Probabilistic Derivation of Some Generating Functions for the Laguerre Polynomials

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Abstract—A well-known generating function of the classical Laguerre polynomials was recently rederived probabilistically by Lee. In this paper, some other (presumably new) generating functions for the Laguerre polynomials are derived by means of probabilistic considerations. A direct (analytical) proof of each of these generating functions is also presented for the sake of completeness. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

The classical Laguerre polynomials $L_{n}^{(\alpha)}(x)$, of order $\alpha$ and degree $n$ in $x$, defined by

$$L_{n}^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^{k}}{k!},$$

are orthogonal over the interval $(0, \infty)$ with respect to the weight function $x^{\alpha} e^{-x}$; in fact, we have (cf., e.g., [1])

$$\int_{0}^{\infty} x^\alpha e^{-x} L_{m}^{(\alpha)}(x) L_{n}^{(\alpha)}(x) dx = \Gamma(\alpha+1) \binom{n+\alpha}{n} \delta_{m,n},$$

$$\Re(\alpha) > -1; \quad m, n \in \mathbb{N}_{0} := \mathbb{N} \cup \{0\}; \quad \mathbb{N} := \{1, 2, 3, \ldots\},$$

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where $\delta_{m,n}$ denotes the Kronecker delta. Just as the other members of the family of classical orthogonal polynomials (e.g., the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the Hermite polynomials $H_n(x)$, the Gegenbauer (or ultraspherical) polynomials $C_n^{(\alpha)}(x)$, and the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds), the Laguerre polynomials can be expressed as a hypergeometric function

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} _1F_1(-n; \alpha+1; x),$$

where $_1F_1$ is the (Kummer’s) confluent hypergeometric function which corresponds to the special case $u = v = 1$ of the generalized hypergeometric function $_uF_v$ (with $u$ numerator and $v$ denominator parameters). Furthermore, since [2, p. 42, equation 1.4(3)]

$$\binom{n+\alpha}{n} = \frac{(\alpha_1)\cdots(\alpha_n)n}{(\beta_1)\cdots(\beta_n)n} (-x)^n \times \binom{n+\alpha}{n}$$

(4)

which follows upon reversing the order of terms in the finite sum for either side of (4), the Laguerre polynomials in (3) can also be expressed in the form

$$L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!} 2F_0 (-n, -\alpha; -; - \frac{1}{x}).$$

Here, and throughout this paper, $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1, & (n = 0; \lambda \neq 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n \in \mathbb{N}). \end{cases}$$

(6)

Recently, Lee [3] gave a probabilistic derivation of the following generating function for the Laguerre polynomials:

$$\sum_{n=0}^{\infty} L_k^{(\nu+n-1)}(x-y) \frac{y^n}{n!} = e^y L_k^{(\nu-1)}(x-y),$$

(7)

$\nu > 0; \ x, y \text{ arbitrary}.$

In his derivation of (7), the index parameter $\nu + t$ of the Negative Binomial (NB) distribution is considered with $t$ assumed to vary as a Poisson random variable (r.v.) $T$ with parameter $\lambda_1 + \lambda_2$. This turns out to be the NB mixture formulation of the Noncentral Negative Binomial (NNB) distribution (cf. [4]) with the Poisson as mixing distribution.

Let $X \mid t$ (X conditional on t) be a NB r.v. with parameters $p$ and $\nu + t$, where $t$ is a Poisson r.v. $T$ with parameter $\lambda$. Then the unconditional r.v. $X$ has the NNB probability mass function (p.m.f.)

$$\mathcal{P}(k) := \text{Prob}(X = k) = e^{-\lambda p} p^k q^\nu L_k^{(\nu-1)}(-\lambda q),$$

(8)

$k \in \mathbb{N}_0; \ \nu, \lambda > 0; \ 0 < p < 1; \ q = 1 - p.$

Furthermore, the probability generating function (p.g.f.) is given by

$$G(z) = \sum_{k=0}^{\infty} \mathcal{P}(k) z^k = \left( \frac{q}{1-pz} \right)^\nu \exp \left( \lambda \left\{ \frac{q}{1-pz} - 1 \right\} \right),$$

(9)

$|z| < p^{-1}.$

The generating function (7) is a well-known (rather classical) result (cf., e.g., [5, p. 348, equation (27); 6, p. 142, equation (18); 7, p. 319, entry 48.19.2; 2, p. 172, Problem 22(ii)]). In fact, since

$$\frac{\partial^n}{\partial y^n} \left\{ e^y L_k^{(\nu-1)}(x-y) \right\} = e^y L_k^{(\nu+n-1)}(x-y), \ \ n, k \in \mathbb{N}_0,$$
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the generating function (7) is an immediate consequence of the Taylor expansion of

\[ e^y L_k^{(\nu-1)}(x - y) \]

in powers of \( y \). The work of Lee [3] stems essentially from a paper of Chatterjea [8] in which the generating function (7) was derived by applying the familiar group-theoretic method of Louis Weisner (1899–1988), which is described and illustrated fairly adequately in the works of Miller [9], McBride [10], and Srivastava and Manocha [2, Chapter 6]. On the other hand, Hubbell and Srivastava [11] (and, subsequently, Rassias and Srivastava [12]) considered a number of applications of the generating function (7) in the theory of bilinear, bilateral, and mixed multilateral generating functions for the Laguerre polynomials.

The application of probabilistic methods in the theory of special functions is an interesting topic which has been discussed, among others, by Lyusternik [13] and Dickey [14]. Probabilistic derivations of various formulas and identities for special functions are sometimes elementary in comparison with the use of some other mathematical techniques. More importantly, such a derivation may be easily generalized to obtain possibly new identities.

In this paper, we shall first consider an alternative probabilistic derivation of the generating function (7) based upon the elementary technique of p.g.f. This technique is then used to derive a bilinear generating function for the Laguerre polynomials. A special case of this bilinear generating function is a well-known generating function, namely (cf. [6, p. 138, equation (11); 2, p. 132, equation 2.5(5)])

\[ \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\mu + 1)_k} L_k^{(\nu)}(z)(-s)^k = (1 + s)^{-\gamma} \left( \frac{\gamma}{\mu + 1}; \frac{s z}{1 + s} \right). \quad (11) \]

In our present investigation, we shall denote by \( X \sim \text{NNB}(\nu, \lambda, p) \) a NNB r.v. with parameters \( (\nu, \lambda, p) \) as shown.

2. DERIVATION OF (7) BY PROBABILITY GENERATING FUNCTION TECHNIQUES

Let \( X | t \) be a NNB r.v. with parameters \( (\nu + t, \lambda_1, p) \), where \( t \) is a Poisson r.v. \( T \) with parameter \( \lambda_2 \). In this case, the p.g.f. of \( X | t \) is given by

\[ G_{X|t}(z) = \left( \frac{q}{1 - pz} \right)^{\nu + t} \exp \left( \lambda_1 \left\{ \frac{q}{1 - pz} - 1 \right\} \right). \quad (12) \]

The unconditional p.g.f. of \( X \) is

\[ G_X(z) = \sum_{t=0}^{\infty} G_{X|t}(z) e^{-\lambda_2} \frac{\lambda_2^t}{t!} = \left( \frac{q}{1 - pz} \right)^\nu \exp \left( \lambda_1 \left\{ \frac{q}{1 - pz} - 1 \right\} \right) \exp \left( \lambda_2 \left\{ \frac{q}{1 - pz} - 1 \right\} \right) \quad (13) \]

\[ = \left( \frac{q}{1 - pz} \right)^\nu \exp \left( (\lambda_1 + \lambda_2) \left\{ \frac{q}{1 - pz} - 1 \right\} \right), \]

since

\[ \sum_{t=0}^{\infty} \left( \frac{q}{1 - pz} \right)^t = e^{-\lambda_2} \frac{\lambda_2^t}{t!} = \exp \left( \lambda_2 \left\{ \frac{q}{1 - pz} - 1 \right\} \right). \quad (14) \]

Observe that (13) is the NNB p.g.f. with parameters \( (\nu, \lambda_1 + \lambda_2, p) \). If the p.g.f.s are replaced by their respective p.m.f.s in (13), we obtain

\[ \sum_{n=0}^{\infty} e^{-\lambda_1 p} q^{\nu + n} L_k^{(\nu + n - 1)} \frac{\lambda_2^n}{n!} = e^{-(\lambda_1 + \lambda_2) p} q^\nu L_k^{(\nu - 1)} \left( - (\lambda_1 + \lambda_2) q \right), \quad (15) \]
which readily simplifies to (cf. [3, p. 154, equation (9)])

\[
\sum_{n=0}^{\infty} L_k^{(\nu+n-1)}(-\lambda_1,q) \frac{(\lambda_2 q)^n}{n!} = e^{\lambda_2 q} L_k^{(\nu-1)}(- (\lambda_1 + \lambda_2) q). \tag{16}
\]

A suitable change of variables in (16) leads to the generating function (7).

3. A GENERALIZATION OF THE GENERATING FUNCTION (7)

The probabilistic derivation of the preceding section may be generalized as follows.

Let \( X \mid t \) be a NNB r.v. with parameters \((\nu + \alpha + t, \lambda_1, p)\), where \( t \) is another NNB r.v. \( T \) with parameters \((\lambda, \lambda_2, \theta)\). As before, the unconditional r.v. \( X \) has the p.g.f. given by

\[
G_X(z) = \sum_{t=0}^{\infty} G_{X|t}(z) e^{-\lambda_1 t} \theta^t (1 - \theta)^\nu L_t^{(\nu-1)}(-\lambda_2(1 - \theta))
\]

\[
= \left( \frac{q}{1 - pz} \right)^{\nu + \alpha} \exp \left( \lambda_1 \left\{ \frac{q}{1 - pz} - 1 \right\} \right) \left( \frac{1 - \theta}{1 - \theta (q/(1 - pz))} \right)^\nu \\
\times \exp \left( \lambda_2 \left\{ \frac{1 - \theta}{1 - \theta (q/(1 - pz))} - 1 \right\} \right) \tag{17}
\]

\[
= \left( \frac{q}{1 - pz} \right)^{\alpha} \exp \left( \lambda_1 \left\{ \frac{q}{1 - pz} - 1 \right\} \right) \left( \frac{q(1 - \theta)/(1 - \theta q)}{1 - \theta (q/(1 - \theta q))} \right)^\nu \\
\times \exp \left( \lambda_2 \theta \left( \frac{q(1 - \theta)/(1 - \theta q)}{1 - \theta (q/(1 - \theta q))} - 1 \right) \right).
\]

Note that the right-hand side of (17) shows that \( X \) is the sum of two NNB r.v.s, that is, \( X = X_1 + X_2 \), where

\( X_1 \sim \text{NNB}(\alpha, \lambda_1, p) \)

and

\( X_2 \sim \text{NNB}\left(\nu, \lambda_2 \theta, \frac{p}{1 - \theta q}\right). \)

Extracting the \( n \)th probability \( P(n) \) from the p.g.f.s in (17), the following (presumably new) bilinear generating function for the Laguerre polynomials is obtained:

\[
\sum_{k=0}^{\infty} L_n^{(\nu + \alpha + k - 1)}(-\lambda_1 q) L_k^{(\nu-1)}(-\lambda_2(1 - \theta)) (\theta q)^k
\]

\[
= (1 - \theta q)^{-\nu - n} \exp \left( \frac{\lambda_2 \theta(1 - \theta) q}{1 - \theta q} \right) \times \sum_{j=0}^{n} L_j^{(\alpha - 1)}(-\lambda_1 q) L_{n-j}^{(\nu - 1)} \left( - \frac{\lambda_2 \theta(1 - \theta) q}{1 - \theta q} \right) (1 - \theta q)^j \tag{18}
\]

or equivalently,

\[
\sum_{k=0}^{\infty} L_n^{(\nu + \alpha + k - 1)}(-\lambda_1 q) L_k^{(\nu-1)}(-\lambda_2(1 - \theta)) (\theta q)^k = (1 - \theta q)^{-\nu - n} \exp \left( \frac{\lambda_2 \theta(1 - \theta) q}{1 - \theta q} \right) \\
\times \sum_{j=0}^{n} L_j^{(\alpha - 1)} \left( \frac{\lambda_1 (1 - \theta q)}{\theta} \right) L_{n-j}^{(\nu + \alpha + j - 1)} \left( - \frac{\lambda_2 \theta(1 - \theta) q}{1 - \theta q} \right) (-\theta q)^j. \tag{19}
\]

Upon making some obvious notational changes, the generating functions (18) and (19) can be rewritten in the following equivalent forms:

\[
\sum_{k=0}^{\infty} L_n^{(\nu + \alpha + k - 1)}(x) L_k^{(\nu-1)}(y) t^k
\]

\[
= (1 - t)^{-\nu - n} \exp \left( -\frac{yt}{1 - t} \right) \times \sum_{j=0}^{n} L_j^{(\alpha - 1)}(x) L_{n-j}^{(\nu - 1)} \left( \frac{yt}{1 - t} \right) (1 - t)^j, \quad |t| < 1, \tag{20}
\]
and
\[
\sum_{k=0}^{\infty} L_n^{(\nu+\alpha+k-1)}(x) L_k^{(\nu-1)}(y) t^k = (1 - t)^{-\nu-n} \exp \left(-\frac{yt}{1-t}\right)
\]
\times \sum_{j=0}^{n} L_j^{(\alpha-1)} \left(-\frac{x(1-t)}{t}\right) I_{n-j}^{(\nu+\alpha+j-1)} \left(\frac{yt}{1-t}\right) (-t)^j, \quad |t| < 1,
\tag{21}
\]
respectively.

Each of the generating functions (18) to (21) does not seem to have been recorded earlier. An interesting special case of (18) or (19), when \(\alpha = 0\)

\[
\sum_{k=0}^{\infty} \frac{\nu+n}{(\nu)_k} L_k^{(\nu-1)} (-\lambda_2(1-\theta)) (\theta q)^k
\]
\[
= (1 - \theta q)^{-\nu-n} \exp \left(\frac{\lambda_2\theta(1-\theta)q}{1-\theta q}\right) \frac{n!}{(\nu)_n} \times L_n^{(\nu-1)} \left(-\frac{\lambda_2\theta(1-\theta)q}{1-\theta q}\right),
\tag{22}
\]
since
\[
(\nu + k) = \frac{(\nu+k+n)}{(\nu)_k} \quad \text{and} \quad L_n^{(\alpha)}(0) = \frac{\alpha + 1}{n!}.
\tag{23}
\]

The application of Kummer's transformation [2, p. 37, equation 1.3(7)]
\[
\text{1}_F(\alpha; \beta; z) = e^z \text{1}_F(\beta - \alpha; \beta; -z)
\tag{24}
\]
and the confluent hypergeometric definition (3) of \(L_n^{(\alpha)}(z)\) yields the following equivalent form of (22):
\[
\sum_{k=0}^{\infty} \frac{\nu+n}{(\nu)_k} L_k^{(\nu-1)} (-\lambda_2(1-\theta)) (\theta q)^k = (1 - \theta q)^{\nu+n} \cdot \text{1}_F(\nu+n; \nu; \frac{\lambda_2\theta(1-\theta)q}{1-\theta q}).
\tag{25}
\]
Substituting \(s = -\theta q, z = \lambda_2(1-\theta), \gamma = \nu+n, \text{ and } \mu+1 = \nu\) in (25) leads to the known generating function (11).

4. ANOTHER GENERATING FUNCTION FOR THE LAGUERRE POLYNOMIALS

Let \(X_1 \sim \text{Poisson} (\lambda_1)\) and \(X_2 \sim \text{NB}(\nu, P)\) denote, respectively, the Poisson and NB r.v.s. Consider \(T = X_1 - X_2\) which has the p.g.f.
\[
H(z) = \sum_{n=0}^{\infty} \text{Prob}(T = n) z^n = e^{\lambda_1(z-1)} \left(\frac{Q}{1-P/z}\right)^\nu,
\tag{26}
\]
\(|z| > P; \quad Q = 1 - P.
\]
Now let \(t\) in the index parameter \(\nu+t\) of the NB distribution (Section 1) be this r.v. \(T = X_1 - X_2\). The unconditional r.v. \(X\) has the p.g.f.
\[
G_X(z) = \left(\frac{q}{1-pz}\right)^\nu \sum_{i=0}^{\infty} \left(\frac{q}{1-pz}\right)^t \text{Prob}(T = t)
\]
\[
= \left(\frac{q}{1-pz}\right)^\nu \exp \left(\lambda_1 \left\{ \frac{q}{1-pz} - 1 \right\}\right) \left(\frac{Q}{1-P(1-pz)/q}\right)^\nu
\]
\[
= \left(\frac{q}{1-pz}\right)^\nu \exp \left(\lambda_1 \left\{ \frac{q}{1-pz} - 1 \right\}\right) \left(\frac{Q}{(1-P/q) + Ppz/q}\right)^\nu,
\]
that is,
\[ G_X(z) = \left( \frac{q}{1-pz} \right)^\nu \exp \left( \lambda_1 \left\{ \frac{q}{1-pz} - 1 \right\} \right) \cdot \left( \frac{Q/(1-P/q)}{1 - (-Pp/q)/(1-P/q)} \right)^z, \] (27)

which may be regarded as the p.g.f. of a convolution of NNB and a pseudo-binomial r.v. (see [15]). Therefore, the p.m.f. of (27) is
\[ P(k) = \sum_{j=0}^{k} e^{-\lambda_1 p} q^{\nu} \binom{\nu}{j} (-\lambda_1 q) \left( \frac{Pp/q}{1-P/q} \right)^{k-j} \left( \frac{Q}{1-P/q} \right)^{j}. \] (28)

Now set \( t = m - n \) with \( m \) varying as a Poisson \( (\lambda_1) \) r.v. and \( n \) as a NB\((\nu, P)\) r.v. This gives another expression for \( P(k) \) as follows:
\[ P(k) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(k \mid m, n) \Theta_1(m) \Theta_2(n) \]
\[ = e^{-\lambda_1 p} p^k q^\nu \sum_{n=0}^{\infty} q^{-n} L_k^{(\nu-n-1)} (-\lambda_1 q) \frac{(\nu)_n}{n!} P^n Q^\nu \]
\[ = e^{-\lambda_1 p} p^k q^\nu Q^\nu \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left( \frac{P}{q} \right)^n L_k^{(\nu-n-1)} (-\lambda_1 q), \]
where
\[ b(k \mid m, n) = \binom{k + m - n + \nu - 1}{k} p^k q^{m-n+\nu}, \quad \Theta_1(m) = e^{-\lambda_1 \frac{m}{m!}}, \]
and
\[ \Theta_2(n) = \frac{(\nu)_n}{n!} P^n Q^\nu. \]

Equating (28) and (29), and simplifying, we have
\[ \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} L_k^{(\nu-n-1)} (-\lambda_1 q) \left( \frac{P}{q} \right)^n = \left( 1 - \frac{P}{q} \right)^{-\nu} \left( -\frac{P/q}{1-P/q} \right)^k \times \sum_{j=0}^{k} \frac{(\nu)_{k-j}}{(k-j)!} L_j^{(\nu-1)} (-\lambda_1 q) \left( -\frac{P(1-P/q)}{P/q} \right)^j, \] (30)

which, under the substitutions \( t = P/q \) and \( x = -\lambda_1 q \), yields the following generating function for the Laguerre polynomials:
\[ \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} L_k^{(\nu-n-1)}(x) t^n = (1-t)^{-\nu} \left( \frac{1}{1-t} \right)^k \times \sum_{j=0}^{k} \frac{(\nu)_{k-j}}{(k-j)!} L_j^{(\nu-1)}(x) \left( \frac{1-t}{1-t} \right)^j = (1-t)^{-\nu} \sum_{j=0}^{k} \frac{(\nu)_j}{j!} L_{k-j}^{(\nu-1)}(x) \left( \frac{1-t}{1-t} \right)^j. \] (31)

It should be noted that (31) is valid for \( t > 0 \) and \( x < 0 \) due to probabilistic considerations. These constraints can indeed be waived by proving the generating function (31) directly, that is, without using probabilistic considerations.

Denote, for convenience, the first member of the generating function (31) by \( \Omega(x, t) \). Then, applying definition (5) and the Kummer transformation (24), we have
\[ \Omega(x, t) := \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} L_k^{(\nu-n-1)}(x) t^n \]
\[ = (-x)^k \sum_{\ell=0}^{k} \frac{(1-\nu-k)_{\ell}}{(k-\ell)!} \frac{t^{\ell}}{x^\ell} \times \text{2F1}(\nu, 1-\nu-k+\ell; 1-\nu-k; t). \] (32)
Now we make use of the Pfaff-Kummer transformation \([2, p. 33, equation 1.2(19)]\)

\[
2F_1(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} 2F_1 \left( \alpha, \gamma - \beta; \gamma; \frac{z}{1 - z} \right),
\]

\(|\arg(1 - z)| \leq \pi - \epsilon, \quad 0 < \epsilon < \pi.\)

We thus find from (32) that

\[
\Omega(x, t) = (-x)^k(1 - t)^{-\nu} \sum_{\ell=0}^{k} \frac{1}{(k - \ell)!} \left( \frac{1}{x} \right)^{\ell} \times 2F_1 \left( \nu, \nu - \ell; 1 - \nu - k; -\frac{t}{1 - t} \right)
\]

\[
= (-x)^k(1 - t)^{-\nu} \sum_{\ell=0}^{k} \frac{(-x)^{-\ell}}{(k - \ell)!} \times 2F_0 \left( -k + j, 1 - \nu - k + j; -\frac{1}{x} \right)
\]

\[
= (1 - t)^{-\nu} \sum_{j=0}^{k} \frac{(\nu)_j}{j!} L_{k-j}(x) \left( -\frac{t}{1 - t} \right)^j, \quad |t| < 1,
\]

where we have applied definition (5) once again.

Thus we have proved the generating function (31) in its obviously equivalent form

\[
\sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} L^{(\nu-n-1)}_k(x) t^n = (1 - t)^{-\nu} \sum_{j=0}^{k} \frac{(\nu)_j}{j!} L^{(\nu-1)}_{k-j}(x) \left( -\frac{t}{1 - t} \right)^j
\]

for all \(x\) and for \(|t| < 1\).

For the sake of completeness, we conclude this paper by first giving a direct proof of the bilinear generating function (21) without using probabilistic considerations. First of all, we recall the following known generating function \([2, p. 131, equation 2.5(4)]\):

\[
\sum_{k=0}^{\infty} \frac{(\nu)_k}{(\mu)_k} L^{(\nu)}_k(x) t^k = (1 - t)^{-\nu} \Phi_1 \left[ \lambda, \mu - \alpha - 1; \mu; -\frac{t}{1 - t}; -\frac{xt}{1 - t} \right], \quad |t| < 1,
\]

where \(\Phi_1\) denotes one of Humbert's confluent hypergeometric functions in two variables, defined by (cf. \([2, p. 58, equation 1.6(36)]\))

\[
\Phi_1[\alpha, \beta; \gamma; x, y] := \sum_{\ell, m=0}^{\infty} \frac{(\alpha)_{\ell+m} (\beta)_\ell x^\ell y^m}{(\gamma)_{\ell+m} \ell! m!},
\]

\(|x| < 1; \quad |y| < \infty.\)
In view of definition (37), if we apply transformations (33) and (24) successively, a special case of the generating function (36) when \( \lambda = \mu + n \) \((n \in \mathbb{N}_0)\) can be written in the form

\[
\sum_{k=0}^{\infty} \frac{\binom{\mu + n}{k}}{\binom{\mu}{k}} L_k^{(\alpha)}(x) t^k = (1 - t)^{-\alpha - n - 1} \exp \left( -\frac{x t}{1 - t} \right) 
\]

\[
\times \frac{n!}{\binom{\mu}{n}} \sum_{j=0}^{n} \frac{(-\alpha - 1)_j}{j!} L_{n-j}^{(\mu+j-1)} \left( \frac{x t}{1 - t} \right) (-t)^j, \quad |t| < 1.
\]

Now we let \( \Lambda(x, y, t) \) denote the left-hand side of the bilinear generating function (21) and replace the Laguerre polynomial in \( x \) by its confluent hypergeometric form given by (3). We thus find that

\[
\Lambda(x, y, t) := \sum_{k=0}^{\infty} L_n^{(\nu + \alpha + k - 1)}(x) L_k^{(\nu - 1)}(y) t^k = \frac{(-x)^n}{(n - \ell)! \ell! (\nu + \alpha + \ell)_{\ell}} \sum_{k=0}^{\infty} \frac{\binom{\nu + \alpha + n}{k}}{\binom{\nu + \alpha + \ell}{k}} L_k^{(\nu - 1)}(y) t^k.
\]

By appealing to the generating function (38) with \( \alpha = \nu - 1 \) and \( \mu = \nu + \alpha + \ell \) (and with \( n \) replaced by \( n - \ell \)), we obtain

\[
\Lambda(x, y, t) = (1 - t)^{-\nu - n} \exp \left( -\frac{yt}{1 - t} \right) \sum_{\ell=0}^{n} \frac{(-x(1 - t))^\ell}{\ell!} \sum_{m=0}^{n-\ell} \frac{(\alpha)_{\ell + m}}{\ell!} L_n^{(\nu + \alpha + \ell + m - 1)} \left( \frac{yt}{1 - t} \right) (-t)^m m! |\ell| < 1.
\]

Finally, upon setting

\[
m = j - \ell, \quad 0 \leq \ell \leq j; \quad 0 \leq j \leq n
\]

in (40), if we interpret the resulting \( \ell \)-sum as a Laguerre polynomial, we have

\[
\Lambda(x, y, t) = (1 - t)^{-\nu - n} \exp \left( -\frac{yt}{1 - t} \right) \sum_{j=0}^{n} L_j^{(\alpha - 1)} \left( -\frac{x(1 - t)}{t} \right) L_n^{(\nu + \alpha + j - 1)} \left( \frac{yt}{1 - t} \right) (-t)^j, \quad |t| < 1,
\]

which is precisely the right-hand side of the bilinear generating function (21).

Alternatively, in addition to the transformations (33) and (24), if we also apply the following special case of a well-known analytic continuation formula for the Gauss hypergeometric function:

\[
\binom{c-b}{n} \binom{b}{c} = \frac{1}{\Gamma(c) \Gamma(b)} \binom{c-b}{n} \binom{b}{c} \Gamma(n+1) = \frac{n!}{\Gamma(c) \Gamma(b)} \binom{c-b}{n} \binom{b}{c}, \quad n \in \mathbb{N}_0; \quad c \neq 0, -1, -2, \ldots,
\]

the aforementioned special case of the generating function (36) when \( \lambda = \mu + n \) \((n \in \mathbb{N}_0)\) can be rewritten in the form [cf. equation (38)]

\[
\sum_{k=0}^{\infty} \frac{\binom{\mu + n}{k}}{\binom{\mu}{k}} L_k^{(\alpha)}(x) t^k = (1 - t)^{-\alpha - n - 1} \exp \left( -\frac{x t}{1 - t} \right) \sum_{j=0}^{n} \frac{(-\alpha - 1)_j}{j!} L_{n-j}^{(\alpha)} \left( \frac{x t}{1 - t} \right) (1 - t)^j, \quad |t| < 1.
\]
Thus, by appealing to the generating function (43) with \( \alpha = \nu - 1 \) and \( \mu = \nu + \alpha + \ell \) (and with \( n \) replaced by \( n - \ell \)), we find from (39) that (cf. equation (40))

\[
\Lambda(x,y,t) = (1 - t)^{-\nu-n} \exp \left( \frac{-yt}{1-t} \right) \times \sum_{\ell=0}^{n} \frac{(-x(1-t))^{\ell}}{\ell!} \sum_{m=0}^{n-\ell} \frac{(\alpha)_{\ell+m}}{(\alpha)_{\ell}} L_{n-\ell-m}^{(\nu-1)} \left( \frac{yt}{1-t} \right) \frac{(1-t)^m}{m!}, \quad |t| < 1,
\]

which, upon setting

\[
m = j - \ell, \quad 0 \leq \ell \leq j; \quad 0 \leq j \leq n,
\]

and interpreting the resulting \( \ell \)-sum as a Laguerre polynomial, yields the right-hand side of the bilinear generating function (20).

It is not difficult to verify that, in view of the Chu-Vandermonde summation theorem (cf., e.g., [2, p. 30, equation 1.2(8)]), (42) is an immediate consequence of the following rather obvious polynomial identity:

\[
\sum_{j=0}^{n} \Omega_j z^j j! = \sum_{k=0}^{n} \frac{-(1-z)^k}{k!} \sum_{j=0}^{n-k} \frac{\Omega_{j+k}}{j!}.
\]

where \( \{\Omega_n\}_{n=0}^{\infty} \) is any sequence of complex numbers.

REFERENCES