{2} \ln((\omega - \omega_k)^2 + \varepsilon^2)
\]

\[
= \ln(\varepsilon + \varepsilon^2) - 2 \ln(\varepsilon).
\]

Hence

\[
\int_{B_k} \varepsilon |\langle e_j(i\omega), e_k(i\omega)\rangle| \, d\omega \leq \frac{M^2 \|U^{-1}_{k}\|_\infty^2 (\sqrt{\varepsilon} \ln(\varepsilon + \varepsilon^2) - 2 \sqrt{\varepsilon} \ln(\varepsilon))}{(1 - M_k \sqrt{\varepsilon})^2} \to 0 \text{ as } \varepsilon \downarrow 0
\]

independently of \(r_0\) and \(d_0\).

(d) We have

\[
e_k(s) = (I_p + F_{e,k}(s)) \frac{U_{e,k}(s)^{-1}}{s - i\omega_k + \varepsilon} (r_k - P(s) d_k)
\]

\[
= \frac{U_{e,k}(s)^{-1}}{s - i\omega_k + \varepsilon} (r_k - P(s) d_k) + \frac{F_{e,k}(s) U_{e,k}(s)^{-1}}{s - i\omega_k + \varepsilon} (r_k - P(s) d_k).
\]

Since \(r_k - P(i\omega) d_k = r_k - P(i\omega_k) d_k - (P(i\omega) - P(i\omega_k)) d_k\) we can write \(e_k(i\omega) = \tilde{Q}_{e,k}(i\omega) + L(i\omega)\), where

\[
\tilde{Q}_{e,k}(i\omega) = \frac{U_{e,k}(i\omega)^{-1}}{i\omega - i\omega_k + \varepsilon} (r_k - P(i\omega_k) d_k)
\]

\[
L(i\omega) = \frac{F_{e,k}(i\omega) U_{e,k}(i\omega)^{-1}}{i\omega - i\omega_k + \varepsilon} (r_k - P(i\omega) d_k) - \frac{U_{e,k}(i\omega)^{-1}}{i\omega - i\omega_k + \varepsilon} (P(i\omega) - P(i\omega_k)) d_k.
\]
Hence
\[ \| e_k(i\omega) \|_2^2 = \| \tilde{Q}_{\varepsilon,k}(i\omega) \|_2^2 + 2 \text{Re}(\tilde{Q}_{\varepsilon,k}(i\omega), L(i\omega)) + \| L(i\omega) \|_2^2. \]

We have
\[ \| L(i\omega) \|_2 \leq \frac{MMk\| U^{-1}_k \|_\infty \sqrt{\varepsilon}}{(1 - M_k\sqrt{\varepsilon})|i\omega - i\omega_k + \varepsilon|} + b_{d_k} \| U^{-1}_k \|_\infty \| P(i\omega) - P(i\omega_k) \|_2 \]

and
\[ \| \tilde{Q}_{\varepsilon,k}(i\omega) \|_2 \leq \frac{M \| U^{-1}_k \|_\infty}{|i\omega - i\omega_k + \varepsilon|}. \]

Since \( P \) is continuous at \( i\omega_k \)
\[ \lim_{\varepsilon \to 0} \sup_{\omega \in B_k} \| P(i\omega) - P(i\omega_k) \|_2 = 0, \]
and since
\[ \int_{B_k} \frac{\varepsilon d\omega}{(\omega - \omega_k)^2 + \varepsilon^2} = 2 \arctan(1/\sqrt{\varepsilon}) \]
we have
\[ \int_{B_k} \varepsilon |\text{Re}(\tilde{Q}_{\varepsilon,k}(i\omega), L(i\omega))| d\omega \]
\[ \leq \int_{B_k} \varepsilon \| \tilde{Q}_{\varepsilon,k}(i\omega) \|_2 \| L(i\omega) \|_2 d\omega \]
\[ \leq 2\| U^{-1}_k \|_\infty \left( \frac{M_kM^2\| U^{-1}_k \|_\infty \sqrt{\varepsilon}}{1 - M_k\sqrt{\varepsilon}} + b_{d_k} \sup_{\omega \in B_k} \| P(i\omega) - P(i\omega_k) \|_2 \right) \arctan(1/\sqrt{\varepsilon}) \]
\[ \to 0 \quad \text{as} \quad \varepsilon \to 0, \]
and
\[ \int_{B_k} \varepsilon \| L(i\omega) \|_2^2 d\omega \leq 4\| U^{-1}_k \|_\infty^2 \left( \frac{M_k^2M^2\varepsilon}{1 - M_k\sqrt{\varepsilon}} + b_{d_k}^2 \sup_{\omega \in B_k} \| P(i\omega) - P(i\omega_k) \|_2^2 \right) \arctan(1/\sqrt{\varepsilon}) \to 0, \]
as \( \varepsilon \to 0 \) independently of \( r_k \) and \( d_k \). Using Eqs. (13) and (14) we have \( (s - i\omega_k + \varepsilon)^{-1}U_{\varepsilon,k}(s)^{-1} = \tilde{Q}_{\varepsilon,k}(s)^{-1}/\varepsilon \) and hence
\[ \lim_{\varepsilon \to 0} \sup_{r_k,d_k} \int_{B_k} \varepsilon \| e_k(i\omega) \|_2^2 d\omega = \lim_{\varepsilon \to 0} \sup_{r_k,d_k} \int_{B_k} \varepsilon \| \tilde{Q}_{\varepsilon,k}(i\omega) \|_2^2 d\omega \]
\[ = \lim_{\varepsilon \to 0} \sup_{r_k,d_k} \int_{B_k} \frac{1}{\varepsilon} \| \tilde{Q}_{\varepsilon,k}(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega \]
\[ = \lim_{\varepsilon \to 0} \sup_{r_k,d_k} \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega. \]

Clearly
\[ \lim_{\varepsilon \to 0} \sup_{r_k,d_k} \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega \leq \sup_{r_k,d_k} \int_{-\infty}^{\infty} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega. \]

On the other hand
\[ \sup_{r_k,d_k} \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega \geq \sup_{r_k,d_k} \int_{-\infty}^{\infty} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega \]
\[ - M^2 \int_{|\omega| > 1/\sqrt{\varepsilon}} \| Q_k(i\omega)^{-1} \|_2^2 d\omega, \]
and thus
\[ \lim_{\varepsilon \to 0} \sup_{r_k,d_k} \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega \geq \sup_{r_k,d_k} \int_{-\infty}^{\infty} \| Q_k(i\omega)^{-1}(r_k - P(i\omega_k)d_k) \|_2^2 d\omega. \]
The next Theorem shows that as \( \varepsilon \downarrow 0 \) the cost function decomposes into a sum of the simpler cost functions \( J_k \), given in (21), which we call modal cost functions in the sequel.

**Theorem 4.** The cost function (11) can be written as

\[
J(K) = \sum_{k=-n}^{n} J_k(K_k).
\]

**Proof.** Let \( B_k \) and \( B_c \) be as in Lemma 3. According to Lemma 3 (a) we have

\[
J(K) = \lim_{\varepsilon \downarrow 0} \sup_{r_0, d_0} \sum_{k=-n}^{n} \int_{B_k} \varepsilon ||e(\iota \omega)||_2^2 d\omega.
\]

Writing \( ||e(\iota \omega)||_2^2 \) in the form

\[
||e(\iota \omega)||_2^2 = (e(\iota \omega), e_k(\iota \omega)) + \sum_{l \neq k} (e(\iota \omega), e_l(\iota \omega))
\]

\[
= ||e_k(\iota \omega)||_2^2 + 2 \sum_{l \neq k} \text{Re}(e_l(\iota \omega), e_k(\iota \omega)) + \sum_{l \neq k} \sum_{j \neq k} \langle e_j(\iota \omega), e_l(\iota \omega) \rangle.
\]

and using Lemma 3 (b)–(d) we get

\[
J(K) = \lim_{\varepsilon \downarrow 0} \sup_{r_0, d_0} \sum_{k=-n}^{n} \int_{B_k} \varepsilon ||e_k(\iota \omega)||_2^2 d\omega = \lim_{\varepsilon \downarrow 0} \sup_{r_k, d_k} \sum_{k=-n}^{n} \int_{B_k} \varepsilon ||e_k(\iota \omega)||_2^2 d\omega = \sum_{k=-n}^{n} J_k(K_k).
\]

In the sequel we need the following definitions. Let \( P_k = P(\iota \omega_k) \) have the singular values

\[
\sigma_{1k} \geq \cdots \geq \sigma_{pk}
\]

and the singular value decomposition \( P_k = W_k [\Lambda_k \ 0] V_k^* \), where \( W_k \in \mathbb{C}^{p \times p} \) and \( V_k \in \mathbb{C}^{m \times m} \) are unitary, and \( \Lambda_k = \text{diag}(\sigma_{1k}, \ldots, \sigma_{pk}) \in \mathbb{R}^{p \times p} \). It follows from the rank condition (6) that \( \sigma_{pk} > 0 \). Partition the matrix \( V_k^* K_k W_k \) as

\[
V_k^* K_k W_k = \begin{bmatrix} K_{k,1} \\ K_{k,2} \end{bmatrix},
\]

where \( K_{k,1} \in \mathbb{C}^{p \times p} \) and \( K_{k,2} \in \mathbb{C}^{(m-p) \times p} \). Then

\[
Q_k(\iota \omega)^{-1} = (\iota \omega I_p + W_k \Lambda_k K_{k,1} W_k^*)^{-1} = W_k (\iota \omega \Lambda_k^{-1} + K_{k,1})^{-1} \Lambda_k^{-1} W_k^*,
\]

\[
r_k - P_k d_k = r_k - W_k [\Lambda_k \ 0] V_k^* d_k = W_k \Lambda_k (\Lambda_k^{-1} W_k r_k - [I_p \ 0] V_k^* d_k).
\]

Let \( \hat{r}_k = W_k^* r_k \) and \( \hat{d}_k = [I_p \ 0] V_k^* d_k \). Clearly the suprema over \( r_k \) and \( d_k \) are the same as the suprema over \( \hat{r}_k \) and \( \hat{d}_k \). Using this and the fact that the euclidean norm is invariant under unitary transformations, we get an alternative form for the modal cost function \( J_k \).

\[
J_k(K_k) = \sup_{r_k, d_k} \int_{-\infty}^{\infty} \| Q_k(\iota \omega)^{-1} (r_k - P_k d_k) \|^2 d\omega = \sup_{\hat{r}_k, \hat{d}_k} \int_{-\infty}^{\infty} \| (\iota \omega \Lambda_k^{-1} + K_{k,1})^{-1} (\Lambda_k^{-1} \hat{r}_k - \hat{d}_k) \|^2 d\omega.
\]

It is clear that \( \|K_\|_2 \leq b_{K,k} \) implies \( \|K_{1,k}\|_2 \leq b_{K,k} \). Let us define

\[
K_{ka} = \sigma_{pk} b_{K,k} P_k^* (P_k P_k^*)^{-1}
\]

\[
K_{kb} = b_{K,k} P_k^* (P_k P_k^*)^{-1/2}.
\]

The next Theorem proves that these minimize the modal cost function \( J_k \) in the special case of single output \( p = 1 \). It turns out that these are also good values in the general case \( p > 1 \).
Theorem 5. If \( p = 1 \), i.e., the plant has only one output, then
\[
\min_{K_k \in S_k} J_k(K_k) = \frac{\pi (b_{rk} + \| P_k \| b_{dk})^2}{\| P_k \| b_{kk}},
\]
which is attained with the choice \( K_k = K_{ka} = K_{kb} = b_{kk} P_k^* / \| P_k \| \).

Proof. Clearly \( P_k \) has only one singular value \( \sigma_{1k} = \| P_k \| \). Since \( P_k K_k \in \mathbb{C} \) the stability condition (5) becomes \( \text{Re}[P_k K_k] > 0 \) and we have
\[
J_k(K_k) = \sup_{r_k,d_k} |r_k - P_k d_k|^2 \int_{-\infty}^{\infty} \frac{d\omega}{|i\omega + P_k K_k|^2} = (b_{rk} + \| P_k \| b_{dk})^2 \int_{-\infty}^{\infty} \frac{d\omega}{(\omega + \text{Im}[P_k K_k])^2 + (\text{Re}[P_k K_k])^2} = \frac{\pi (b_{rk} + \| P_k \| b_{dk})^2}{\text{Re}[P_k K_k]}.
\]
Since
\[
\frac{1}{\text{Re}[P_k K_k]} \geq \frac{1}{\| P_k \| / \| K_k \|} \geq \frac{1}{b_{kk} \| P_k \|}
\]
and setting \( K_k = K_{ka} \) we have \( \text{Re}[P_k K_{ka}] = b_{kk} \| P_k \| \), which proves the claim.

Unfortunately it is not known if Theorem 5 holds in the case of multiple outputs. Therefore we next find lower and upper bounds for the modal costs \( J_k \). First note that
\[
\| r_k - P_k d_k \|_2 \leq b_{rk} + \sigma_{1k} b_{dk},
\]
and choosing \( r_k = b_{rk} W_k e_1, d_k = -b_{dk} V_k \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \), where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^p \), we get
\[
\| r_k - P_k d_k \|_2 = \| b_{rk} W_k e_1 + b_{dk} W_k \Lambda_k e_1 \|_2 = (b_{rk} + \sigma_{1k} b_{dk}) \| W_k e_1 \|_2 = b_{rk} + \sigma_{1k} b_{dk}.
\]
Hence
\[
\sup_{r_k,d_k} \| r_k - P_k d_k \|_2 = b_{rk} + \sigma_{1k} b_{dk}.
\]
Similarly
\[
\sup_{\tilde{r}_k,\tilde{d}_k} \| \Lambda_k^{-1} \tilde{r}_k - \tilde{d}_k \|_2 = \frac{b_{rk}}{\sigma_{pk}} + b_{dk}.
\]
The next Lemma gives a lower bound for the modal cost function \( J_k \) in the case of multiple outputs, \( p > 1 \).

Lemma 6. Let \( c_{1k} = b_{rk} + \sigma_{1k} b_{dk} \) and \( c_{pk} = b_{rk} + \sigma_{pk} b_{dk} \). For every \( K_k \in S_k \) we have the lower bound
\[
J_k(K_k) \geq \frac{2}{b_{kk}} \max \left\{ \frac{c_{1k}^2}{\sigma_{pk}}, \frac{c_{pk}^2}{\sigma_{1k}} \right\}
\]
Proof. Using the inequality \( \| A^{-1} x \| \geq \| x \| / \| A \| \) we get
\[
J_k(K_k) \geq \sup_{r_k,d_k} \| r_k - P_k d_k \|_2 \int_{\mathbb{R}} \frac{d\omega}{\| i\omega I + P_k K_k \|^2} \geq \frac{c_{1k}^2}{\| \Lambda_k^{-1} + K_k \|} \int_{\mathbb{R}} \frac{d\omega}{(|\omega| + \sigma_{1k} b_{kk})^2} = \frac{2}{b_{kk}} \frac{c_{1k}^2}{\sigma_{1k}}.
\]
Similarly, using (22)
\[
J_k(K_k) \geq \sup_{\tilde{r}_k,\tilde{d}_k} \| \Lambda_k^{-1} \tilde{r}_k - \tilde{d}_k \|_2 \int_{\mathbb{R}} \frac{d\omega}{\| i\omega \Lambda_k^{-1} + K_k \|^2} \geq \frac{c_{pk}^2}{\| \Lambda_k^{-1} + K_k \|} \int_{\mathbb{R}} \frac{d\omega}{(|\omega|/\sigma_{pk} + b_{kk})^2} = \frac{2}{b_{kk}} \frac{c_{pk}^2}{\sigma_{pk}}.
\]
We have the upper bounds
\[ J_k(K_k) \leq \sup_{r_k, d_k} \| r_k - P_k d_k \|^2 \int_{\mathbb{R}} \| (i\omega I + P_k K_k)^{-1} \|^2 d\omega = c_{k,1}^2 J_{k,1}(K_k), \] (23)
where
\[ J_{k,1}(K_k) = \int_{\mathbb{R}} \| (i\omega I + P_k K_k)^{-1} \|^2 d\omega, \]
and from (22)
\[ J_k(K_k) \leq \frac{c_{pk}^2}{\sigma_{pk}} J_{k,2}(K_k), \] (24)
where
\[ J_{k,2}(K_k) = \int_{\mathbb{R}} \| (i\omega \Lambda_k^{-1} + K_{k,1})^{-1} \|^2 d\omega. \]

Next we minimize \( J_{k,1} \) and \( J_{k,2} \).

**Lemma 7.** We have
\[ \min_{K_k \in S_k} J_{k,1}(K_k) = \frac{\pi}{\sigma_{pk} b_{K_k}}, \]
where the minimum is attained with \( K_k = K_{ka} \) and \( K_k = K_{kb} \), and
\[ \min_{K_k \in S_k} J_{k,2}(K_k) = \frac{\pi \sigma_{1k}}{b_{K_k}}, \]
where the minimum is attained with \( K_k = K_{kb} \).

**Proof.** Let us first minimize \( J_{k,1} \). Using \( \| A^{-1} \| = 1/\sigma_{\min}(A) = 1/\sigma_{\min}(A^*) \) we get
\[ \| (i\omega I + P_k K_k)^{-1} \|^2 = \frac{1}{\sigma_{\min}(-i\omega I + K_k^* P_k^*)^2} \]
and
\[ \sigma_{\min}(-i\omega I + K_k^* P_k^*)^2 = \min_{\| x \|=1} \|(-i\omega I + K_k^* P_k^*) x \|^2 = \min_{\| x \|=1} (\omega^2 \| x \|^2 - 2 \omega \text{Re}(ix, K_k^* P_k^* x) + \| K_k^* P_k^* x \|^2). \]
Let \( x_0 \) be such that \( \| P_k^* x_0 \| = \sigma_{pk} \) and \( \| x_0 \| = 1 \). Then, letting \( \alpha = \text{Re}(ix_0, K_k^* P_k^* x_0) \),
\[ \sigma_{\min}(-i\omega I + K_k^* P_k^*)^2 \leq \omega^2 - 2\omega \alpha + \| K_k^* P_k^* x_0 \|^2 \leq (\omega - \alpha)^2 + \sigma_{pk}^2 b_{K_k}^2 - \alpha^2 \leq (\omega - \alpha)^2 + \sigma_{pk}^2 b_{K_k}^2. \]
Hence
\[ J_{k,1}(K_k) \geq \int_{\mathbb{R}} \frac{d\omega}{(\omega - \alpha)^2 + \sigma_{pk}^2 b_{K_k}^2} = \frac{\pi}{\sigma_{pk} b_{K_k}}. \]
Since \( P_k K_{ka} = \sigma_{pk} b_{K_k} I \) we have
\[ J_{k,1}(K_{ka}) = \int_{\mathbb{R}} \| (i\omega I + \sigma_{pk} b_{K_k} I)^{-1} \|^2 d\omega = \int_{\mathbb{R}} \frac{d\omega}{\omega^2 + \sigma_{pk}^2 b_{K_k}^2} = \frac{\pi}{\sigma_{pk} b_{K_k}}. \]
Because \( P_k K_{kb} = b_{K_k} W_k \Lambda_k W_k^* \) we get
\[ J_{k,1}(K_{kb}) = \int_{\mathbb{R}} \| (i\omega I + b_{K_k} W_k \Lambda_k W_k^*)^{-1} \|^2 d\omega = \int_{\mathbb{R}} \| (i\omega I + b_{K_k} \Lambda_k)^{-1} \|^2 d\omega \]
\[ = \int_{\mathbb{R}} \frac{1}{(\omega^2 + b_{K_k}^2)^2} d\omega = \int_{\mathbb{R}} \max_{1 \leq j \leq p} \frac{1}{(\omega^2 + \sigma_{jk}^2 b_{K_k}^2)^2} d\omega \]
\[ = \int_{\mathbb{R}} \frac{d\omega}{\omega^2 + \sigma_{pk}^2 b_{K_k}^2} = \frac{\pi}{\sigma_{pk} b_{K_k}}. \]
Let us next minimize $J_{k,2}$. As above we have
\[
\| (i\omega \Lambda_k^{-1} + K_{k,1})^{-1} \|_2^2 = \frac{1}{\sigma_{\text{min}}(i\omega \Lambda_k^{-1} + K_{k,1})^2}.
\]

Let $\beta = \text{Re}\langle i e_1, K_{k,1} e_1 \rangle$. Then
\[
\sigma_{\text{min}}(i\omega \Lambda_k^{-1} + K_{k,1})^2 = \min_{\| x \|_1 = 1} \{ \omega^2 \| \Lambda_k^{-1} x \|_2^2 + 2\omega \text{Re}\langle i\Lambda_k^{-1} x, K_{k,1} x \rangle + \| K_{k,1} x \|_2^2 \}
\leq \omega^2 \| \Lambda_k^{-1} e_1 \|_2^2 + 2\omega \text{Re}\langle i\Lambda_k^{-1} e_1, K_{k,1} e_1 \rangle + b_{Kk}^2
\]
\[
= \frac{\omega^2}{\sigma_{1k}^2} + \frac{2\omega\beta}{\sigma_{1k}} + b_{Kk}^2 \leq (\omega/\sigma_{1k} + \beta)^2 + b_{Kk}^2.
\]

Thus
\[
J_{k,2}(K_k) \geq \int_{\mathbb{R}} \frac{d\omega}{(\omega/\sigma_{1k} + \beta)^2 + b_{Kk}^2} = \frac{\pi \sigma_{1k}}{b_{Kk}}.
\]

If $K_k = K_{kb}$ we have $K_{k,1} = b_{Kk} I$ and
\[
J_{k,2}(K_{kb}) = \int_{\mathbb{R}} \| i\omega \Lambda_k^{-1} + b_{Kk} I \|_2^2 d\omega = \int_{\mathbb{R}} \max_{1 \leq j \leq p} \frac{1}{\| i\omega/\sigma_{jk} + b_{Kk} \|_2^2} d\omega
\]
\[
= \int_{\mathbb{R}} \frac{\omega^2}{\sigma_{jk}^2} \frac{d\omega}{\omega^2/\sigma_{1k}^2 + b_{Kk}^2} = \frac{\pi \sigma_{1k}}{b_{Kk}}.
\]

Hence the lower bound is attained. \(\square\)

The next Theorem gives upper and lower bounds for the optimal value of $J_k(K_k)$ in the general case $p > 1$.

**Theorem 8.** For $k = -n, \ldots, n$ let $c_{1k}$ and $c_{pk}$ be as in Lemma 6. Then we have the bounds
\[
\frac{2}{b_{Kk}} \max \left\{ \frac{c_{pk}^2}{\sigma_{pk}}, \frac{c_{1k}^2}{\sigma_{1k}} \right\} \leq \min_{K_k \in S_k} J_k(K_k) \leq \frac{\pi}{b_{Kk}} \min \left\{ \frac{c_{1k}^2}{\sigma_{1k}}, \frac{\sigma_{1k} c_{pk}^2}{\sigma_{pk}} \right\}
\]
\[(25)\]

The first upper bound is attained with $K_k = K_{ka}$ and the second with $K_k = K_{kb}$.

**Proof.** It follows from inequalities (23) and (24) and from Lemma 7 that
\[
\min_{K_k \in S_k} J_k(K_k) \leq \frac{c_{1k}^2}{\sigma_{1k}} \min_{K_k \in S_k} J_{k,1}(K_k) = \frac{\pi c_{1k}^2}{\sigma_{1k} b_{Kk}}
\]
\[
\min_{K_k \in S_k} J_k(K_k) \leq \frac{c_{pk}^2}{\sigma_{pk}} \min_{K_k \in S_k} J_{k,2}(K_k) = \frac{\pi \sigma_{1k} c_{pk}^2}{b_{Kk} \sigma_{pk}^2}.
\]

Combining these with Lemma 6 gives the result. \(\square\)

The next Corollary gives tight bounds in the case where either the reference signal or the disturbance signal is absent.

**Corollary 9.** If $r_k = 0$, i.e., the reference signal is absent, we have the bounds
\[
\frac{2b_{dk}^2 \sigma_{1k}}{b_{Kk}} \leq \min_{K_k \in S_k} J_k(K_k) \leq \frac{\pi b_{dk}^2 \sigma_{1k}}{b_{Kk}}.
\]

If $d_k = 0$, i.e., the disturbance signal is absent, we have the bounds
\[
\frac{2b_{rk}^2}{\sigma_{pk} b_{Kk}} \leq \min_{K_k \in S_k} J_k(K_k) \leq \frac{\pi b_{rk}^2}{\sigma_{pk} b_{Kk}}.
\]

The upper bounds are attained with $K_k = K_{kb}$.

**Proof.** The results follow from Theorem 8 by setting $b_{rk} = 0$ and $b_{dk} = 0$, respectively, and from the inequality $\sigma_{pk} \leq \sigma_{1k}$. \(\square\)
References


