IDENTIFIABILITY OF PIECEWISE CONSTANT CONDUCTIVITY IN A HEAT CONDUCTION PROCESS

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Abstract. We study the identification and identifiability problems for heat conduction in a nonhomogeneous rod. The identifiability results are established for two different sets of observations. Given a sequence of distributed type observations, the identifiability is proved for conductivities in a piecewise smooth class of functions. In the case of observations taken at finitely many points the identifiability is established for piecewise constant conductivities. Such conductivities can be uniquely identified using the proposed marching algorithm.

Key words. identification, identifiability, piecewise constant conductivity

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1. Introduction. Consider the heat conduction in a nonhomogeneous insulated rod of a unit length, with the ends kept at zero temperature at all times. Our main interest is in the identification and identifiability of the discontinuous conductivity (thermal diffusivity) coefficient \(a(x), 0 \leq x \leq 1\). The identification problem consists of finding a conductivity \(a(x)\) in an admissible set \(K\) for which the temperature \(u(x,t)\) fits given observations in a prescribed sense. Under a wide range of conditions one can establish the continuity of the objective function \(J(a)\) representing the best fit to the observations. Then the existence of the best fit to data conductivity follows if the admissible set \(K\) is compact in the appropriate topology. However, such an approach usually does not guarantee the uniqueness of the found conductivity \(a(x)\). Establishing such a uniqueness is referred to as the identifiability problem. If the conductivity is identifiable and one can design an algorithm for its reconstruction, then we say that \(a\) is constructively identifiable.

From physical considerations the conductivity coefficients \(a(x)\) are assumed to be in

\[
A_{ad} = \{a \in L^\infty(0,1) : 0 < \nu \leq a(x) \leq \mu\}.
\]

The temperature \(u(a) = u(x,t;a)\) inside the rod satisfies

\[
\begin{align*}
    &u_t - (a(x)u_x)_x = 0, \quad Q = (0,1) \times (0,T), \\
    &u(0,t) = u(1,t) = 0, \quad t \in (0,T), \\
    &u(x,0) = g(x), \quad x \in (0,1),
\end{align*}
\]

where \(g \in L^2(0,1)\). In general, the solution of (1.2) is understood in the weak sense. According to [11] for any \(a \in A_{ad}\) there exists a unique weak solution \(u(a) \in L^2(0,1;H^1_0(0,1)) \cap C([0,1];L^2(0,1))\), and so the map \(a \rightarrow u(a)\) is well defined. Moreover, this map is continuous from \(A_{ad}\) equipped with the \(L^2(0,1)\) topology into
The identification (parameter estimation) problem for (1.2) is as follows: Find a conductivity $a \in A_{ad}$ such that the solution $u(a)$ of (1.2) fits a given observation $z$ of the heat conduction process. For example, given $z \in L^2(0,1)$ one defines
\begin{equation}
J(a) = \|u(x,T;a) - z(x)\|_{L^2(0,1)}.
\end{equation}

Then the parameter estimation problem for (1.2) is reduced to the minimization of the objective function $J$ over the admissible set $A_{ad}$ or its subset $K_{ad}$: Find $\bar{a} \in K_{ad} \subset A_{ad}$ such that
\begin{equation}
J(\bar{a}) = \inf\{J(a) : a \in K_{ad}\}.
\end{equation}

The above-mentioned properties of the solutions $u(a)$ imply that the objective function $J(a)$ is continuous on $A_{ad} \cap L^2(0,1)$. Therefore the identification problem (1.4) has a solution if $A_{ad}$ is compact in $L^2(0,1)$. One such choice is $K_{ad} = \{a \in A_{ad} \cap H^1(0,1) : \|a\|_{H^1} \leq \text{constant}\}$; see [8]. However, $a \in H^1(0,1)$ implies that the conductivity is continuous. Therefore this choice of $K_{ad}$ is not suitable for the study of the identification problems with discontinuous coefficients.

To overcome this difficulty we have shown in [5] (in a multidimensional case) that one can take for $K_{ad}$ the set of functions in $A_{ad}$ which have a uniformly bounded variation. Such a set $K_{ad}$ is compact in $L^2(0,1)$, and the existence of solutions to the identification problem (1.4) follows. See [5] for additional details and numerical experiments for 2D parameter identification problems. A variety of identification problems is studied in [1] under very general assumptions on the problem’s parameters.

The identifiability questions for partial differential equations are much more difficult, and there are just a few available results. Suppose that one is given an observation $z(t) = u(p,t;a)$ of the heat conduction process (1.2) for $t_1 < t < t_2$ at some observation point $0 < p < 1$. From the series solution for (1.2) and the uniqueness of the Dirichlet series expansion (see section 2), one can, in principle, recover all of the eigenvalues of the associated Sturm–Liouville problem. If one also knows the eigenvalues for the heat conduction process with the same coefficient $a$ and different boundary conditions, then the classical results of Gelfand and Levitan [4] show that smooth coefficients $a(x)$ can be uniquely identified from the knowledge of the two spectral sequences. Also, if the entire spectral function is known (i.e., the eigenvalues and the values of the derivatives of the normalized eigenfunctions at $x = 0$), then the conductivity is identifiable as well. However, such results have little practical value, since the observation data $z(t)$ always contain some noise, and therefore one cannot hope to adequately identify more than just the few first eigenvalues of the problem.

A different approach is taken in [6, 12, 13, 14]. These works show that one can identify a constant conductivity $a$ in (1.2) from the measurement $z(t)$ taken at one point $p \in (0,1)$. These works also discuss problems more general than (1.2), including problems with a broad range of boundary conditions, nonzero forcing functions, as well as elliptic and hyperbolic problems. In [7, 3] and references therein identifiability results are obtained for elliptic and parabolic equations with discontinuous parameters in a multidimensional setting. A typical assumption there is that one knows the normal derivative of the solution at the boundary of the region for every Dirichlet boundary input.

The main result of this paper is contained in Theorem 4.6. This theorem describes and justifies the marching algorithm for the unique identification of piecewise constant conductivities from observations of (1.2) given at finitely many points $p_k \in (0,1)$. We
start by recalling some basic properties of (1.2) in section 2. Identifiability results for countably many distributed observations are given in section 3. Identifiability of piecewise constant conductivities a is discussed in section 4. Numerical results for the identifiability algorithms described in this paper require an extensive exposition, and they will be presented elsewhere.

2. Auxiliary results. In this section we collect some well-known results for the solutions \( u(x, t; a) \) of (1.2), as well as for its associated Sturm–Liouville problem. Since such results are scattered in the literature, some brief proof outlines are included as well. See [2, 9, 10, 11] for a detailed discussion.

**Definition 2.1.** Function \( a(x) \) is said to belong to the class \( \mathcal{PS}_N \) if

(i) \( a \in A_{ad} = \{ a \in L^\infty(0, 1) : 0 < \nu \leq a(x) \leq \mu \} \) for some positive constants \( \nu \) and \( \mu \);

(ii) function \( a \) is piecewise smooth; that is, there exists a finite sequence of points

\( 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1 \) such that both \( a(x) \) and \( a'(x) \) are continuous on every open subinterval \( (x_i, x_{i+1}) \), \( i = 0, \ldots, N - 1 \), and both can be continuously extended to the closed intervals \( [x_i, x_{i+1}] \), \( i = 0, \ldots, N - 1 \). For definiteness, we assume that \( a \) and \( a' \) are continuous from the right, i.e., \( a(x) = a(x+) \) and \( a'(x) = a'(x+) \) for all \( x \in [0, 1) \). Also let \( a(1) = a(1-) \).

**Definition 2.2.** \( \mathcal{PS} = \bigcup_{N=1}^\infty \mathcal{PS}_N \).

Everywhere in the following the conductivities \( a \) are assumed to be in \( \mathcal{PS} \). If \( a \in \mathcal{PS}_N \), then the regularity conditions on \( a \) and the uniqueness of the weak solutions imply that for any \( t > 0 \) the weak solution \( u(x, t; a) \) of (1.2) satisfies the equation in the classical sense on any subinterval \( (x_i, x_{i+1}) \), \( i = 0, \ldots, N - 1 \). Also \( u \) satisfies the matching conditions for the continuity of the solution and its conormal derivative at \( x_i \in (0, 1) \), \( i = 1, 2, \ldots, N - 1 \):

\[
\begin{align*}
& u_t - (a(x)u_x)_x = 0, \quad x \neq x_i, \quad t \in (0, T), \\
& u(0, t) = u(1, t) = 0, \quad t \in (0, T), \\
& u(x_i +, t) = u(x_i -, t), \\
& a(x_i +)u_x(x_i +, t) = a(x_i -)u_x(x_i -, t), \\
& u(x, 0) = g(x), \quad x \in (0, 1),
\end{align*}
\]

(2.1)

where \( g \in L^2(0, 1) \); see [11, 16].

Denote by \( \| \cdot \| \), \( \langle \cdot, \cdot \rangle \) the norm and the inner product, respectively, in \( H = L^2(0, 1) \).

**Theorem 2.3.** Let \( a \in \mathcal{PS} \). Then

(i) the associated Sturm–Liouville problem

\[
\begin{align*}
& (a(x)v'(x))' = -\lambda v(x), \quad x \neq x_i, \\
& v(0) = v(1) = 0,
\end{align*}
\]

(2.2)

has infinitely many eigenvalues

\[
0 < \lambda_1 < \lambda_2 < \cdots \to \infty.
\]

The eigenvalues \( \{ \lambda_k \}_{k=1}^\infty \) and the corresponding orthonormal set of eigenfunctions \( \{ \psi_k \}_{k=1}^\infty \) satisfy

\[
\lambda_k = \inf \left\{ \frac{\int_0^1 a(x)[v'(x)]^2 dx}{\int_0^1 [v(x)]^2 dx} : v \in H_0^1(0, 1), \quad \langle v, \psi_j \rangle = 0, \quad j = 1, 2, \ldots, k - 1 \right\},
\]

(2.3)
The normalized eigenfunctions \( \{v_k\}_{k=1}^{\infty} \) form a basis in \( L^2[0,1] \).

(ii) Each eigenvalue is simple. For each eigenvalue \( \lambda_k \) there exists a unique continuous, piecewise smooth normalized eigenfunction \( v_k(x) \) such that \( v'_k(0+) > 0 \), and the function \( a(x)v'_k(x) \) is continuous on \([0,1] \).

(iii) Eigenvalues \( \{\lambda_k\}_{k=1}^{\infty} \) satisfy the inequality

\[
\nu \pi^2 k^2 \leq \lambda_k \leq \mu \pi^2 k^2.
\]

(iv) The first eigenfunction \( v_1 \) satisfies \( v_1(x) > 0 \) for any \( x \in (0,1) \).

(v) The first eigenfunction \( v_1 \) has a unique point of maximum \( q \in (0,1) : v_1(x) < v_1(q) \) for any \( x \neq q \).

(vi) For any fixed \( t > 0 \) the solution \( u \) of (2.1) is given by

\[
u(t,a) = \sum_{k=1}^{\infty} (g,v_k)e^{-\lambda_k t}v_k(x),
\]

and the series converges uniformly and absolutely on \([0,1] \).

(vii) For any \( p \in (0,1) \) the function

\[
zip(t) = u(p,t,a), \quad t > 0,
\]

is real analytic on \((0,\infty)\).

Proof. (i) The proof is standard; see, e.g., [10].

(ii) On any subinterval \((x_1,x_1+1)\) the coefficient \( a(x) \) has a bounded continuous derivative. Therefore, on any such interval the initial value problem \((a(x)v'(x))' + \lambda v = 0, v(x_1) = A, v'(x_1) = B \) has a unique solution. Suppose that two eigenfunctions \( w_1(x) \) and \( w_2(x) \) correspond to the same eigenvalue \( \lambda_k \). Then they both satisfy the condition \( w_1(0) = w_2(0) = 0 \). Therefore their Wronskian is equal to zero at \( x = 0 \). Consequently, the Wronskian is zero throughout the interval \((x_0,x_1)\), and the solutions are linearly dependent there. Thus \( w_2(x) = Cw_1(x) \) on \((x_0,x_1)\), \( w_2(x_1-) = Cw_1(x_1-) \), and \( w_2'(x_1-) = Cw_1'(x_1-) \). The linear matching conditions imply that \( w_2(x_1+) = Cw_1(x_1+) \) and \( w_2'(x_1+) = Cw_1'(x_1+) \). The uniqueness of solutions implies that \( w_2(x) = Cw_1(x) \) on \((x_1,x_2)\), etc. Thus \( w_2(x) = Cw_1(x) \) on \((0,1)\), and each eigenvalue \( \lambda_k \) is simple. In particular \( \lambda_1 \) is a simple eigenvalue. The uniqueness and the matching conditions also imply that any solution of \((a(x)v'(x))' + \lambda v = 0, v(0) = 0, v'(0) = 0 \) must be identically equal to zero on the entire interval \((0,1)\). Thus no eigenfunction \( v_k(x) \) satisfies \( v_k'(0) = 0 \). Assuming that the eigenfunction \( v_k \) is normalized in \( L^2(0,1) \), it leaves us with the choice of its sign for \( v_k'(0) \). Letting \( v_k'(0) > 0 \) makes the eigenfunction unique.

(iii) The eigenvalues of (2.3) satisfy the min-max principle

\[
\lambda_k = \min_{V_k} \max_{v \in V_k} \left\{ \frac{\int_0^1 a(x)[v'(x)]^2 dx}{\int_0^1 [v(x)]^2 dx} : v \in V_k \right\},
\]

where \( V_k \) varies over all subspaces of \( H_0^1(0,1) \) of finite dimension \( k \); see [10]. Therefore \( a(x) \leq b(x), x \in [0,1] \) implies \( \lambda_k^{(a)} \leq \lambda_k^{(b)} \). Since the eigenvalues of (2.3) with \( a(x) = 1 \) are \( \pi^2 k^2 \), the required inequality follows.
(iv) Recall that \( v_1(x) \) is a continuous function on \([0, 1]\). Suppose that there exists \( p \in (0, 1) \) such that \( v_1(p) = 0 \). Let \( w_l(x) = v_1(x) \) for \( 0 \leq x < p \) and \( w_l(x) = 0 \) for \( p \leq x \leq 1 \). Let \( w_r(x) = v_1(x) - w_l(x) \), \( x \in [0, 1] \). Then \( w_l, w_r \) are continuous, and, moreover, \( w_l, w_r \in H^1_0(0, 1) \). Also
\[
\int_0^1 w_l(x) w_r(x) \, dx = 0 \quad \text{and} \quad \int_0^1 a(x) w'_l(x) w'_r(x) \, dx = 0.
\]
Suppose that \( w_l \) is not an eigenfunction for \( \lambda_1 \). Then
\[
\int_0^1 a(x) [w_l'(x)]^2 \, dx > \lambda_1 \int_0^1 [w_l(x)]^2 \, dx.
\]
Since
\[
\int_0^1 a(x) [w_l'(x)]^2 \, dx \geq \lambda_1 \int_0^1 [w_r(x)]^2 \, dx,
\]
we have
\[
\lambda_1 = \frac{\int_0^1 a(x) [v'_l(x)]^2 \, dx}{\int_0^1 [v_l(x)]^2 \, dx} = \frac{\int_0^1 a(x) (|v'_l(x)|^2 + [w_r(x)]^2) \, dx}{\int_0^1 (|w_l(x)|^2 + [w_r(x)]^2) \, dx} > \frac{\int_0^1 (\lambda_1 |w_l(x)|^2 + \lambda_1 |w_r(x)|^2) \, dx}{\int_0^1 (|w_l(x)|^2 + [w_r(x)]^2) \, dx} = \lambda_1.
\]
This contradiction implies that \( w_l \) (and \( w_r \)) must be an eigenfunction for \( \lambda_1 \). However, \( w_l(x) = 0 \) for \( p \leq x \leq 1 \), and as in (ii) it implies that \( w_l(x) = 0 \) for all \( x \in [0, 1] \), which is impossible. Since \( v'_1(0) > 0 \) the conclusion is that \( v_1(x) > 0 \) for \( x \in (0, 1) \).

(v) From part (i), any eigenfunction \( v_k \) is continuous and satisfies
\[
(a(x)v'_k(x))' = -\lambda_k v_k(x)
\]
for \( x \neq x_i \). Also the function \( a(x)v'_k(x) \) is continuous on \([0, 1]\) because of the matching conditions at the points of discontinuity \( x_i, i = 1, 2, \ldots, N - 1 \) of \( a \). The integration gives
\[
a(x)v'_k(x) = a(p)v'_k(p) - \lambda_k \int_p^x v_k(s) \, ds
\]
for any \( x, p \in (0, 1) \).

Let \( p \in (0, 1) \) be a point of maximum of \( v_k \). If \( p \neq x_i \), then \( v'_k(p) = 0 \). If \( p = x_i \), then \( v'_k(x_i-) \geq 0 \) and \( v'_k(x_i+) \leq 0 \). Therefore \( \lim_{x \to p} a(x)v'_k(x) = 0 \) and \( v'_k(p+) = v'_k(p-) = 0 \) since \( a(x) \geq \nu > 0 \). In any case for such a point \( p \) we have
\[
(2.5) \quad a(x)v'_k(x) = -\lambda_k \int_p^x v_k(s) \, ds, \quad x \in (0, 1).
\]
Since \( v_1(x) > 0 \), \( a(x) > 0 \) on \((0, 1)\), (2.5) implies that \( v'_1(x) > 0 \) for any \( 0 \leq x < p \) and \( v'_1(x) < 0 \) for any \( p < x \leq 1 \). Since the derivative of \( v_1 \) is zero at any point of maximum, we have to conclude that such a maximum \( p \) is unique.
(vi) We prove only the convergence part. Note that
\[ \nu \| v_k' \|^2 \leq \int_0^1 a(x) |v_k'(x)|^2 \, dx = \lambda_k \| v_k \|^2 = \lambda_k. \]
Thus
\[ \| v_k' \| \leq \frac{\sqrt{\lambda_k}}{\sqrt{\nu}} \]
and
\[ |v_k(x)| \leq \int_0^x |v_k'(s)| \, ds \leq \| v_k' \| \leq \frac{\sqrt{\lambda_k}}{\sqrt{\nu}}. \]
Bessel’s inequality implies that the sequence of Fourier coefficients \( \langle g, v_k \rangle \) is bounded. Therefore, denoting by \( C \) various constants and using the fact that the function \( s \to \sqrt{s} e^{-\sigma s} \) is bounded on \([0, \infty)\) for any \( \sigma > 0 \), one gets
\[ |\langle g, v_k \rangle e^{-\lambda_k t} v_k(x) | \leq C \frac{\sqrt{\lambda_k}}{\sqrt{\nu}} e^{-\lambda_k t} \leq C e^{-\frac{\lambda_k t}{\nu}}. \]
From (iii) of this theorem \( \lambda_k \geq \nu \pi^2 k^2 \). Thus
\[ \sum_{k=1}^{\infty} |\langle g, v_k \rangle e^{-\lambda_k t} v_k(x) | \leq C \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2}{\nu}} \leq C \sum_{k=1}^{\infty} \left( e^{-\frac{\pi^2 t}{\nu}} \right)^k < \infty. \]
(vii) Let \( t_0 > 0 \) and \( p \in (0, 1) \). From (vi), the series \( \sum_{k=1}^{\infty} \langle g, v_k \rangle e^{-\lambda_k t_0} v_k(p) \) converges absolutely. Therefore \( \sum_{k=1}^{\infty} \langle g, v_k \rangle e^{-\lambda_k t} v_k(p) \) is analytic in the part of the complex plane \( \{ s \in C : \Re s > t_0 \} \), and the result follows.

Lemma 2.4. Let \( \mu_k > 0, k = 1, 2, \ldots, \) be a strictly increasing sequence. Suppose that \( T_1 \geq 0 \) and \( \sum_{k=1}^{\infty} |C_k| < \infty \). If
\[ \sum_{k=1}^{\infty} C_k e^{-\mu_k t} = 0 \quad \text{for all} \quad t \in (T_1, T_2), \]
then \( C_k = 0 \) for \( k = 1, 2, \ldots, \).

The result follows at once from the observation that the series \( \sum_{k=1}^{\infty} C_k e^{-\mu_k z} \) converges uniformly in the \( \Re z > 0 \) region of the complex plane, implying that it is an analytic function there. See Chapter 9 of [15] for additional results on Dirichlet series.

Remark. According to Theorem 2.3(vi) for each fixed \( p \in (0, 1) \) the solution \( z(t) = u(p, t; a) \) of (2.1) is given by a Dirichlet series. However, Lemma 2.4 is not directly applicable since the coefficients \( C_k = \langle g, v_k \rangle v_k(p) \) are only square summable. Nevertheless, the conclusion of Lemma 2.4 remains valid, since the exponents \( \mu_k \) in the Dirichlet series are the eigenvalues \( \lambda_k \) which satisfy the growth condition stated in Theorem 2.3(iii). This allows one to conclude (Theorem 2.3(vii)) that the solution \( z(t) \) is a real analytic function on \((0, \infty)\), and the uniqueness of such a representation follows. Thus it would be a mistake to simply refer to the standard results such as Lemma 2.4 for the uniqueness of the Dirichlet series representation to justify the paper’s conclusions.
3. Identifiability by distributed measurements. Suppose that one is given some observations of the heat conduction process (2.1) with an unknown conductivity \( a(x) \) and that they coincide with the observations of the model process

\[
\begin{align*}
\frac{d^m}{dt^m} u^m(x) - (a^m(x)u^m_x)_x &= 0, \quad x \neq x_i^m, \quad t \in (0, T), \\
u^m(0, t) &= u^m(1, t) = 0, \quad t \in (0, T), \\
u^m(x, 0) &= g(x), \quad x \in (0, 1),
\end{align*}
\]

where \( g \) is the same as in (2.1). The conductivity \( a \) is said to be identifiable in some class of functions \( \mathcal{M} \) if the coincidence of the measurements of the observed and the model processes implies that \( a = a^m \), provided \( a, a^m \in \mathcal{M} \).

**Theorem 3.1.** Let \( \{\psi_n\}_{n=1}^\infty \) be a complete orthonormal set in \( H = L^2(0, 1) \). Suppose that nonzero initial data \( g \in H \) and the observations \( z_n(t) = \langle u(x, t; a), \psi_n \rangle \) for \( n = 1, 2, \ldots \) and \( 0 \leq T_1 < t < T_2 \) of the heat conduction process (2.1) are given. Then the conductivity \( a(x) \in A_{ad} \) is constructively identifiable in the class of piecewise smooth functions \( PS \).

**Proof.** To show the identifiability of \( a \) we give an algorithm for its reconstruction from the data \( z_n(t), \ n = 1, 2, \ldots, \) guaranteeing the uniqueness in each step. Using Theorem 2.3(vi) we have

\[
z_n(t) = \sum_{k=1}^\infty \langle g, v_k \rangle e^{-\lambda_k t} \langle v_k, \psi_n \rangle
\]

for each \( n = 1, 2, \ldots \) and \( 0 \leq T_1 < t < T_2 \).

Fix an \( n > 0 \). Since \( \psi_n \in H \) and \( \{v_k\}_{k=1}^\infty \) form a basis in \( H \), the Bessel inequality implies that the sequence of the Fourier coefficients \( \{\langle v_k, \psi_n \rangle\}_{k=1}^\infty \) is bounded. From Theorem 2.3(vi) one concludes that the above series converges absolutely. Note that some products \( \langle g, v_k \rangle \langle v_k, \psi_n \rangle \) may be equal to zero. The zero value products present a difficulty, since we would not know how to associate the sequence of exponents recovered from (3.2) with the (unknown) eigenvalues \( \lambda_k \): Some eigenvalues may be missing from the sequence. Define (possibly empty) subsets \( Q_n \subset \mathbb{N} \) by

\[
Q_n = \{k \in \mathbb{N} : \langle g, v_k \rangle \langle v_k, \psi_n \rangle \neq 0\}, \quad n = 1, 2, \ldots
\]

For \( Q_n \neq \emptyset \) reindex (3.2) so that there would be no vanishing coefficients:

\[
z_n(t) = \sum_{l=1}^\infty C_{l,n} e^{-\mu_{l,n} t}, \quad t \in (T_1, T_2).
\]

By Theorem 2.3(vii) the solutions \( z_n(t) \) are real analytic. Therefore, since all of the coefficients \( C_{l,n} \neq 0 \), one can uniquely determine them and the sequences \( \mu_{l,n}, \ l = 1, 2, \ldots \). Recall that \( \{\mu_{l,n}\}_{l=1}^\infty \subset \{\lambda_k\}_{k=1}^\infty \) for any \( n \) with a nonempty \( Q_n \) so each \( \mu_{l,n} \geq \lambda_1 > 0 \). For each \( n \) such that \( Q_n \neq \emptyset \) let

\[
\gamma_n = \min\{\mu_{l,n} : l \in \mathbb{N}\}.
\]

Define

\[
\gamma = \min\{\gamma_n : Q_n \neq \emptyset\}
\]
and

\[
A_n = \begin{cases} 
C_{1,n} & \text{if } \gamma_n = \gamma, \quad Q_n \neq \emptyset, \\
0 & \text{if } \gamma_n > \gamma, \quad Q_n \neq \emptyset, \\
0 & \text{if } Q_n = \emptyset.
\end{cases}
\]

(3.4)

Let

\[
w(x) = \sum_{n=1}^{\infty} A_n \psi_n(x).
\]

(3.5)

We claim that \(w\) is a nonzero multiple of some eigenfunction \(v_J\) of (2.1). Indeed, let \(J\) be the smallest index for which \(\langle g, v_J \rangle \neq 0\). Such an index exists since \(g \neq 0\), and the eigenfunctions form a basis in \(H\). Now, since \(v_J \neq 0\) and \(\{\psi_n\}_{n=1}^{\infty}\) also form a basis in \(H\), there exists \(n \in \mathbb{N}\) such that \(\langle v_J, \psi_n \rangle \neq 0\). Thus \(\langle g, v_J \rangle \langle v_J, \psi_n \rangle \neq 0\) and \(\gamma \leq \lambda_J\). The choice of \(J\) implies that \(\gamma = \lambda_J\). By the definition \(A_n = \langle g, v_J \rangle \langle v_J, \psi_n \rangle\) for nonzero products. Therefore

\[
w(x) = \sum_{n=1}^{\infty} \langle g, v_J \rangle \langle v_J, \psi_n \rangle \psi_n(x)
\]

\[
= \langle g, v_J \rangle \sum_{n=1}^{\infty} \langle v_J, \psi_n \rangle \psi_n(x) = \langle g, v_J \rangle v_J(x)
\]

as claimed.

Now we show that the set of points \(y \in (0, 1)\) where \(w'(y) = 0\) is finite. Assuming the opposite, and since \(w\) is a nonzero multiple of \(v_J\), there exists a sequence \(y_j \in (0, 1)\) such that \(v_J'(y_j) = 0\) and \(\lim_{j \to \infty} y_j = c \in [0, 1]\). The continuity of \(a(x)v_J'(x)\) implies that \(v_J'(c) = 0\). From \((a(x)v_J'(x))' = -\gamma v_J(x)\) one gets

\[
0 = a(y_{j+1})v_J'(y_{j+1}) - a(y_j)v_J'(y_j) = -\gamma \int_{y_j}^{y_{j+1}} v_J(s) ds
\]

(3.6)

and concludes that \(v_J\) cannot be strictly positive or strictly negative on \((y_j, y_{j+1})\). Let \(\zeta_j \in (y_j, y_{j+1})\) be such that \(v_J(\zeta_j) = 0\). Then \(\lim_{j \to \infty} \zeta_j = c \in [0, 1]\) and \(v_J(c) = 0\). Now we have both \(v_J(c) = 0\) and \(v_J'(c) = 0\). But then the uniqueness of the Cauchy problem for the second order linear equations, and the matching conditions (see the proof of Theorem 2.3(ii)) imply that \(v_J(x) = 0\) for all \(x \in [0, 1]\), which is impossible.

Let \(q\) be a point of maximum of \(w\). Note that \(w\) may have several maxima, unless it is a multiple of \(v_1\). In any case, (2.5) implies

\[
a(x)w'(x) = -\gamma \int_{q}^{x} w(s) ds, \quad x \in (0, 1).
\]

(3.7)

Then, outside of the finite sets \(\{x_i\}\) and \(\{y_j\}\), the conductivity \(a(x)\) is uniquely determined from (3.7) by

\[
a(x) = -\frac{\int_{q}^{x} w(s) ds}{w'(x)}.
\]

Because \(a\) is assumed to be in \(\mathcal{PS}\), it can be uniquely extended to the entire interval \([0, 1]\). \(\Box\)
Remark 1. In an application one can choose \( \psi_n = \sqrt{2} \sin \pi nx \) and the initial condition \( g(x) > 0 \) on \((0, 1)\). Then \( \langle g, v_1 \rangle \langle v_1, \psi_1 \rangle \neq 0 \), since \( v_1(x) > 0 \) on \((0, 1)\) by Theorem 2.3(iv). Thus \( J = 1 \) in this case, and \( w(x) = \langle g, v_1 \rangle v_1(x) \) in the above algorithm. Also, one can see from (3.6) that there is only one point \( y_1 = q \in (0, 1) \) where \( v'_1(q) = 0 \), and it is the point of maximum of \( v_1(x) \) on \((0, 1)\). Indeed, if there were two such points, then by (3.6) \( v_1(x) \) would have to become negative between them, which would contradict \( v_1(x) > 0 \) on \((0, 1)\).

Remark 2. Since the system \( \{ \psi_n \}_{n=1}^\infty \) is complete, the conditions of Theorem 3.1 imply that for any \( t > 0 \) one knows \( u(x, t; a) \) almost everywhere on \([0, 1]\). Since \( u(x, t; a) \) is continuous in \( x \) and analytic in \( t \), Theorem 3.1, in fact, assumes that the solution is known in \([0, 1] \times (0, T)\) or (equivalently) in \([0, 1] \times (0, \infty)\). Thus, Theorem 3.1 can be stated under any of these conditions. However, a practical reconstruction of the conductivity \( a \) directly from the equation \( u_t = (au_x)_x \) is extremely unstable. The algorithm presented in the above theorem does not reconstruct the entire solution \( u \) but just the first eigenfunction of the associated Sturm–Liouville problem. Its stability properties will be studied elsewhere.

4. Identifiability of piecewise constant conductivities from finitely many observations. The identifiability is sought within the following set. Let \( \mathcal{PC} \subset \mathcal{PS} \) be the class of piecewise constant conductivities, and let \( \mathcal{PC}_N = \mathcal{PC} \cap \mathcal{PS}_N \).

Functions \( a \in \mathcal{PC}_N \) have the form \( a(x) = a_i \) for \( x \in [x_{i-1}, x_i) \), \( i = 1, 2, \ldots, N \). In this case the governing system (2.1) is

\[
\begin{align*}
    u_t - a_i u_{xx} & = 0, \quad x \in (x_{i-1}, x_i), \quad t \in (0, T), \\
    u(0, t) & = u(1, t) = 0, \quad t \in (0, T), \\
    u(x_i, t) & = u(x_{i-1}, t), \\
    a_i u_x(x_i, t) & = a_{i-1} u_x(x_{i-1}, t), \\
    u(x, 0) & = g(x), \quad x \in (0, 1),
\end{align*}
\]

(4.1)

where \( g \in L^2(0, 1) \) and \( i = 1, 2, \ldots, N-1 \). The associated Sturm–Liouville problem is

\[
\begin{align*}
    a_i v''(x) & = -\lambda v(x), \quad x \in (x_{i-1}, x_i), \\
    v(0) & = v(1) = 0, \\
    v(x_i) & = v(x_{i-1}), \\
    a_i v'(x_i) & = a_{i-1} v'(x_{i-1})
\end{align*}
\]

(4.2)

for \( i = 1, 2, \ldots, N-1 \).

We are interested only in the first eigenfunction \( v_1 \) of (4.2). Let \( \lambda_1 \) be the first eigenvalue. Suppose that \( p^* \in (x_{i-1}, x_i) \). Then \( v_1 \) can be expressed on \((x_{i-1}, x_i)\) as

\[
v_1(x) = A \cos \left( \sqrt{\frac{\lambda_1}{a_i}} (x - p^*) + \gamma \right), \quad A > 0, \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2}.
\]

The range for \( \gamma \) in the above representation follows from the fact that \( v_1(p^*) = A \cos \gamma > 0 \) by Theorem 2.3(iv).

The identifiability of piecewise constant conductivities is based on the following three lemmas.

Lemma 4.2. Suppose that \( \delta > 0 \). Assume \( Q_1, Q_3 \geq 0, \quad Q_2 > 0, \quad \text{and} \quad 0 < Q_1 + Q_3 < 2Q_2 \). Let

\[
    \Gamma = \left\{ (A, \omega, \gamma) : A > 0, \quad 0 < \omega < \frac{\pi}{2\delta}, \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \right\}.
\]
Then the system of equations
\[ A \cos(\omega \delta - \gamma) = Q_1, \quad A \cos \gamma = Q_2, \quad A \cos(\omega \delta + \gamma) = Q_3 \]
has a unique solution \((A, \omega, \gamma) \in \Gamma\) given by
\[
\omega = \frac{1}{\delta} \arccos \frac{Q_1 + Q_3}{2Q_2}, \quad \gamma = \arctan \left( \frac{Q_1 - Q_3}{2Q_2 \sin \omega \delta} \right), \quad A = \frac{Q_2}{\cos \gamma}.
\]

Proof. For \((A, \omega, \gamma) \in \Gamma\) one has \(A > 0\) and \(\cos \gamma > 0\). Therefore
\begin{align*}
(4.3) \quad \cos(\omega \delta - \gamma) = \cos(\omega \delta) \frac{\sin \gamma}{\cos \gamma} = \frac{Q_1}{Q_2}, \\
(4.4) \quad \cos(\omega \delta + \gamma) = \cos(\omega \delta) \frac{\sin \gamma}{\cos \gamma} = \frac{Q_3}{Q_2}.
\end{align*}
Adding (4.3) and (4.4) yields
\[ \cos \omega \delta = \frac{Q_1 + Q_3}{Q_2} \]
Since \(0 < \omega \delta < \frac{\pi}{2}\) and \(0 < (Q_1 + Q_3)/2Q_2 < 1\) the above equation is uniquely solvable. Now subtracting (4.4) from (4.3) yields
\[ \tan \gamma = \frac{Q_1 - Q_3}{2Q_2 \sin \omega \delta}, \]
which is also uniquely solvable, since \(-\pi/2 < \gamma < \pi/2\). Finally, we have \(A = Q_2/\cos \gamma\) \(\square\)

Lemma 4.3. Suppose that \(\delta > 0\), \(0 < p \leq x_1 < p + \delta < 1\), \(0 < \omega_1, \omega_2 < \pi/2\).
Let \(w(x), v(x), x \in [p, p + \delta]\) be such that
\[ w(x) = A_1 \cos \omega_1 x + B_1 \sin \omega_1 x, \]
\[ v(x) = A_2 \cos \omega_2 x + B_2 \sin \omega_2 x. \]
Suppose that
\[ v(x_1) = w(x_1), \quad \omega_1^2 v'(x_1) = \omega_2^2 w'(x_1), \]
\[ v'(x_1) > 0, \quad v(x_1) > 0. \]
Then
(i) conditions \(v(p + \delta) = w(p + \delta), v'(p + \delta) \geq 0\), and \(\omega_1 \leq \omega_2\) imply \(\omega_1 = \omega_2\); (ii) conditions \(v(p + \delta) = w(p + \delta), w'(p + \delta) \geq 0\), and \(\omega_1 \geq \omega_2\) imply \(\omega_1 = \omega_2\).
Proof. Since \(v(x_1) > 0, v'(x_1) > 0\) we have
\[ v(x) = A \sin [\omega_2 (x - x_1) + \gamma], \quad 0 < \gamma < \frac{\pi}{2}, \]
where \(A > 0\). The matching conditions for \(w(x)\) at \(x_1\) imply
\begin{align*}
w(x) &= A \sin \gamma \cos \omega_1 (x - x_1) + A \frac{\omega_1}{\omega_2} \cos \gamma \sin \omega_1 (x - x_1) \\
&= A \sin [\omega_1 (x - x_1) + \gamma] + A \left[ \frac{\omega_1}{\omega_2} - 1 \right] \cos \gamma \sin \omega_1 (x - x_1).
\end{align*}
Thus
\[ v(p + \delta) - w(p + \delta) = A \left[ 1 - \frac{\omega_1}{\omega_2} \right] \cos \gamma \sin \omega_1(p + \delta - x_1) \]
\[ + A \sin[\omega_2(p + \delta - x_1) + \gamma] - A \sin[\omega_1(p + \delta - x_1) + \gamma] \]
\[ = A \frac{\omega_2 - \omega_1}{\omega_2} \cos \gamma \sin \omega_1(p + \delta - x_1) \]
\[ + 2A \sin \frac{\omega_2 - \omega_1}{2} (p + \delta - x_1) \cos \left[ \frac{\omega_2 + \omega_1}{2}(p + \delta - x_1) + \gamma \right]. \]

Observe that \(0 < p + \delta - x_1 \leq \delta\). Thus \(\sin \omega_1(p + \delta - x_1) > 0\).

For \(\omega_2 > \omega_1\) and \(v'(p + \delta) \geq 0\) one has
\[ \cos[\omega_2(p + \delta - x_1) + \gamma] = \frac{1}{\omega_2} v'(p + \delta) \geq 0. \]

Therefore
\[ \frac{\omega_2 + \omega_1}{2}(p + \delta - x_1) + \gamma < \omega_2(p + \delta - x_1) + \gamma \leq \frac{\pi}{2} \]
and
\[ \cos \left[ \frac{\omega_2 + \omega_1}{2}(p + \delta - x_1) + \gamma \right] > \cos[\omega_2(p + \delta - x_1) + \gamma] \geq 0. \]

Thus \(v(p + \delta) - w(p + \delta) > 0\), and the conclusion (i) of the lemma follows.

The case \(\omega_2 < \omega_1\) and \(w'(p + \delta) \geq 0\) is reduced to the already established one by interchanging \(\omega_1\) with \(\omega_2\) and \(w\) with \(v\).

Lemma 4.4. Let \(\delta > 0\), \(0 < \eta \leq 2\delta\), \(\omega_1 \neq \omega_2\), with \(0 < \omega_1 \delta, \omega_2 \delta < \pi/2\). Also let \(A, B > 0\), \(0 \leq p < p + \eta \leq 1\), and
\[ w(x) = A \cos[\omega_1(x - p) + \gamma_1], \]
\[ v(x) = B \cos[\omega_2(x - p - \eta) + \gamma_2], \]

with \(|\gamma_1|, |\gamma_2| < \pi/2\).

Then the system
\[(4.5) \quad w(q) = v(q), \]
\[(4.6) \quad \omega_2^2 w'(q) = \omega_1^2 v'(q), \]
\[(4.7) \quad w(q) > 0, \quad v(q) > 0 \]

admits at most one solution \(q\) on \([p, p + \eta]\). This unique solution \(q\) can be computed as follows:

If \(\gamma_1 \geq 0\), then
\[(4.8) \quad q = p + \frac{1}{\omega_1} \left[ \arctan \left( \omega_1 \sqrt{\frac{B^2 - A^2}{A^2 \omega_2^2 - B^2 \omega_1^2}} \right) - \gamma_1 \right]. \]

If \(\gamma_2 \leq 0\), then
\[(4.9) \quad q = p + \eta + \frac{1}{\omega_2} \left[ \arctan \left( \omega_2 \sqrt{\frac{B^2 - A^2}{A^2 \omega_2^2 - B^2 \omega_1^2}} \right) - \gamma_2 \right]. \]
Otherwise, compute \( q_1 \) and \( q_2 \) according to (4.8) and (4.9) and discard the one that does not satisfy the conditions of the lemma.

**Proof.** Let \( \alpha > 0 \) and

\[
\mathbf{c}(t; \alpha) = \left( \begin{array}{c} \cos t \\ \alpha \sin t \end{array} \right), \quad t \in \mathbb{R}.
\]

Vector function \( \mathbf{c}(t, \alpha) \) traverses the ellipse \( E(1, \alpha) \) centered in the origin with the \( x \) semiaxis equal to 1 and the \( y \) semiaxis equal to \( \alpha \). This function can be rewritten as

\[
\mathbf{c}(t; \alpha) = \mathbf{P}(\alpha) \mathbf{M}(t) \mathbf{e}_1,
\]

where

\[
\mathbf{M}(t) = \left( \begin{array}{cc} \cos t & -\sin t \\ \sin t & \cos t \end{array} \right), \quad \mathbf{P}(\alpha) = \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha \end{array} \right), \quad \mathbf{e}_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]

Note that \( \mathbf{M}(t) \) is the counterclockwise rotation in \( \mathbb{R}^2 \) by the angle \( t \), \( \mathbf{P}(\alpha) \) is the \( \alpha \)-contraction (expansion) in \( \mathbb{R}^2 \) along the \( y \) axis, and \( \mathbf{e}_1 \) is the standard basis vector along the \( x \) axis. Furthermore

(4.10) \[
\mathbf{c}(t_1 + t_2; \alpha) = \mathbf{P}(\alpha) \mathbf{M}(t_1) \mathbf{M}(t_2) \mathbf{e}_1.
\]

With this notation system (4.5)–(4.6) is

\[
\mathbf{A} \mathbf{e}(\omega_1(q - p) + \gamma_1; 1/\omega_1) = \mathbf{B} \mathbf{e}(\omega_2(q - p - \eta) + \gamma_2; 1/\omega_2)
\]

or

(4.11) \[
\mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1(q - p) + \gamma_1] \mathbf{A} \mathbf{e}_1 = \mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2(q - p - \eta) + \gamma_2] \mathbf{B} \mathbf{e}_1.
\]

If \( q \) is a solution of (4.5)–(4.6), then the vectors in the right and the left sides of (4.11) are identical. Thus they belong to the intersection of the ellipses \( E(A, A/\omega_1) \) and \( E(B, B/\omega_2) \), and this intersection is not empty. In general the ellipses intersect in four points: one in each quadrant.

Suppose that \( q^* \neq q \) is another solution of (4.5)–(4.6) on \( [p, p + \eta] \). We can assume that \( q^* = q + \tau \) for some \( \tau > 0 \), \( 0 < \omega_1 \tau, \omega_2 \tau < \pi \). System (4.5)–(4.6) at \( x = q^* \) is

(4.12) \[
\mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1(q + \tau - p) + \gamma_1] \mathbf{A} \mathbf{e}_1 = \mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2(q + \tau - p - \eta) + \gamma_2] \mathbf{B} \mathbf{e}_1.
\]

Using (4.10) and \( \mathbf{P}(\alpha) \mathbf{P}(\alpha^{-1}) = \mathbf{I} \) the right side of (4.12) can be written as

\[
\mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2(q + \tau - p - \eta) + \gamma_2] \mathbf{B} \mathbf{e}_1 = \mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2 \tau] \mathbf{M}[\omega_2(q - p - \eta) + \gamma_2] \mathbf{B} \mathbf{e}_1
\]

\[
= \mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2 \tau] \mathbf{P}(\omega_2) \mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2(q - p - \eta) + \gamma_2] \mathbf{B} \mathbf{e}_1
\]

\[
= \mathbf{P}(\omega_2^{-1}) \mathbf{M}[\omega_2 \tau] \mathbf{P}(\omega_2) \mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1(q - p) + \gamma_1] \mathbf{A} \mathbf{e}_1.
\]

Similarly the left side of (4.12) can be written as

\[
\mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1(q + \tau - p) + \gamma_1] \mathbf{A} \mathbf{e}_1 = \mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1 \tau] \mathbf{M}[\omega_1(q - p) + \gamma_1] \mathbf{A} \mathbf{e}_1
\]

\[
= \mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1 \tau] \mathbf{P}(\omega_1) \mathbf{P}(\omega_1^{-1}) \mathbf{M}[\omega_1(q - p) + \gamma_1] \mathbf{A} \mathbf{e}_1.
\]
Let \( v = P(\omega_1^{-1})M[\omega_1(q - p) + \gamma_1]Ae_1 \) and
\[
D = P(\omega_2^{-1})M[\omega_2\tau]P(\omega_2) - P(\omega_1^{-1})M[\omega_1\tau]P(\omega_1).
\]

Then (4.12) is \( Dv = 0 \). Since \( v \neq 0 \) we must have \( \det(D) = 0 \). Note that
\[
\det(D) = \frac{1}{\omega_1\omega_2} \left[ 2\omega_1\omega_2 - (\omega_1^2 + \omega_2^2)\sin\omega_1\tau \sin\omega_2\tau - 2\omega_1\omega_2 \cos\omega_1\tau \cos\omega_2\tau \right]
\begin{align*}
&= \frac{1}{\omega_1\omega_2} \left[ 2\omega_1\omega_2 - \frac{1}{2} (\omega_1 + \omega_2)^2 \cos(\omega_1 - \omega_2)\tau + \frac{1}{2} (\omega_1 - \omega_2)^2 \cos(\omega_1 + \omega_2)\tau \right].
\end{align*}
\]

Let us define \( f(\tau) \) on \([0, \pi/\omega_1] \cap [0, \pi/\omega_2] \) as
\[
f(\tau) = 2\omega_1\omega_2 + \frac{1}{2} (\omega_1 - \omega_2)^2 \cos(\omega_1 + \omega_2)\tau - \frac{1}{2} (\omega_1 + \omega_2)^2 \cos(\omega_1 - \omega_2)\tau.
\]

Function \( f \) is smooth on \((0, \pi/\omega_1) \cap (0, \pi/\omega_2)\), and its first and second derivatives are
\[
\begin{align*}
f' (\tau) &= -\frac{1}{2} (\omega_1 - \omega_2)^2 (\omega_1 + \omega_2) \sin(\omega_1 + \omega_2)\tau + \frac{1}{2} (\omega_1 + \omega_2)^2 (\omega_1 - \omega_2) \sin(\omega_1 - \omega_2)\tau, \\
f''(\tau) &= (\omega_1 - \omega_2)^2 (\omega_1 + \omega_2)^2 \sin(\omega_1\tau) \sin(\omega_2\tau).
\end{align*}
\]

Since \( f(0) = f'(0) = 0 \), \( f''(\tau) > 0 \), and \( f'(\tau) > 0 \) on \((0, \pi/\omega_1) \cap (0, \pi/\omega_2)\), we conclude that \( f(\tau) > 0 \) for all \( \tau \in (0, \pi/\omega_1) \cap (0, \pi/\omega_2) \). Thus \( \det(D) = 0 \) if and only if \( \tau = 0 \). This contradicts the assumption \( \tau > 0 \). Therefore the solution \( q \) of (4.5)–(4.7) is unique on \([p, p + \eta] \).

To obtain formulas (4.8) and (4.9) notice that the ellipses \( \mathcal{E}(A, A/\omega_1) \) and \( \mathcal{E}(B, B/\omega_2) \) are given by
\[
x^2 + \omega_1^2 y^2 = A^2 \quad \text{and} \quad x^2 + \omega_2^2 y^2 = B^2.
\]

At the intersection points we have
\[
y^2 = \frac{B^2 - A^2}{\omega_2^2 - \omega_1^2} \quad \text{and} \quad x^2 = \frac{A^2 \omega_2^2 - B^2 \omega_1^2}{\omega_2^2 - \omega_1^2}.
\]

The polar angle of the intersection point in the first quadrant is
\[
\zeta = \arctan \sqrt{\frac{B^2 - A^2}{A^2 \omega_2^2 - B^2 \omega_1^2}}.
\]

Since \( w(q) = v(q) > 0 \) the intersection points corresponding to the solution \( q \) are in either the first or the fourth quadrants, and \( 0 \leq \zeta < \pi/2 \).

If \( w'(p) \leq 0 \), then \( \gamma_1 \geq 0 \). Therefore \( 0 \leq \gamma_1 \leq \omega_1(q - p) + \gamma_1 \). In this case the intersection point is in the first quadrant. Accordingly \( \tan[\omega_1(q - p) + \gamma_1] = \omega_1 \tan \zeta \). Thus
\[
q = p + \frac{1}{\omega_1} \left[ \arctan \left( \omega_1 \sqrt{\frac{B^2 - A^2}{A^2 \omega_2^2 - B^2 \omega_1^2}} \right) - \gamma_1 \right].
\]

If \( v'(p+\eta) \geq 0 \), then \( \gamma_2 \leq 0 \) and \( \omega_2(q - p - \eta) + \gamma_2 \leq \gamma_2 \leq 0 \). In this case the intersection point is in the fourth quadrant. Accordingly \( \tan[\omega_2(q - p - \eta) + \gamma_2] = -\omega_2 \tan \zeta \) and one gets (4.9). \( \blacksquare \)
Now we would like to define a class of piecewise constant conductivities with sufficiently separated points of discontinuity.

**Definition 4.5.** By the definition of \( a \in \mathcal{PC} \) there exists \( N \in \mathbb{N} \) and a finite sequence \( 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1 \) such that \( a \) is a constant on each subinterval \( (x_{n-1}, x_n), n = 1, \ldots, N \). Let \( \sigma > 0 \). Define

\[
\mathcal{PC}(\sigma) = \{ a \in \mathcal{PC} : x_n - x_{n-1} \geq \sigma, \ n = 1, 2, \ldots, N \}.
\]

Note that \( a \in \mathcal{PC}(\sigma) \) attains at most \( N = \lfloor 1/\sigma \rfloor \) distinct values \( a_i, 0 < \nu \leq a_i \leq \mu \).

The following theorem is our main result. It describes and justifies the marching algorithm for the unique identification of piecewise constant conductivities in the class \( \mathcal{PC}(\sigma) \).

**Theorem 4.6.** Given \( \sigma > 0 \) let an integer \( M \) be such that

\[
M \geq \frac{3}{\sigma} \quad \text{and} \quad M > 2\sqrt{\frac{\mu}{\nu}}.
\]

Suppose that the initial data \( g(x) > 0, \ 0 < x < 1 \), and the observations \( z_m(t) = u(p_m, t; a), p_m = m/M \) for \( m = 1, 2, \ldots, M - 1 \) and \( 0 \leq T_1 < t < T_2 \) of the heat conduction process (4.1) are given. Then the conductivity \( a \in \mathcal{A}_{ad} \) is constructively identifiable in the class of piecewise constant functions \( \mathcal{PC}(\sigma) \).

First, we present the marching algorithm for the unique identification of the conductivity \( a \) and then justify it. The algorithm marches from the left end \( x = 0 \) to a certain observation point \( p_1 \in (0, 1) \) and identifies the values \( a_n \) and the discontinuity points \( x_n \) of the conductivity \( a \) on \( [0, p_1] \). Then the algorithm marches from the right end point \( x = 1 \) to the left until it reaches the observation point \( p_{M+1} \in (0, 1) \) identifying the values and the discontinuity points of \( a \) on \( [p_{M+1}, 1] \). Finally, the values of \( a \) and its discontinuity are identified on the interval \( [p_{M+1}, 1] \). The overall goal of the algorithm is to determine the number \( N - 1 \) of the discontinuities of \( a \) on \( [0, 1] \), the discontinuity points \( x_n, \ n = 1, 2, \ldots, N - 1 \), and the values \( a_n \) of \( a \) on \( [x_{n-1}, x_n], \ n = 1, 2, \ldots, N \) \((x_0 = 0, x_N = 1)\). As a part of the process the algorithm determines certain functions \( H_n(x) \) defined on intervals \( [x_{n-1}, x_n], \ n = 1, 2, \ldots, N \). The resulting function \( H(x) \) defined on \([0, 1]\) is a multiple of the first eigenfunction \( v_1 \).

**Marching Algorithm.**

(i) Represent the data \( z_m(t) \) as

\[
z_m(t) = \sum_{k=1}^{\infty} c_{k,m} e^{-\lambda_k t}, \quad m = 1, 2, \ldots, M - 1, \quad 0 \leq T_1 < t < T_2,
\]

and use it to uniquely identify the first eigenvalue \( \lambda_1 \) and the coefficients \( G_m = c_{1,m}, \ m = 1, 2, \ldots, M - 1 \). Let \( G_0 = G_M = 0 \).

(ii) Find \( l, \ 0 < l < M \), such that \( G_l = \max\{G_m : m = 1, 2, \ldots, M - 1\} \) and \( G_m < G_l \) for any \( 0 \leq m < l \).

(iii) Let \( i = 1, \ m = 0 \).

(iv) Use Lemma 4.2 to find \( A_i, \ \omega_i, \) and \( \gamma_i \) from the system

\[
A_i \cos(\omega_i \delta - \gamma_i) = G_m,
\]

\[
A_i \cos \gamma_i = G_{m+1},
\]

\[
A_i \cos(\omega_i \delta + \gamma_i) = G_{m+2}.
\]

Let

\[
H_i(x) = A_i \cos(\omega_i(x - p_{m+1}) + \gamma_i).
\]
(v) If \( m + 3 \geq l \), then go to step (viii). If \( H_i(p_{m+3}) \neq G_{m+3} \), or \( H_i(p_{m+3}) = G_{m+3} \) and \( H'_i(p_{m+3}) \leq 0 \), then \( a \) has a discontinuity \( x_i \) on interval \([p_{m+2}, p_{m+3})\). Proceed to the next step (vi). If \( H_i(p_{m+3}) = G_{m+3} \) and \( H'_i(p_{m+3}) > 0 \), then let \( m := m + 1 \) and repeat this step (v).

(vi) Use Lemma 4.2 to determine the nonzero coefficients in (4.16) and the corresponding exponents. In (4.17) each observation \( v_i \) determines the corresponding functions \( H_i \) on \([0, 1]\), and the above parameters \( \Omega_1, \Omega_2, \gamma_1, \gamma_2 \) required in Lemma 4.4.

(vii) Use the formulas in Lemma 4.4 to find the unique discontinuity point \( x_i \in [p_{m+2}, p_{m+3}) \). The parameters and functions used in Lemma 4.4 are defined as follows. Let \( p = p_{m+2} \), \( \eta = \delta \). To avoid confusion we are going to use the notation \( \Omega_1, \Omega_2, \gamma_1, \gamma_2 \) for the corresponding parameters \( \omega_1, \omega_2, \gamma_1, \gamma_2 \) required in Lemma 4.4. Let \( \Omega_1 = \omega_1, \Omega_2 = \omega_i + 1 \). For \( w(x) \) use function \( H_i(x) \) recentered at \( p = p_{m+2} \); i.e., rewrite \( H_i(x) \) in the form

\[
H_i(x) = A_{i+1} \cos(\omega_i(x - p_{m+4}) + \gamma_i + 1).
\]

Let

\[
H_{i+1}(x) = A_{i+1} \cos(\omega_i + 1(x - p_{m+4}) + \gamma_i + 1).
\]

For \( v(x) \) use function \( H_{i+1} \) centered at \( p + \eta = p_{m+3} \); i.e.,

\[
v(x) = H_{i+1}(x) = B \cos(\omega_2(x - p_{m+3}) + \Gamma_2), \quad |\Gamma_2| < \pi/2.
\]

Let \( i := i + 1, m := m + 3 \). If \( m < l \), then return to step (v). If \( m \geq l \), then go to the next step (viii).

(viii) Do steps (iii)–(vii) in the reverse direction of \( x \), advancing from \( x = 1 \) to \( x = p_{l+1} \). Identify the values and the discontinuity points of \( a \) on \([p_{l+1}, 1]\), and determine the corresponding functions \( H_i \).

(ix) Using the notation introduced in (vii) let \( H_j \) be the previously determined function \( H \) on interval \([p_{l-2}, p_{l-1}]\). Recenter it at \( p = p_{l-1} \); i.e., \( w(x) = H_j(x) = A \cos(\Omega_1(x - p_{l-1}) + \Gamma_1) \). Let \( H_{j+1}(x) \) be the previously determined function \( H \) on interval \([p_{l+1}, p_{l+2}]\). Recenter it at \( p_{l+1} \); i.e., \( v(x) = H_{j+1}(x) = B \cos(\Omega_2(x - p_{l+1}) + \Gamma_2) \). If \( \Omega_1 = \Omega_2 \), then stop; otherwise, use Lemma 4.4 with \( \eta = 2\delta \) and the above parameters to find the discontinuity \( x_j \in (p_{l-1}, p_{l+1}] \). Stop.

Proof. To prove Theorem 4.6 we need to justify the marching algorithm and to show the uniqueness of the identification in each step.

(i) Using Theorem 2.3(vi) we get

\[
z_m(t) = \sum_{k=1}^{\infty} g_k e^{-\lambda_1 t} v_k(p_m), \quad m = 1, 2, \ldots, M - 1, \quad 0 \leq T_1 < t < T_2,
\]

where \( g_k = (g, v_k) \) for \( k = 1, 2, \ldots \). By Theorem 2.3(iv) \( v_1(x) > 0 \) on interval \((0, 1)\).

Since \( g \) is positive on \((0, 1)\) we conclude that \( g_1 v_1(p_m) > 0 \). According to Theorem 2.3(vii) each observation \( z_m(t) \) is a real analytic function. Thus one can uniquely determine the nonzero coefficients in (4.16) and the corresponding exponents. In particular, one determines the first eigenvalue \( \lambda_1 \) and the values of

\[
G_m = g_1 v_1(p_m) > 0, \quad p_m = m/M, \quad m = 1, 2, \ldots, M - 1.
\]
Because of the zero boundary conditions we can let \( G_0 = G_M = 0 \). The crucial point is that the numbers \( \{G_m\}_{m=1}^{M-1} \) are not arbitrary but are the values (up to a nonzero multiplicative constant \( g_1 \)) of the still-undetermined eigenfunction \( v_1 \).

(ii) Let index \( l \) be defined as in (ii) of the marching algorithm. By Theorem 2.3(iii) there exists a unique point \( q^* \) of maximum of \( v_1 \) on \((0, 1)\). Note that \( q^* \in (p_{l-1}, p_{l+1}) \). Thus \( G_{l+1} \leq G_l \) and \( G_m < G_l \) for \( m > l + 1 \). Also \( v_1'(p_m -) > 0 \) for \( m = 1, 2, \ldots, l - 1 \) and \( v_1'(p_m -) < 0 \) for \( m = l + 1, l + 2, \ldots, M - 1 \).

(iii) Start at the left end point \( p_0 = 0 \) and work on interval \([0, x_1]\), where \( x_1 \) is the first discontinuity point of \( a \).

(iv) Let \( \delta = 1/M \). Since \( \sigma \geq 3\delta \) and \( a \in \mathcal{PC}(\sigma) \) we conclude that \( [0, p_2] \subset [0, x_1] \) and \( a = a_1 \) on \([0, x_1]\). To apply Lemma 4.2 we just need to check the conditions for \( Q_1, Q_2, Q_3 \) required there.

We have \( Q_1 = G_0 = 0 \), \( Q_2 = G_1 = g_1 v_1(p_1) > 0 \), \( Q_3 = G_2 = g_1 v_1(p_2) > 0 \). Let

\[
\omega_1 = \sqrt{\frac{\lambda_1}{a_1}}
\]

By Theorem 2.3(iii) \( 0 < \lambda_1 \leq \mu \pi^2 \). Since \( 0 < \nu \leq a_1 \) we have

\[
0 < \omega_1 \delta < \sqrt{\frac{\mu \pi^2}{\nu} \frac{1}{2}} \sqrt{\frac{\nu}{\mu}} = \frac{\pi}{2}.
\]

This inequality and \( v_1(x) > 0 \) on \((0, 1)\) imply that the first eigenfunction \( v_1 \) of (4.2) can be represented on \((0, x_1)\) as

\[
v_1(x) = C_1 \cos(\omega_1(x - p_1) + \gamma_1)
\]

for some \((C_1, \omega_1, \gamma_1) \in \Gamma\), where \( \Gamma \) was defined in Lemma 4.2.

Also \( Q_1 + Q_3 = g_1 C_1 (\cos(\omega_1 \delta + \gamma_1) + \cos(\omega_1 \delta - \gamma_1)) = 2g_1 C_1 \cos(\omega_1 \delta) \cos \gamma_1 < 2G_1 = 2Q_2 \) since \( 0 < \omega_1 \delta < \pi/2 \); hence \( 0 < \cos(\omega_1 \delta) < 1 \). Now Lemma 4.2 guarantees a unique solution of the system

\[
\begin{align*}
&g_1 C_1 \cos(\omega_1 \delta - \gamma_1) = G_0, \\
&g_1 C_1 \cos \gamma_1 = G_1, \\
&g_1 C_1 \cos(\omega_1 \delta + \gamma_1) = G_2.
\end{align*}
\]

It also gives formulas for the computation of \( A_1 = g_1 C_1, \ \gamma_1, \ \omega_1 \) from the known values of \( G_0, G_1, \) and \( G_2 \). Thus one can determine \( a_1 = \lambda_1/\omega_1^2 \) and obtain

\[
H_1(x) = g_1 C_1 \cos(\omega_1(x - p_1) + \gamma_1) = g_1 v_1(x)
\]

for \( x \in [0, x_1] \).

(v) Let \( a_i \) be the value of \( a \) on the part of the interval \([p_{m+1}, p_{m+2})\) adjacent to \([p_{m+2}, p_{m+3})\). By construction this value and the associated function \( H_i(x) = g_1 v_1(x) \) are already determined by the algorithm. If there is no discontinuity of \( a \) on interval \([p_{m+2}, p_{m+3})\), then \( a \) has the same value \( a_i \) on interval \([p_{m+2}, p_{m+3})\) as well. Therefore \( H_i(x) = g_1 v_1(x) \) on this interval, and we must have \( G_{m+3} = H_i(p_{m+3}) \) by (4.17). If one has \( G_{m+3} \neq H_i(p_{m+3}) \), then the implication is that there is a discontinuity of \( a \) on \([p_{m+2}, p_{m+3})\), and one proceeds to step (vi).

On the other hand, if \( G_{m+3} = H_i(p_{m+3}) \), then one cannot, in general, conclude that there is no discontinuity of \( a \) on \([p_{m+2}, p_{m+3})\). However, since we have \( m + 3 < l \)
then (ii) of the proof implies that \( v_i'(p_{m+3}^-) > 0 \). Then the assumption \( a = a_i \) on \( [p_{m+2}, p_{m+3}] \) implies \( H'_i(p_{m+3}) = g_1 v_i'(p_{m+3}^-) \). Therefore the equality \( G_{m+3} = H'_i(p_{m+3}) \) together with \( H'_i(p_{m+3}) \leq 0 \) lead to a contradiction. The conclusion is that \( G_{m+3} = H_i(p_{m+3}) \) and \( H'_i(p_{m+3}) \leq 0 \) imply a discontinuity of \( a \) on \( [p_{m+2}, p_{m+3}] \), and one proceeds to step (vi).

Finally, one uses Lemma 4.3 to conclude that \( m + 3 < l \), \( G_{m+3} = H_i(p_{m+3}) \), and \( H'_i(p_{m+3}) > 0 \) imply that there is no discontinuity of \( a \) on \( [p_{m+2}, p_{m+3}] \). Indeed, suppose that there is a discontinuity point \( x_i \) of \( a \) on interval \( [p_{m+2}, p_{m+3}] \). Then \( a = a_i \) on \( [p_{m+1}, x_i] \) and \( a = a_{i+1} \) on \( [x_i, p_{m+3}] \). We are going to use the notation \( x_i, \Omega, \Omega' \) for the corresponding variables \( x_1, \omega_1 \), and \( \omega_2 \) used in Lemma 4.3. Let \( p = p_{m+2}, p + \delta = p_{m+3} \), \( \Omega_1 = \sqrt{\lambda_1/a_1} \), \( \Omega_2 = \sqrt{\lambda_1/a_{i+1}} \), and

\[
\begin{align*}
    w(x) &= H_i(x) = g_1 v_1(x) = A_1 \cos \Omega_1 x + B_1 \sin \Omega_1 x, \quad x \in [p, x_i), \\
    v(x) &= g_1 v_1(x) = A_2 \cos \Omega_2 x + B_2 \sin \Omega_2 x, \quad x \in [x_i, p + \delta].
\end{align*}
\]

Note that the condition \( \Omega_1^2 v'(x_i) = \Omega_2^2 w'(x_i) \) is just the matching condition (4.2) at \( x = x_i \). Since \( m + 3 < l \), the maximum \( q^* \) of \( v_1 \) satisfies \( q^* > p_{m+3} \). Because \( w \) is a positive multiple of \( v_1 \), it implies \( w(x_i) > 0 \) and \( w'(x_i) > 0 \). Therefore \( v(x_i) > 0 \) and \( v'(x_i) > 0 \). Because \( v \) is a positive multiple of \( v_1 \), we have \( v'(p + \delta) > 0 \). The condition \( v(p + \delta) = w(p + \delta) \) means \( v(p_{m+3}) = g_1 v_1(p_{m+3}) = G_m = H_i(p_{m+3}) = w(p_{m+3}) \).

Suppose that \( \Omega_1 < \Omega_2 \). We have \( v(p + \delta) = w(p + \delta) \) and \( v'(p + \delta) > 0 \). According to Lemma 4.3(i), this is impossible.

Suppose that \( \Omega_1 > \Omega_2 \). We have \( v(p + \delta) = w(p + \delta) \) and \( v'(p + \delta) = H_i'(p_{m+3}) > 0 \). According to Lemma 4.3(ii), this is also impossible.

Thus the conclusion is that there is no point of discontinuity of \( a \) on \( [p_{m+2}, p_{m+3}] \) in this case. By assigning \( m := m + 1 \) one advances to the next observation interval \( [p_{m+3}, p_{m+4}] \) and repeats the analysis of (v).

(vi) Since it is already determined that there is a discontinuity point on interval \( [p_{m+2}, p_{m+3}] \), the assumption \( a \in \mathcal{PC}(\sigma) \) implies that \( a \) is constant on \( [p_{m+3}, p_{m+5}] \). This value \( a_{i+1} \) of \( a \) can be uniquely determined from the system in (vi) similarly to the argument presented in (iv). Note that \( H_i+1(x) = g_1 v_1(x) \) on \( [p_{m+3}, p_{m+5}] \).

(vii) One knows that the discontinuity \( x_i \in [p_{m+2}, p_{m+3}] \) as well as the values \( a_i \) and \( a_{i+1} \) of \( a \) on the adjacent intervals \( [p_{m+1}, p_{m+2}] \) and \( [p_{m+3}, p_{m+4}] \) together with the corresponding functions \( H_i(x) \) and \( H_{i+1}(x) \). According to Lemma 4.4 one can determine the unique location of the discontinuity \( x_i \) by the formulas given there.

(viii) The advance of the algorithm from \( x = 1 \) to \( x = p_1+1 \) is justified by reducing it to (iii)–(vii) using the change of variables \( z = 1 - x \).

(ix) Lemma 4.4 is applicable with \( \eta = 2\delta \). Note that there can be only one discontinuity of \( a \) on \( [p_{i-1}, p_{i+1}] \), since \( 2\delta < \sigma \). The values of \( a \) as well as the corresponding functions \( H_j \) and \( H_{j+1} \) are already known on the adjacent intervals. The discontinuity of \( a \) exists on \( [p_{i-1}, p_{i+1}] \) if \( \omega_j \neq \omega_{j+1} \).

The marching algorithm of Theorem 4.6 requires measurements of the system at a possibly large number of observation points. Our next theorem shows that if a piecewise constant conductivity \( a \) is known to have just one point of discontinuity \( x_1 \), and its values \( a_1 \) and \( a_2 \) are known beforehand, then the discontinuity point \( x_1 \) can be determined from just one measurement of the heat conduction process.

**THEOREM 4.7.** Let \( p \in (0, 1) \) be an observation point, \( g(x) > 0 \) on \((0, 1)\), and the observation \( z_p(t) = u(x_p, t; a), t \in (T_1, T_2), \) of the heat conduction process (4.1) be given. Suppose that the conductivity \( a \in A_{ad} \) is piecewise constant and has only one (unknown) point of discontinuity \( x_1 \in (0, 1) \). Given positive values \( a_1 \neq a_2 \) such that
a(x) = a₁ for 0 ≤ x < x₁ and a(x) = a₂ for x₁ ≤ x < 1, the point of discontinuity x₁ is constructively identifiable.

Proof. Arguing as in the previous theorem,

$$z_p(t) = \sum_{k=1}^{\infty} g_k e^{-\lambda_k t} v_k(p), \quad 0 \leq T_1 < t < T_2,$$

where $g_k = \langle g, v_k \rangle$ for $k = 1, 2, \ldots$. Since $g_1 v_1(p) > 0$, the uniqueness of the Dirichlet series representation implies that one can uniquely determine the first eigenvalue $\lambda_1$ and the value of $G_p = g_1 v_1(p)$.

Without loss of generality one can assume that $a_1 > a_2$. In this case we show that the first eigenvalue $\lambda_1$ is strictly increasing as a function of $x_1 \in [0, 1]$. Indeed, suppose that

$$0 \leq x_1^a < x_1^b \leq 1;$$

that is,

$$a(x) = \begin{cases} a_1, & 0 < x < x_1^a \\ a_2, & x_1^a < x < 1 \end{cases} \quad \text{and} \quad b(x) = \begin{cases} a_1, & 0 < x < x_1^b \\ a_2, & x_1^b < x < 1 \end{cases}.$$

By Theorem 2.3(i)

$$\lambda_1^a = \frac{\int_0^1 b(x) [v_{1,b}^a(x)]^2 dx}{\int_0^1 [v_{1,b}^a(x)]^2 dx} > \frac{\int_0^1 a(x) [v_{1,b}^a(x)]^2 dx}{\int_0^1 [v_{1,b}^a(x)]^2 dx} \geq \inf_{v \in H_2^1(0,1)} \frac{\int_0^1 a(x) [v'(x)]^2 dx}{\int_0^1 [v(x)]^2 dx} = \lambda_1^a$$

provided that the derivative $v_{1,b}^a(x)$ of the first eigenfunction $v_{1,b}(x)$ is not identically zero on $(x_1^a, x_1^b)$. But, from $(b(x)v_{1,b}^a(x)')' = -\lambda_1^a v_{1,b}(x)$, the assumption $v_{1,b}^a(x) = 0$ on $(x_1^a, x_1^b)$ implies $v_{1,b}(x) = 0$ on $(x_1^a, x_1^b)$, and this is impossible, since $v_{1,b}(x) > 0$ on $(0, 1)$. Thus there exists a unique conductivity of the type sought in the theorem for which its first eigenvalue is equal to $\lambda_1$; i.e., $a$ is identifiable.

Now the unique discontinuity point $x_1$ of $a$ can be determined as follows. Let

$$\omega_1 = \sqrt{\frac{\lambda_1}{a_1}}, \quad \omega_2 = \sqrt{\frac{\lambda_1}{a_2}}.$$

Then the first eigenfunction $v_1$ is given by

$$(4.20) \quad v_1(x) = \begin{cases} A \sin \omega_1 x, & 0 < x < x_1, \\ B \sin \omega_2 (1 - x), & x_1 < x < 1, \end{cases}$$

for some $A, B > 0$. The matching conditions at $x_1$ give

$$A \sin \omega_1 x_1 = B \sin \omega_2 (1 - x_1) \quad \text{and} \quad \frac{A}{\omega_1} \cos \omega_1 x_1 = \frac{B}{\omega_2} \cos \omega_2 (1 - x_1).$$

Since $v_1(x_1) > 0$ we have $0 < \omega_1 x_1 < \pi$ and $0 < \omega_2 (1 - x_1) < \pi$. Therefore $x_1$ satisfies

$$\frac{1}{\omega_1} \cot \omega_1 x_1 = \frac{1}{\omega_2} \cot \omega_2 (1 - x).$$

The existence and the uniqueness of the solution $x_1$ of the above nonlinear equation follows from the monotonicity and the continuity of the cotangent functions. Practically, the value of $x_1$ can be found by a numerical method. \qed
5. Conclusions. The prevalent approach to parameter identification (estimation) problems is to find such parameters from the best fit to data minimization. However, such an approach usually does not guarantee the uniqueness of the identified parameters. The identifiability problem consists of finding sufficient conditions assuring such a uniqueness, and there have been just a few results for the identifiability in distributed parameter systems.

In this paper we have shown that in some cases a variable conductivity in a 1D heat conduction process can be uniquely identified from observations of this process. The identifiability has been established for two sets of observations. In one case it is assumed that the conductivity is piecewise smooth, and we are given a sequence of distributed observations of the form \( z_n(t) = \langle u(x, t; a), \psi_n \rangle \) for \( n = 1, 2, \ldots \) on a finite time interval, where functions \( \{ \psi_N \}_{n=1}^{\infty} \) form a basis in \( H = L^2(0, 1) \). An algorithm for the conductivity identification is proposed. Its numerical study will be reported elsewhere.

In the second case it is assumed that the conductivity is piecewise constant with sufficiently separated points of discontinuity. The observations of the process are taken at equidistant points \( p_m \in (0, 1) \). The total number of points needed for the unique conductivity identification can be computed from a priori known parameters of the process. A marching algorithm for the conductivity identification is presented and justified.

In both cases the plant does not require a special external input for its identifiability; i.e., it is modeled by \( u_t = (au_x)_x \) rather than by \( u_t = (au_x)_x + f(x, t) \). It will be of interest to extend the developed methods to vibration and steady-state processes.

Our current research shows that the methods described in this paper can be extended to identifiability problems for heat conduction processes admitting various boundary (e.g., periodic) inputs and to other cases. A numerical implementation shows that the marching algorithm achieves a perfect identification for observations with low noise levels. These results will be presented elsewhere.

REFERENCES


