Two theorems concerning the Bannai–Ito conjecture

S. Bang\textsuperscript{a}, J.H. Koolen\textsuperscript{b}, V. Moulton\textsuperscript{c}

\textsuperscript{a} Com\textsuperscript{2}MaC, Pohang University of Science and Technology, Hyoja-dong, Namgu, Pohang 790-784, Republic of Korea
\textsuperscript{b} Department of Mathematics, Pohang University of Science and Technology, Hyoja-dong, Namgu, Pohang 790-784, Republic of Korea
\textsuperscript{c} School of Computing Sciences, University of East Anglia, Norwich NR4 7TJ, UK

Available online 2 October 2006

Abstract

In 1984 Bannai and Ito conjectured that there are finitely many distance-regular graphs with fixed valencies greater than two. In a series of papers, they showed that this is the case for valency 3 and 4, and also for the class of bipartite distance-regular graphs. To prove their result, they used a theorem concerning the intersection array of a triangle-free distance-regular graph, a theorem that was subsequently generalized in 1994 by Suzuki to distance-regular graphs whose intersection numbers satisfy a certain simple condition. More recently, Koolen and Moulton derived a more general version of Bannai and Ito’s theorem which they used to show that the Bannai–Ito conjecture holds for valencies 5, 6 and 7, and which they subsequently extended to triangle-free distance-regular graphs in order to show that the Bannai–Ito conjecture holds for such graphs with valencies 8, 9 and 10. In this paper, we extend the theorems of Bannai and Ito, and Koolen and Moulton to arbitrary distance-regular graphs.

1. Introduction

In [2, p. 237], Bannai and Ito conjectured that there are finitely many distance-regular graphs with a fixed valency \( k \), \( k \geq 3 \), and in the series of papers [3–6] they showed that their conjecture holds for \( k = 3, 4 \), and also for the class of bipartite distance-regular graphs. In [10], it was shown that there are finitely many distance-regular graphs with \( k = 5, 6, 7 \), and, extending this result, it was shown in [11] that there are finitely many triangle-free distance-regular graphs with \( k = 8, 9, 10 \).
Suppose that $G$ is a distance-regular graph with valency $k \geq 3$, diameter $D \geq 2$ and intersection numbers $a_i, b_i, c_i$, $i = 0, 1, \ldots, D$. Let $h = h(G)$ be the number of terms at the beginning of this sequence that equal $(1, a_1, b_1)$, and $t = t(G)$ be the number of terms at the end of this sequence that equal $(b_1, a_1, 1)$. Note that $h \geq 1$, and $t \geq 0$. Bannai and Ito showed that their conjecture holds for $k = 3, 4$ by proving that there are finitely many triangle-free distance-regular graphs with $k$ and $D - h - t$ fixed $[5, 6]$. In $[12]$, this was generalized by Suzuki to include distance-regular graphs that are not necessarily triangle-free, but that satisfy $k \geq (a_1 + 1)(a_1 + 2)$. In this paper, we will prove that this latter condition is unnecessary. In particular, we will prove the following theorem.

**Theorem 1.1.** Suppose that $k \geq 3$ and $C \geq 1$ are fixed integers. Then there are finitely many distance-regular graphs $G$ with valency $k$, head $G$, tail $G$, and diameter $D$ that satisfy

$$D - (h(G) + t(G)) \leq C.$$ 

The proof of the Bannai–Ito conjecture for $k = 5, 6, 7$ in $[10]$ employs a generalization of this last result (Theorem 8.1), whose proof uses a combinatorial argument that is based on the Terwilliger Tree bound $[13]$. In $[11]$, an even more general result is shown to hold for triangle-free distance-regular graphs (Theorem 1.1), whose proof is based on algebraic arguments. Here, we extend this result to arbitrary distance-regular graphs. In particular, using algebraic arguments and Theorem 1.1 stated above, we will prove the following theorem.

**Theorem 1.2.** Suppose that $k \geq 3$ is a fixed integer. Then there exists a positive number $\epsilon = \epsilon(k)$, depending only on $k$, so that there are finitely many distance-regular graphs with valency $k$, head $G$, tail $G$, and diameter $D$ that satisfy

$$D - (h(G) + t(G)) \leq \epsilon h(G).$$

In forthcoming papers, we will use this last result to prove that there are finitely many distance-regular graphs with valency 10 or less, and also to prove that there are finitely many regular near polygons. We will also explore whether it is possible to use Theorem 1.2 to prove the Bannai–Ito conjecture for valencies greater than 10.

The proof of Theorem 1.2 is similar in spirit to that of $[11$, Theorem 1.1]. One of the main differences is that we will use the fact that $\beta_i := a_i + 2\sqrt{b_i c_i} \geq a_i + 2\sqrt{b_i}$ with equality holding if and only if $(c_i, a_i, b_i) \in \{(1, a_1, b_1), (b_1, a_1, 1)\}$ (cf. Lemma 2.2). For the case $a_1 = 0$, it is easy to show that this holds, but for $a_1 > 0$ it is less obvious. An important consequence of this fact is that the sequence $(\beta_1, \beta_2, \ldots, \beta_D)$ is unimodular, a fact that will be important for proving the future results mentioned above (including the triangle-free case).

In addition, we also have to overcome the fact that we are not able to bound the multiplicity of an eigenvalue of $G$ that is near to $-k/(a_1 + 1)$ in the case $k < (a_1 + 1)(a_1 + 2)$. We circumvent this problem using two different strategies, that are presented in the proofs of Theorems 1.1 and 1.2. Note that the strategy that we use in the proof of Theorem 1.2 can also be applied to prove Theorem 1.1, but not vice versa. Even so, we believe that it provides some useful insights for describing two different methods. Note that the polynomial that was used in $[11$, Section 3] is of a much simpler form than the one in Section 7. This is because we require $-k/(a_1 + 1)$ to be a root of the polynomial, and also for the polynomial to be symmetric around $a_1$.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries concerning distance-regular graphs and some new results concerning their intersection numbers.
In Section 3, we show that eigenvalues of a distance-regular graph with certain properties exist. In Section 4 we present some results concerning three-term relations, and in Section 5 we use these to bound multiplicities of eigenvalues of distance-regular graphs. We have kept the arguments presented in these last two sections as self-contained as possible, since we have to be very careful to check which variables certain constants depend upon. In Sections 6 and 7 we conclude the paper with the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

We begin this section by recalling some facts concerning distance-regular graphs (for more details see [7]). Suppose that $\Gamma$ is a connected graph. The distance $d(u, v)$ between any two vertices $u, v$ in the vertex set $V \Gamma$ of $\Gamma$ is the length of a shortest path between $u$ and $v$ in $\Gamma$. For any $v \in V \Gamma$, define $\Gamma_i(v)$ to be the set of vertices in $\Gamma$ at distance precisely $i$ from $v$, where $i$ is any non-negative integer not exceeding the diameter $D$ of $\Gamma$. In addition, define $\Gamma_{i-1}(v) = \Gamma_{D+1}(v) := \emptyset$.

Following [7], we call $\Gamma$ distance-regular if there are integers $b_i, c_i, i = 0, 1, \ldots, D$, such that for any two vertices $u, v \in V \Gamma$ at distance $i = d(u, v)$, there are precisely $c_i$ neighbors of $v$ in $\Gamma_{i-1}(u)$ and $b_i$ neighbors of $v$ in $\Gamma_{i+1}(u)$. In particular, $\Gamma$ is regular with valency $k := b_0$.

The numbers $c_i, b_i$ and

$$a_i := k - b_i - c_i \quad (i = 0, 1, \ldots, D),$$

the number of neighbors of $v$ in $\Gamma_i(u)$ for $d(u, v) = i$, are called the intersection numbers of $\Gamma$. Clearly $b_D = c_0 = a_0 = 0$ and $c_1 = 1$ and, as is shown in [7, Section 4.1], $\Gamma_i(u)$ contains exactly $k_i$ elements, where

$$k_0 := 1, \quad k_1 := k, \quad k_{i+1} := k_i b_i / c_i \quad (i = 0, \ldots, D - 1).$$

Moreover, as is shown in [7, Proposition 4.1.6], the following inequalities always hold

$$k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{D-1} > b_D = 0 \quad \text{and} \quad 1 = c_1 \leq c_2 \leq \cdots \leq c_D \leq k. \tag{2}$$

Note that a distance-regular graph $\Gamma$ is triangle-free (i.e. contains no 3-cycles) if and only if $a_1 = 0$.

Now, suppose that $k$ is an integer with $k \geq 3$ and that $\Gamma$ is a distance-regular graph with valency $k$, diameter $D \geq 2$ and intersection numbers $a_i, b_i, c_i, i = 0, 1, \ldots, D$. We call the sequence $((c_i, a_i, b_i) \mid i = 1, \ldots, D - 1)$ the tridiagonal sequence of $\Gamma$, and put

$$\Omega(\Gamma) := \{(c_i, a_i, b_i) \mid i = 1, \ldots, D - 1\},$$

the terms that appear in the tridiagonal sequence, and

$$\Omega^*(\Gamma) := \Omega(\Gamma) \setminus \{(1, a_1, b_1), (b_1, a_1, 1)\}.$$  

Also, given integers $a \geq 0$ and $b, c \geq 1$ with $a + b + c = k$, we define

$$\ell_{(c,a,b)}(\Gamma) := |\{i \mid i = 1, \ldots, D - 1 \text{ and } (c_i, a_i, b_i) = (c, a, b)\}|,$$

and put

$$h = h(\Gamma) := \ell_{(1,a_1,b_1)} \quad \text{and} \quad t = t(\Gamma) := \ell_{(b_1,a_1,1)}.$$  

Note that the first $h$ terms of the tridiagonal sequence of $\Gamma$ are all equal to $(1, a_1, b_1)$, and therefore we sometimes refer to $h$ as the head of $\Gamma$. Moreover, if $t \geq 1$ then, by Lemma 2.1(iii)
below, the last t terms of the tridiagonal sequence are equal to \((b_1, a_1, 1)\), and so we call t the tail of \(\Gamma\).

In the following lemma, we collect together several relationships between the intersection numbers of distance-regular graphs. Note that (i) in Lemma 2.1 is a restatement of [7, Proposition 4.1.6(ii)].

**Lemma 2.1.** Let \(\Gamma\) be a distance-regular graph with valency \(k \geq 3\), head \(h\), tail \(t\), and diameter \(D \geq 2\). Then the following hold.

(i) If \(i + j \leq D\), then \(c_i \leq b_j\).
(ii) For all \(i = 1, \ldots, D - 1\), we have \(a_i \geq a_1 + 1 - \min\{b_i, c_i\}\).
(iii) If \(t \geq 1\) then
\[
(c_{D-t}, a_{D-t}, b_{D-t}) = \cdots = (c_{D-1}, a_{D-1}, b_{D-1}) = (b_1, a_1, 1).
\]
(iv) \(h \geq t\).
(v) If \(t \geq 1\), then \(a_D \in \{a_1 + 1, 0\}\).

**Proof.** (ii) Let \(i \in \{1, \ldots, D - 1\}\), and let \(u, v, w\) be vertices of \(\Gamma\) with \(d(u, v) = i\), \(d(u, w) = i + 1\), and \(d(v, w) = 1\). Let \(\Delta\) be the set of common neighbors of \(v\) and \(w\). Then \(|\Delta| = a_1\). Since there are at most \(b_i - 1\) elements in \(\Delta\) at distance \(i + 1\) from \(u\), and at most \(a_i\) elements in \(\Delta\) at distance \(i\) from \(u\), it follows that \(a_i \geq a_1 + 1 - b_i\). It can be shown that \(a_i \geq a_1 + 1 - c_i\) holds in a similar fashion, from which (ii) immediately follows.

(iii) Suppose \(t \geq 1\) and \(i = 1, \ldots, t\). Then by (2) it follows that \(c_{D-t} \geq b_1\) and \(b_{D-t} \leq 1\), and therefore \(b_{D-i} = 1\) and \(c_{D-i} \geq b_1\). Hence \(a_{D-i} \leq a_1\). But by (ii), \(a_{D-i} \geq a_1\). (iii) now follows immediately.

(iv) Suppose \(t \geq 1\). By (i) and (iii) it follows that \(c_t \leq b_{D-t} = c_1\) and \(b_1 = c_{D-t} \leq b_t\). By (2), it follows that \((c_t, a_t, b_t) = (c_1, a_1, b_1)\), and hence \(h \geq t\).

(v) Suppose \(a_D \neq 0\). Then there are vertices \(u, v, w\) of \(\Gamma\) with \(d(u, v) = d(u, w) = D\) and \(d(v, w) = 1\). Since \(t \geq 1\), it follows that \(b_{D-1} = 1\). Therefore the common neighbors of \(v\) and \(w\) cannot have distance \(D - 1\) to \(u\), and thus \(a_D \geq a_1 + 1\). On the other hand, \(a_D \leq a_1 + 1\), since \(c_D \geq c_{D-1} = b_1\). Hence \(a_D = a_1 + 1\). \(\blacksquare\)

For \(k \geq 3\) an integer, define
\[
V_k := \{(c, a, b) \in \mathbb{N}_0^3 \mid b, c \geq 1, a \geq 0 \text{ and } a + b + c = k\},
\]
where \(\mathbb{N}_0\) denotes the set of non-negative integers. Note that if \(\Gamma\) is a distance-regular graph with diameter \(D \geq 2\) and valency \(k\), then \(\Omega(\Gamma) \subseteq V_k\). Now, for \(\lambda \in \mathbb{N}_0\) with \(\lambda \leq k - 2\) let \(V_{k,\lambda}\) be the subset of \(V_k\) defined by
\[
V_{k,\lambda} := \{(c, a, b) \in V_k \mid a \geq \max\{0, \lambda + 1 - b, \lambda + 1 - c\}\},
\]
and put
\[
V_{k,\lambda}^* := V_{k,\lambda} \setminus \{(1, \lambda, k - \lambda - 1), (k - \lambda - 1, \lambda, 1)\}.
\]
Note that \(V_{k,0} = V_k\), and that if \(\Gamma\) is a distance-regular graph with valency \(k\) and intersection number \(a_1 = \lambda\), then by Lemma 2.1(ii) it follows that \(\Omega(\Gamma) \subseteq V_{k,\lambda}\).

We now prove a further lemma concerning the intersection numbers of a distance-regular graph.
Lemma 2.2. Let $\Gamma$ be a distance-regular graph with valency $k \geq 3$, head $h$, tail $t$, diameter $D \geq 2$, and intersection numbers $a_i, b_i, c_i, i = 0, 1, \ldots, D$. For $i = 1, \ldots, D - 1$, let
\[ \beta_i := a_i + 2\sqrt{b_i c_i}. \]

Then the following statements hold:

(i) If $b_{i+1} \geq c_{i+1}$, then $\beta_i \leq \beta_{i+1}$ for $i = 1, \ldots, D - 2$, and if $b_{i+1} \leq c_{i+1}$, then $\beta_{i+2} \leq \beta_{i+1}$ for $i = 1, \ldots, D - 3$.

(ii) The inequality $\beta_i \geq \beta_1$ holds for all $i = 1, \ldots, D - 1$, with equality holding if and only if $(c_i, a_i, b_i) \in \{(1, a_1, b_1), (b_1, a_1, 1)\}$.

(iii) The inequality $\beta_i \geq a_1 + 1 + 2\sqrt{b_1 - 1}$ holds for all $i = h + 1, \ldots, D - t - 1$.

Proof. We begin by making an elementary observation. Suppose that $a, b, c$ are non-negative integers with $b \geq c \geq 1$. Then for any non-negative integers $x, y, z$ with $c \geq x$,
\[ a + x + 2\sqrt{b(c - x)} \leq a + 2\sqrt{bc} \] (3)
and
\[ a - y + 2\sqrt{(b + y)c} \leq a + 2\sqrt{bc}. \] (4)

We now prove that (i) holds. Suppose that $b_{i+1} \geq c_{i+1}$ holds. Then $b_i \geq b_{i+1} \geq c_{i+1} \geq c_i$. Therefore, by applying (3) and (4), it immediately follows that $\beta_i \leq \beta_{i+1}$ holds. Similarly, if $b_{i+1} \leq c_{i+1}$ then it follows that $\beta_{i+2} \leq \beta_{i+1}$. Hence, (i) holds.

We now prove that (ii) and (iii) both hold. For each $(c, a, b) \in V_{k, a_1}$, we will show that
\[
\begin{cases}
    a + 2\sqrt{bc} = a_1 + 2\sqrt{b_1} & \text{if } (c, a, b) \in \{(1, a_1, b_1), (b_1, a_1, 1)\}, \\
    a + 2\sqrt{bc} \geq a_1 + 1 + 2\sqrt{b_1 - 1} & \text{otherwise}.
\end{cases}
\] (5)

Without loss of generality, assume $b \geq c$. Then, since $(c, a, b) \in V_{k, a_1}$, we have $a + c = \min\{a + b, a + c\} \geq a_1 + 1$.

First, suppose $a + c = a_1 + 1$. In the case $c = 1$, it follows that $(c, a, b) = (c_1, a_1, b_1)$, and so (5) holds. If $c \geq 2$, then since $b \geq c$, $b = b_1 \geq 2$, and since $a + c \geq 2$, $a_1 \geq 1$. Therefore, applying (3) with $x = c - 2$,
\[ a + 2\sqrt{bc} \geq a + (c - 2) + 2\sqrt{2b} = a_1 - 1 + 2\sqrt{2b_1} \geq a_1 + 1 + 2\sqrt{b_1 - 1}. \] (6)

Now, suppose $a + c \geq a_1 + 2$. Then $b_1 > 1$ and $b \leq b_1 - 1$. By applying (3) with $x = c - 1$, and (4) with $y = b_1 - b - 1$, we obtain
\[ a + 2\sqrt{bc} \geq a + (c - 1) + 2\sqrt{b} \geq a + (c - 1) - (b_1 - b - 1) + 2\sqrt{b_1 - 1} = a_1 + 1 + 2\sqrt{b_1 - 1}. \] (7)

It follows that (5) holds.

Now, let $i \in \{1, \ldots, D - 1\}$. Then by Lemma 2.1(ii) we have $(c_i, a_i, b_i) \in V_{k, a_1}$. Thus, in the case $b_1 > 1$, by (5) it follows that (ii) and (iii) both hold since $a_1 + 1 + 2\sqrt{b_1 - 1} > \beta_1$. If $b_1 = 1$ then, by [7, Lemma 5.3.1], $D = 2$ holds. This completes the proof of the lemma.

Before concluding this section, we recall some bounds concerning the diameter of a distance-regular graph. The following proposition was shown by Ivanov [9]. Let $\mathbb{N}$ denote the set of positive integers.
Proposition 2.3 (A.A. Ivanov’s Diameter Bound). Let $k$ and $h$ be positive integers with $k \geq 3$. Then there is a function $f : \mathbb{N} \to \mathbb{N}$ so that, for all distance-regular graphs $\Gamma$ with valency $k$, diameter $D$, and head $h(\Gamma) = h$, the inequality

$$D \leq f(k)h$$

holds.

Later on we will make frequent use of the following result that is straightforward to prove using the last proposition.

Corollary 2.4. Suppose $k \geq 3$ and $C \geq 1$ are positive integers. Then there are finitely many distance-regular graphs $\Gamma$ with valency $k$ and $h(\Gamma) \leq C$.

Before concluding this section, we note that in [9], Ivanov showed that one can take $f(k) = 4^k$. This was improved by Hiraki and Koolen [8] to $f(k) = k^3$, and subsequently further improved by Bang, Hiraki and Koolen [1] to $f(k) = 2k^r$ if $h \geq 2$, where $r := \min\{x > 0 \mid 4^{1/x} - 2^{1/x} \leq 1\}(1.44 < r < 1.441)$. A.V. Ivanov conjectured that one can in fact take $f(k) = 2k$.

3. Interlacing

In this section we will derive some results concerning the eigenvalues of a distance-regular graph. First we recall some well-known facts concerning these eigenvalues.

Suppose that $\Gamma$ is a distance-regular graph with valency $k$, diameter $D \geq 2$ and intersection numbers $a_i, b_i, c_i, i = 0, 1, \ldots, D$. Define the tridiagonal matrix $L_1 = L_1(\Gamma)$ by

$$L_1 := \begin{pmatrix}
0 & k \\
1 & a_1 & b_1 \\
& c_2 & a_2 & b_2 \\
& & \ddots & \ddots & \ddots \\
& & & c_D & a_{D-1} & b_{D-1} \\
& & & & c_{D-1} & a_D & b_D \\
& & & & & c_D & a_D
\end{pmatrix}.$$  

Then, for any number $\theta$, $\theta$ is an eigenvalue of $\Gamma$ if and only if $\theta$ is an eigenvalue of $L_1(\Gamma)$ [7, p. 129]. Moreover, if $\theta$ is an eigenvalue of $\Gamma$ (and hence of $L_1$) then, for $(u_0 = 1, u_1, \ldots, u_D)^T$ a right eigenvector of $L_1$ associated with $\theta$, the multiplicity $m(\theta)$ of $\theta$ is given by

$$m(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^{D} k_i u_i^2}.$$  

This is known as Biggs’ formula, and the sequence $u_0, u_1, \ldots, u_D$ is called the standard sequence associated with $\theta$ (cf. [7, Theorem 4.1.4]). Note that the standard sequence can also be computed by setting $u_0 = 1, u_1 = \theta/k$ and

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i$$  

for $i = 1, 2, \ldots, D$ [7, p. 128].
The following lemma will help providing some relationships between the eigenvalues of \( \Gamma \). It generalizes the well-known Interlacing Theorem [7, Theorem 3.3.1], from which it immediately follows.

**Lemma 3.1.** Suppose that \( A \) is a real \( n \times n \) matrix for which there exists a non-singular matrix \( Q \) such that the matrix \( Q^{-1}AQ \) is real and symmetric. If \( \eta_1 \leq \cdots \leq \eta_n \) are the eigenvalues of \( A \) and \( \theta_1 \leq \cdots \leq \theta_{n-1} \) are the eigenvalues of the matrix obtained by removing the \( i \)th row and \( i \)th column of \( A \), with \( i = 1, \ldots, n \), then

\[
\eta_1 \leq \theta_1 \leq \eta_2 \leq \cdots \leq \eta_{n-1} \leq \theta_{n-1} \leq \eta_n.
\]

In particular, since \( b_l c_{i+1} > 0 \) and \( L_1(\Gamma) \) is tridiagonal, it follows that \( L_1 \) satisfies the conditions on \( A \) given in this lemma, and therefore the eigenvalues of \( \Gamma \) satisfy the inequalities given by this lemma.

We now prove a result that guarantees the existence of eigenvalues of a distance-regular graph lying within certain limits. In this proof we will use the (straightforward to derive) fact that for \( a, b, c \) real numbers and \( \ell \) a positive integer, the spectrum of the tridiagonal \( (\ell \times \ell) \) matrix

\[
B_{(c,a,b)} := \begin{pmatrix}
a & b & & & \\
c & a & b & & \\
  & c & a & b & \\
  &  & \ddots & \ddots & \\
  &  &  & c & a & b \\
  &  &  &  & c & a
\end{pmatrix}
\]

equals

\[
\left\{ a + 2\sqrt{bc} \cos \left( \frac{i\pi}{\ell + 1} \right) \mid i = 1, \ldots, \ell \right\}.
\]

**Theorem 3.2.** Suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 2 \), diameter \( D \geq 2 \), and intersection numbers \( a_i, b_i, c_i, i = 0, 1, \ldots, D \). Let \( (c, a, b) \in \Omega(\Gamma) \) and \( \ell := \ell_{(c,a,b)} \). Then the following statements hold.

(i) If \( \ell \geq 2 \) then there is an eigenvalue \( \theta \) of \( \Gamma \) with

\[
a + 2\sqrt{bc} \cos \left( \frac{2\pi}{\ell + 1} \right) \leq \theta < k.
\]

(ii) If \( \ell \geq 3 \) then there is an eigenvalue \( \theta \) of \( \Gamma \) with

\[
a + 2\sqrt{bc} \cos \left( \frac{i\pi}{\ell + 1} \right) \leq \theta \leq a + 2\sqrt{bc} \cos \left( \frac{(i-2)\pi}{\ell + 1} \right),
\]

where \( i = 3, \ldots, \ell \).

**Proof.** Let \( s \) be the smallest \( i \) in the range \( i \in \{1, 2, \ldots, D - 1\} \) for which \( (c_i, a_i, b_i) = (c, a, b) \) holds. Clearly, the spectrum of the matrix obtained by removing the \( s \)th and \( (s + \ell + 1) \)th rows and columns from \( L_1(\Gamma) \) contains the spectrum of the \( (\ell \times \ell) \) matrix \( B_{(c,a,b)} \). The result now follows by applying Lemma 3.1 twice. \( \blacksquare \)
4. Three-term recurrence relations

To prove our main results we will use some bounds for the multiplicities of eigenvalues of a distance-regular graph. To derive these bounds we will use Biggs’ formula, and the fact that the standard sequence of an eigenvalue of a distance-regular graph can be computed using the three-term recurrence relation given in the previous section. As a first step in this direction, in this section we derive some properties of three-term recurrence relations.

Suppose that \( \alpha, \beta, \gamma, x_0, x_1 \) are real numbers satisfying
\[
\alpha \geq 0, \quad \beta > 0, \quad \gamma > 0 \quad \text{and} \quad (x_0, x_1) \neq (0, 0). \tag{10}
\]
For a real number \( \theta \), define real numbers \( x_2, x_3, \ldots, x_N \) \((N \geq 2)\) by the recurrence relation
\[
\gamma x_{i-1} + \alpha x_i + \beta x_{i+1} = \theta x_i \quad (i = 1, \ldots, N-1).
\tag{11}
\]
Let \( \rho := \rho(\theta), \sigma := \sigma(\theta) \) be the roots of the auxiliary equation
\[
\gamma + (\alpha - \theta)x + \beta x^2 = 0.
\]
Without loss of generality we may assume \( |\rho| \geq |\sigma| \). If \( |\theta - \alpha| \neq 2\sqrt{\beta\gamma} \) then the roots \( \rho \) and \( \sigma \) are distinct, and using the theory of recurrence relations, it follows that
\[
x_i = \omega_1 \rho^i + \omega_2 \sigma^i \quad (i = 0, 1, \ldots, N), \tag{12}
\]
where \( \omega_1, \omega_2 \) are complex numbers. More precisely, the following hold:

(a) If \( |\theta - \alpha| > 2\sqrt{\beta\gamma} \) then the roots \( \rho \) and \( \sigma \) are real numbers with \( |\rho| > \frac{\sqrt{\gamma}}{\beta} > |\sigma| \), and \( \omega_1, \omega_2 \) are real.

(b) If \( |\theta - \alpha| < 2\sqrt{\beta\gamma} \), then the roots \( \rho \) and \( \sigma \) are complex numbers with \( \sigma = \bar{\rho} \) and \( |\rho| = |\sigma| = \frac{\sqrt{\gamma}}{\beta} \), and \( \omega_1, \omega_2 \) are complex numbers with \( \omega_2 = \bar{\omega_1} \) (since \( x_0, x_1 \) are real numbers).

We now present some preliminary observations concerning the numbers \( x_0, x_1, \ldots, x_N \) for the case (b). Put \( \rho = \sqrt{\frac{\gamma}{\beta}} e^{\sqrt{-1} j} \) and \( \omega_1 = \omega e^{\sqrt{-1} \tau} \), where \( j = \sqrt{-1}, 0 \leq \varphi, \tau \leq \pi \) and \( \omega > 0 \). By (12),
\[
x_i = 2\omega \sqrt{\frac{\gamma}{\beta}} \cos(\tau + i \varphi) \quad (i = 0, 1, \ldots, N). \tag{13}
\]
Moreover, by equating the coefficients on either side of the equation
\[
\gamma + (\alpha - \theta)x + \beta x^2 = \beta (\rho - x)(\bar{\rho} - x),
\]
it follows that
\[
\cos(\varphi) = \frac{\theta - \alpha}{2\sqrt{\beta\gamma}}. \tag{14}
\]
Now, note that for all \( i = 1, 2, \ldots, N \),
\[
\frac{1}{4} |\sin(\varphi)| \leq \max \{ |\cos(\tau + (i-1)\varphi)|, |\cos(\tau + i\varphi)| \} \leq 1. \tag{15}
\]
Indeed, suppose \(|\cos(\tau + (i - 1)\varphi)| < \frac{1}{4} |\sin(\varphi)|\). Then clearly \(|\cos(\tau + (i - 1)\varphi)| < \frac{1}{4}\), and so \(|\sin(\tau + (i - 1)\varphi)| > \frac{1}{\sqrt{1 - \frac{1}{16}}} > \frac{3}{4}\). Moreover, \(|\cos(\varphi)| < |\sin(\tau + (i - 1)\varphi)|\), since \(|\cos(\tau + (i - 1)\varphi)| < |\sin(\varphi)|\). Thus

\[
|\cos(\tau + i\varphi)| > \frac{3}{4}|\sin(\tau + (i - 1)\varphi)\sin(\varphi)|
\]

\[
> \frac{9}{16}|\sin(\varphi)| > \frac{1}{4}|\sin(\varphi)|,
\]

from which (15) immediately follows.

In view of (13) and (15) it follows that

\[
\frac{1}{2} \max \left\{|x_0|, \sqrt{\frac{\beta}{\gamma}}|x_1|\right\} \leq \omega \leq \frac{2}{\sin(\varphi)} \max \left\{|x_0|, \sqrt{\frac{\beta}{\gamma}}|x_1|\right\}.
\]

(16)

Moreover, since

\[
1 \geq \frac{|\cos(\tau + i\varphi) - \cos(\tau + (i - 1)\varphi)|}{\varphi} = \left| \frac{1}{2\omega\varphi} \left( \sqrt{\frac{\beta}{\gamma}} x_i - \sqrt{\frac{\beta}{\gamma}} x_{i-1} \right) \right|
\]

for \(i = 1, 2, \ldots, N\), we have

\[
\frac{1}{2\varphi} \left| \sqrt{\frac{\beta}{\gamma}} x_1 - x_0 \right| \leq \omega \leq \frac{2}{\sin(\varphi)} \max \left\{|x_0|, \sqrt{\frac{\beta}{\gamma}}|x_1|\right\}.
\]

(17)

Hence, since

\[
\sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i x_i^2 = \sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i (\omega_1 \rho^i + \omega_1 \rho^i)^2
\]

\[
= 2(N + 1)\omega_1 \omega_1 + \sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i \omega_1^2 \rho^{2i} + \sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i \omega_1^2 \rho^{2i}
\]

\[
= \omega^2 \left( 2(N + 1) + e^{(2\tau-\varphi)j} \left( \frac{1 - e^{2(N+1)\varphi j}}{e^{\varphi j} - e^{-\varphi j}} \right) + e^{(-2\tau+\varphi)j} \left( \frac{1 - e^{-2(N+1)\varphi j}}{e^{\varphi j} - e^{-\varphi j}} \right) \right)
\]

\[
= \omega^2 \left( 2(N + 1) + \frac{1}{\sin(\varphi)} \left( \sin(\varphi - 2\tau) + \sin((2N + 1)\varphi + 2\tau) \right) \right)
\]

and

\[
\left( \frac{\beta}{\gamma} \right)^i x_i^2 = 4\omega^2 \cos^2(\tau + i\varphi) \leq 4\omega^2,
\]

it follows that

\[
\omega^2 \left( 2(N + 1) - \frac{2}{\sin(\varphi)} \right) \leq \sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i x_i^2 \leq 4\omega^2(N + 1).
\]

(18)

Using these observations, we now prove some rather technical bounds.
Proposition 4.1. Suppose $N \geq 2$ is an integer and $T \geq 1$ is a real number. Let $\alpha, \beta, \gamma, x_0$ and $x_1$ be real numbers satisfying (10), and let $\delta$ be a real number satisfying

$$0 < \delta < \frac{\pi}{2} T.$$

Then there exist positive real numbers $C \delta := C_\delta(\delta), s = 1, 2, 3, 4$ (that depend only on $\delta$, and not on $N$, $T$ nor $\theta$), such that, for every real number $\theta$ with

$$|\theta - \alpha| \leq 2\sqrt{\beta} \gamma \cos \left(\frac{\delta}{T}\right),$$

and for all real numbers $x_2, \ldots, x_N$ satisfying (11), we have

$$C_1 \frac{1}{T^2} \max \left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\} \leq \max \left\{\left(\frac{\beta}{\gamma}\right)^{i-1} x_{i-1}^2, \left(\frac{\beta}{\gamma}\right)^i x_i^2\right\} \leq C_2 T^2 \max \left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\}$$

for $i = 1, 2, \ldots, N$, and

$$C_3 N \frac{1}{T^2} \max \left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\} \leq \sum_{i=0}^{N} \left(\frac{\beta}{\gamma}\right)^i x_i^2 \leq C_4 N T^2 \max \left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\}.$$

Proof. We first show that (19) holds. In view of (13) and (14), we have $x_i = 2\omega \sqrt{\frac{\gamma}{\beta}} \cos(\tau + i\varphi)$, where $\cos(\varphi) = \frac{\theta - \alpha}{2\sqrt{\beta} \gamma}$. It follows that $|\sin(\varphi)| \geq |\sin(\delta/T)| \geq \frac{2\delta}{\pi T}$, since $|\cos(\varphi)| = \frac{|\theta - \alpha|}{2\sqrt{\beta} \gamma} < |\cos(\delta/T)|$ and $\sin x \geq \frac{2}{\pi} x$ ($0 \leq x \leq \frac{\pi}{2}$). Now, in view of (16) it follows that

$$\left|\sqrt{\frac{\beta}{\gamma}} x_i\right| \leq 2\omega \leq \frac{2\pi T}{\delta} \max \left\{|x_0|, \sqrt{\frac{\beta}{\gamma}} |x_1|\right\} (i = 0, 1, \ldots, N).$$

By putting $C_2 = C_2(\delta) := \frac{4\pi^2}{\delta^2}$, we thus obtain a number with the required properties for which the rightmost inequality of (19) holds.

That a number $C_1$ exists with the required properties for which the leftmost inequality of (19) holds follows by symmetry. Indeed, for $n = 1, 2, \ldots, N$ put $y_i := x_{n-i} (i = 0, \ldots, n)$. Then

$$\beta y_{i-1} + (\alpha - \theta)y_i + \gamma y_{i+1} = 0 \quad (i = 1, \ldots, n-1).$$

Thus, after applying the rightmost inequality of (19) to the sequence $y_0, \ldots, y_n$, we obtain the leftmost inequality of (19) by putting $n = i$.

To complete the proof of the proposition, note that (20) follows immediately from (19) by putting $C_3 = \frac{1}{4} C_1$ and $C_4 = \frac{3}{2} C_2$, since (19) holds for all $n$ with $n = 1, 2, \ldots, N$. □

The following result immediately follows from the last proposition by putting $T = 1$ and

$$\delta := \arccos \left(\frac{2\sqrt{\beta} \gamma - \epsilon}{2\sqrt{\beta} \gamma}\right).$$
Corollary 4.2. Suppose $N \geq 2$ is an integer, and $\alpha, \beta, \gamma, x_0$ and $x_1$ are real numbers satisfying (10). Let $\epsilon$ be a real number with $0 < \epsilon < 2\sqrt{\beta \gamma}$. Then there exist positive real numbers $C_s := C_s(\beta, \gamma, \epsilon)$, $s = 1, 2, 3, 4$ (that depend only on $\beta, \gamma$ and $\epsilon$, and not on $\theta$), such that, for every real number $\theta$ with

$$|\theta - \alpha| \leq 2\sqrt{\beta \gamma} - \epsilon,$$

and for all real numbers $x_2, \ldots, x_N$ satisfying (11), we have

$$C_1 \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\} \leq \max \left\{ \left( \frac{\beta}{\gamma} \right)^{i-1} x_{i-1}^2, \left( \frac{\beta}{\gamma} \right)^i x_i^2 \right\} \leq 2 \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\}$$

for $i = 1, 2, \ldots, N$, and

$$C_3 N \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\} \leq \sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i x_i^2 \leq C_4 N \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\}.$$

We conclude this section with the following result.

Proposition 4.3. Suppose $N \geq 2$ is an integer, and $\alpha, \beta, \gamma, x_0$ and $x_1$ are real numbers satisfying (10). Let $\kappa, \epsilon$ and $\epsilon'$ be positive real numbers. Then there exist positive numbers $C_1 = C_1(\kappa, \alpha, \beta, \gamma, \epsilon, \epsilon')$, $C_2 = C_2(\kappa, \alpha, \beta, \gamma, \epsilon, \epsilon')$, $C_3 = C_3(\kappa, \alpha, \beta, \gamma, \epsilon, \epsilon')$, $C_4 = C_4(\beta, \gamma, \epsilon, \epsilon')$, $M = M(\kappa, \alpha, \beta, \gamma, \epsilon, \epsilon')$, that depend only on $\kappa, \alpha, \beta, \gamma$ and $\epsilon$ and $\epsilon'$ as indicated (and not on $\theta$), such that, for every real number $\theta$ with $|\theta - \alpha| \geq 2\sqrt{\beta \gamma} + \epsilon$, $|\theta| \leq \kappa$, and

$$|x_1 - x_0 \sigma| > \epsilon' \max \left\{ |x_0|, \sqrt{\frac{\beta}{\gamma}} |x_1| \right\}$$

(with $\rho = \rho(\theta)$ and $\sigma = \sigma(\theta)$ the roots of $\beta x^2 + (\alpha - \theta)x + \gamma = 0$ with $|\rho| \geq |\sigma|$), and for all real numbers $x_2, \ldots, x_N$ satisfying (11), we have

$$C_1 \left( \left( \frac{\beta}{\gamma} \right) \rho^2 \right)^N \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\} \leq \sum_{i=0}^{N} \left( \frac{\beta}{\gamma} \right)^i x_i^2 \leq C_2 \left( \left( \frac{\beta}{\gamma} \right) \rho^2 \right)^N \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\}$$

and, for all $M \leq n \leq N$,

$$C_3 \rho^{2n} \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\} \leq x_n^2 \leq C_4 \rho^{2n} \max \left\{ x_0^2, \left( \frac{\beta}{\gamma} \right) x_1^2 \right\}.$$

Proof. By taking $i = 0, 1$ in (12), it follows that

$$\omega_1 = \frac{x_1 - x_0 \sigma(\theta)}{\rho(\theta) - \sigma(\theta)} \quad \text{and} \quad \omega_2 = x_0 - \omega_1 = -\frac{x_1 - x_0 \rho(\theta)}{\rho(\theta) - \sigma(\theta)}.$$
Since $|\theta| \leq \kappa$ and the function $|\rho(\theta)|$ is increasing for values of $\theta$ in the range $|\theta - \alpha| \geq 2\sqrt{\beta}\gamma + \epsilon$,

$$\frac{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta}\gamma}}{\beta} \leq |\rho(\theta) - \sigma(\theta)| < |\rho(\theta)| \leq \max(|\rho(\kappa)|, |\rho(-\kappa)|)$$

$$=: F(\kappa, \alpha, \beta, \gamma) = F. \quad (25)$$

Moreover, since $|\rho(\theta)| > \sqrt{\frac{\gamma}{\beta}} > |\sigma(\theta)|$,

$$\epsilon' \max \left\{ |x_0|, \sqrt[\beta]{\frac{\beta}{\gamma}} |x_1| \right\} < |x_1 - x_0\sigma(\theta)| \leq |x_1| + |\sigma(\theta)x_0| \leq |x_1| + \sqrt[\beta]{\frac{\beta}{\gamma}} |x_0|.$$ 

Therefore,

$$\frac{\epsilon'}{F} \max \left\{ |x_0|, \sqrt[\beta]{\frac{\beta}{\gamma}} |x_1| \right\} < |\omega_1| \leq \frac{\sqrt{\beta}\gamma \left( \sqrt[\beta]{\frac{\beta}{\gamma}} |x_1| + |x_0| \right)}{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta}\gamma}} \leq \frac{2\sqrt{\beta}\gamma}{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta}\gamma}} \max \left\{ |x_0|, \sqrt[\beta]{\frac{\beta}{\gamma}} |x_1| \right\},$$

and

$$|\omega_2| = |x_0 - \omega_1| \leq |x_0| + |\omega_1| \leq \left( 1 + \frac{2\sqrt{\beta}\gamma}{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta}\gamma}} \right) \max \left\{ |x_0|, \sqrt[\beta]{\frac{\beta}{\gamma}} |x_1| \right\}.$$ 

The existence of some $C_4$ satisfying the conditions in the statement of the proposition now follows by putting $C_4 := (1 + \frac{4\sqrt{\beta}\gamma}{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta}\gamma}})^2$, since

$$|x_i| \leq |\omega_1| |\rho(\theta)|^i + |\omega_2| |\sigma(\theta)|^i \leq (|\omega_1| + |\omega_2|) |\rho(\theta)|^i.$$ 

We now show that there are numbers $M, C_1$ and $C_2$ with the required properties. Since $|\frac{\sigma(\theta)}{\rho(\theta)}|$ is a decreasing function of $\theta$ on $[\alpha + 2\sqrt{\beta}\gamma + \epsilon, \kappa]$, and an increasing function of $\theta$ on $[-\kappa, \alpha - 2\sqrt{\beta}\gamma - \epsilon]$, it follows that

$$\frac{|\sigma(\theta)|}{|\rho(\theta)|} \leq \max \left\{ \frac{\sigma(\alpha + 2\sqrt{\beta}\gamma + \epsilon)}{|\rho(\alpha + 2\sqrt{\beta}\gamma + \epsilon)|}, \frac{\sigma(\alpha - 2\sqrt{\beta}\gamma - \epsilon)}{|\rho(\alpha - 2\sqrt{\beta}\gamma - \epsilon)|} \right\} := G(\beta, \gamma, \epsilon) = G < 1.$$ 

Define $M := M(\kappa, \alpha, \beta, \gamma, \epsilon, \epsilon')$ to be the smallest value of $i$ for which

$$G^i \left( 1 + \frac{2\sqrt{\beta}\gamma}{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta}\gamma}} \right) \leq \frac{\epsilon'}{2F}.$$ 

Then, for all $n \geq M$,

$$|x_n| \geq |\omega_1| |\rho(\theta)|^n - |\omega_2| |\sigma(\theta)|^n$$

$$= \left( |\omega_1| - |\omega_2| \left| \frac{\sigma(\theta)}{\rho(\theta)} \right|^n \right) |\rho(\theta)|^n$$

$$\geq (|\omega_1| - |\omega_2| G^n) |\rho(\theta)|^n.$$
\[
\frac{e'}{F} - \left(1 + \frac{2\sqrt{\beta\gamma}}{\sqrt{\epsilon^2 + 4\epsilon\sqrt{\beta\gamma}}}\right) G^n |\rho(\theta)|^n \max \left\{ |x_0|, \sqrt{\frac{\beta}{\gamma}} |x_1| \right\} \\
\geq \frac{e'}{2F} |\rho(\theta)|^n \max \left\{ |x_0|, \sqrt{\frac{\beta}{\gamma}} |x_1| \right\}.
\]

Therefore, the leftmost inequality of (24) holds for
\[
C_3 := \left(\frac{e'}{2F}\right)^2.
\]

Moreover, using the rightmost inequality of (24), and the fact that
\[
\left(\frac{\beta}{\gamma}\right) |\rho(\theta)|^2 \geq \left(\frac{\beta}{\gamma}\right) \min \left\{ \rho \left(\alpha - 2\sqrt{\beta\gamma} - \epsilon\right)^2, \rho \left(\alpha + 2\sqrt{\beta\gamma} + \epsilon\right)^2 \right\} := H + 1 > 1,
\]
it follows that
\[
\sum_{i=0}^{N} \left(\frac{\beta}{\gamma}\right)^i x_i^2 \leq C_4 \max \left\{ x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2 \right\} \frac{\left(\frac{\beta}{\gamma}\right) |\rho(\theta)|^2} {\left(\frac{\beta}{\gamma}\right) |\rho(\theta)|^2 - 1}^N - 1 < \frac{C_4 F^2}{H} \frac{\beta}{\gamma} \left(\frac{\beta}{\gamma}\right) |\rho(\theta)|^2}^N \max \left\{ x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2 \right\}.
\]

Hence there is some \(C_2\) for which the rightmost inequality of (23) holds.

To show that \(C_1\) exists, and therefore complete the proof of the proposition, note that if \(N \geq M\) then we can put \(C_1 := C_3\), and that if \(N \leq M\), then we can put
\[
C_1 = \left(\frac{\gamma}{\beta}\right) \left(\frac{1}{F^2}\right)^M,
\]
since
\[
\sum_{i=0}^{N} \left(\frac{\beta}{\gamma}\right)^i x_i^2 \geq \max \left\{ x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2 \right\} \geq \frac{1} {\left(\frac{\beta}{\gamma}\right) |\rho(\theta)|^2} \max \left\{ x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2 \right\} \geq \left(\frac{\gamma}{\beta}\right) \left(\frac{1}{F^2}\right)^M \left(\frac{\beta}{\gamma}\right) |\rho(\theta)|^2}^N \max \left\{ x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2 \right\}.
\]

5. Multiplicities

In this section, we make some calculations that will allow us to bound multiplicities of eigenvalues of a distance-regular graph. In particular, using the results on three-term recurrences from the previous section, we will find bounds for the sum appearing in the denominator of Biggs' formula (8).

Before proving the main result of this section, we present two useful lemmas. Let \(\Gamma'\) be a distance-regular graph with valency \(k\), diameter \(D \geq 2\) and intersection numbers \(a_i, b_i, c_i (i = \)
0, 1, . . . , D). Let θ be an eigenvalue of Γ, and (u_i = u_i(θ)) be the standard sequence corresponding to θ. Note that since these values can be computed using the recurrence relation (9), it follows that

\[ 1 \leq \max\{u_0^2, ku_1^2\} \leq k, \]  

(26)

and that

\[ u_{D-1} = \frac{θ - a_D}{c_D} u_D. \]  

(27)

Let \( \rho_1 = \rho_1(θ) \) (respectively \( σ_1 = σ_1(θ) \)) be the largest (respectively smallest) root in absolute value of the equation

\[ 0 = 1 + (a_1 - θ)x + b_1x^2. \]

**Lemma 5.1** ([5, Proposition 7]). Suppose that \( Γ \) is a distance-regular graph with valency \( k \geq 3 \) and diameter \( D \geq 2 \). Let θ be an eigenvalue of Γ and \( (u_i \mid i = 0, 1, \ldots , D) \) be the standard sequence corresponding to θ. Then for \( i = 1, \ldots , D - 1 \) the following hold:

(i)

\[
\frac{1}{3k} \max\{|u_i|, |u_{i+1}|\} \leq \max\{|u_{i-1}|, |u_i|\} \leq 3k \max\{|u_i|, |u_{i-1}|\};
\]

(ii)

\[
\max\{k_iu_i^2, k_{i+1}u_{i+1}^2\} \leq 9k^4 \max\{k_{i-1}u_{i-1}^2, k_iu_i^2\}.
\]

**Proof.** (i) Since \( |θ| \leq k \) and \( b_i, c_i > 0 \) (\( i = 1, \ldots , D - 1 \)), by (9)

\[
|u_{i+1}| = \left| \left( \frac{θ - a_i}{b_i} \right) u_i - \left( \frac{c_i}{b_i} \right) u_{i-1} \right| \leq 2k|u_i| + k|u_{i-1}| \leq 3k \max\{|u_i|, |u_{i-1}|\}
\]

and

\[
|u_{i-1}| = \left| \left( \frac{θ - a_i}{c_i} \right) u_i - \left( \frac{b_i}{c_i} \right) u_{i+1} \right| \leq 2k|u_i| + k|u_{i+1}| \leq 3k \max\{|u_i|, |u_{i+1}|\}.
\]

Part (i) follows immediately.

(ii) Since \( \frac{1}{k} \max_i \leq k_i \leq k \min_i \), the result follows from (i).  

**Lemma 5.2.** Let \( k \geq 3 \) and \( 0 \leq λ \leq k - 2 \) be integers. Suppose that Γ is a distance-regular graph with valency \( k \), head \( h \), tail \( t \), diameter \( D \geq 2 \), and intersection number \( a_1 = λ \), and that θ is an eigenvalue of Γ. Then the following hold:

(i)

\[
\max\{k_{h+i}u_{h+i}^2, k_{h+i}u_{h+i+1}^2\} \leq (9k^4)^i \max\{k_hu_h^2, k_{h+1}u_{h+1}^2\}
\]

for \( i = 0, 1, \ldots , D - h - 1 \);

(ii)

\[
\sum_{i=0}^{D-h-t-1} k_{h+i}u_{h+i}^2 < (9k^4)^{D-h-t} \max\{k_hu_h^2, k_{h+1}u_{h+1}^2\}.
\]
Proof. (i) This follows immediately by applying Lemma 5.1(ii), \(i\) times.

(ii) This follows using (i), since
\[
\sum_{i=0}^{D-h-t-1} k_{h+i}u_{h+i}^2 \leq \left( \sum_{i=0}^{D-h-t-1} (9k^4)^i \right) \max \{ k_h^2, k_{h+1}^2 \} < (9k^4)^{D-h-t-h} \max \{ k_h^2, k_{h+1}^2 \}. \]

We now prove the main result of this section.

Theorem 5.3. Let \(k \geq 3\) and \(0 \leq \lambda \leq k-2\) be integers, and let \(\epsilon, \delta > 0\) be real numbers with \(\epsilon < 2\sqrt{k-1}\). Suppose that \(\Gamma\) is a distance-regular graph with valency \(k\), head \(\mathfrak{h}\), tail \(\mathfrak{t}\), diameter \(D \geq 2\), and intersection number \(a_1 = \lambda\), and that \(\theta\) is an eigenvalue of \(\Gamma\). Then there are positive numbers \(C_1 := C_1(k, a_1, \epsilon), C_2 := C_2(k, a_1, \epsilon), C_3 := C_3(k, a_1), C_4 := C_4(k), C_5 := C_5(k, a_1, \epsilon, \delta), C_6 := C_6(k, a_1, \epsilon, \delta)\) (that depend only on \(k, a_1, \epsilon\) and \(\delta\), not on \(\Gamma\) nor \(\theta\)) such that the following hold:

(i) If \(|\theta - a_1| \leq 2\sqrt{D} - \epsilon\) then
\[
C_1 h \leq \sum_{i=0}^{D} k_i u_i(\theta)^2 \leq C_2 (9k^4)^{D-h-t} h.
\]

(ii) If
\[
a_1 + 2\sqrt{b_1} \cos \left( \frac{3}{h+1} \pi \right) \leq \theta \leq a_1 + 2\sqrt{b_1} \cos \left( \frac{1}{h+1} \pi \right)
\]
then
\[
C_3 h^3 \leq \sum_{i=0}^{D} k_i u_i(\theta)^2 \leq C_4 (9k^4)^{D-h-t} h^5.
\]

(iii) If \(|\theta - a_1| \geq 2\sqrt{b_1} + \epsilon\) and \(|\theta + \frac{k}{a_1+1}| \geq \delta\), then
\[
C_5 (b_1 \rho_1)^2 h \leq \sum_{i=0}^{D} k_i u_i(\theta)^2 \leq C_6 (9k^4)^{D-h-t} (b_1 \rho_1)^2 h
\]
where \(\rho_1\) and \(\sigma_1\) are the roots of \(0 = b_1 x^2 + (a_1 - \theta)x + 1\) with \(|\rho_1| > |\sigma_1|\).

Proof. (i) Since \(h+1 \geq 2\), it follows by (22) of Corollary 4.2 and (26) that there exists a positive number \(C_1 := C_1(k, a_1, \epsilon)\) so that
\[
\sum_{i=0}^{D} k_i u_i^2 \geq \sum_{i=0}^{h+1} k_i u_i^2 \geq \frac{1}{k} \sum_{i=0}^{h+1} b_i^2 u_i^2 \geq C_1 (h+1) \geq C_1 h.
\]

Therefore \(C_1\) exists for which the left-hand inequality of (i) holds.

We now show that there is a number \(C_2\) with the required properties for which the right-hand inequality of (i) holds. Since \(k^{-1} k_i \leq b_i^2 (i = 0, 1, \ldots, h+1)\), by Corollary 4.2 and (26), there exist positive numbers \(F_1 = F_1(k, a_1, \epsilon), F_2 = F_2(k, a_1, \epsilon)\) so that
\[
\frac{1}{k} \sum_{i=0}^{h+1} k_i u_i^2 \leq \sum_{i=0}^{h+1} b_i^2 u_i^2 \leq F_1 (h+1) \max \{ u_0^2, ku_1^2 \} \leq 2kF_2 h
\] (28)
and
\[ \max\{k_h u_h^2, k_{h+1} u_{h+1}^2\} \leq F_2 \max\{u_0^2, k u_1^2\} \leq k F_2. \]  
(29)

By Lemma 5.2 and (29),
\[ \max\{k_{D-t} u_{D-t}^2, k_{D-t+1} u_{D-t+1}^2\} \leq (9k^4)^{D-h-t} \max\{k_h u_h^2, k_{h+1} u_{h+1}^2\} \]
\[ \leq k F_2 (9k^4)^{D-h-t}, \]  
(30)
and
\[ D-h-t-1 \sum_{i=0}^{D-h-t-1} k_{h+i} u_{h+i}^2 < (9k^4)^{D-h-t} \max\{k_h u_h^2, k_{h+1} u_{h+1}^2\} \]
\[ \leq k F_2 (9k^4)^{D-h-t}, \]  
(31)
where \( k_{D+1} = u_{D+1} = 0. \) Since \( k_{D-t+i} \leq k_{D-t} b_{i-1} (i = 0, \ldots, t), \) by (22) and (30), and Lemma 2.1(iv), there is a positive number \( F_3 := F_3(k, a_1, \epsilon) \) such that if \( t \geq 2 \) then
\[ \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq k_{D-t} \sum_{i=0}^{t} b_{i-1} u_{D-t+i}^2 \]
\[ \leq F_3 k_{D-t} \max\{u_{D-t}^2, b_{1} u_{D-t+1}^2\} \]
\[ = F_3 \max\{k_{D-t} u_{D-t}^2, k_{D-t+1} u_{D-t+1}^2\} \]
\[ \leq k F_2 F_3 (9k^4)^{D-h-t} h. \]  
(32)

Note that if \( t < 2 \) then, by Lemma 5.2(i) and (29),
\[ \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq k_{D-1} u_{D-1}^2 + k_D u_D^2 \leq 2 \max\{k_{D-1} u_{D-1}^2, k_D u_D^2\} \]
\[ \leq 2k F_2 (9k^4)^{D-h-t}. \]  
(33)

Therefore, by (28) and (31)–(33) it follows that
\[ \sum_{i=0}^{D} k_i u_i^2 \leq \left( \sum_{i=0}^{h+1} k_i u_i^2 \right) + \left( D-h-t-1 \sum_{i=0}^{D-h-t-1} k_{h+i} u_{h+i}^2 \right) + \left( \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \right) \]
\[ \leq C_2 (9k^4)^{D-h-t} h \]
for some positive constant \( C_2 := C_2(k, a_1, \epsilon), \) and in particular the right-hand inequality of (i) holds.

(ii) We begin by showing that there must be some positive number \( C_4 := C_4(k) \) for which the right-hand inequality holds. If \( h \leq 2 \) then it follows by Proposition 2.3 that \( D \leq 2 f(k) \) for some function \( f, \) and so \( |V \Gamma| \leq g(k) \) for some function \( g \) of \( k. \) By \( m(\theta) \geq 1, (8) \) and \( D \geq h + t + 1, \)
\[ \sum_{i=0}^{D} k_i u_i^2 \leq |V \Gamma| \leq g(k) \leq g(k) (9k^4)^{D-h-t}. \]

Hence there is some number \( C_4 \) depending only on \( k \) for which the right-hand inequality of (ii) holds. Therefore, we assume from now on that \( h > 2. \)
By putting $T := h + 1$ and $\delta := \pi$ in Proposition 4.1, it follows that there exist numbers $G_1 = G_1(\delta)$, $G_2 = G_2(\delta)$ for which
\[
\frac{1}{k} \sum_{i=0}^{h} k_i u_i^2 \leq \sum_{i=0}^{h} b_i^1 u_i^2 \leq 2G_1 \max(u_0^2, k_u^2)h^3 \leq 2kG_1 h^3
\] (34)
and
\[
\max[k_h u_h^2, k_{h+1} u_{h+1}^2] \leq 2kG_2 \max(u_0^2, k_u^2)h^2 \leq 2k^2G_2 h^2
\] (35)
both hold, in view of (26), $k^{-1} b_i \leq b_i^1$ ($i = 0, 1, \ldots, h + 1$), and $T^2 = (h + 1)^2 \leq 2h^2$. By Lemma 5.2 and (35),
\[
\max[k_{D-t} u_{D-t}^2, k_{D-t+1} u_{D-t+1}^2] \leq (9k^4)^{D-h-t} \max(k_h u_h^2, k_{h+1} u_{h+1}^2) 
\leq 2k^2G_2(9k^4)^{D-h-t}h^2
\] (36)
and
\[
\sum_{i=0}^{D-h-t-1} k_{h+i} u_{h+i}^2 < (9k^4)^{D-h-t} \max(k_h u_h^2, k_{h+1} u_{h+1}^2) 
\leq 2k^2G_2(9k^4)^{D-h-t}h^2
\] (37)
both hold, where $k_{D+1} = u_{D+1} = 0$

Using the inequality $k_{D-t+i} \leq k_{D-t}b_i^{-1}$ ($i = 0, \ldots, t$), (20), $T^2 = (h + 1)^2 \leq 2h^2$, (36), and Lemma 2.1(iv), we see that there is a positive number $G_3 := G_3(\delta)$ such that if $t \geq 2$ then
\[
\sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq k_{D-t} \sum_{i=0}^{t} b_i^{-1} u_{D-t+i}^2 
\leq 2G_3 k_{D-t} \max(u_{D-t}^2, b_i^{-1} u_{D-t+i}^2)th^2 
= 2G_3 \max(k_{D-t} u_{D-t}^2, k_{D-t+1} u_{D-t+1}^2)th^2 
\leq 4k^2G_2G_3(9k^4)^{D-h-t}h^5.
\] (38)

Note that if $t < 2$ then, by Lemma 5.2(i) and (35),
\[
\sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq k_{D-1} u_{D-1}^2 + k_{D} u_{D}^2 \leq 2 \max(k_{D-1} u_{D-1}^2, k_{D} u_{D}^2) 
\leq 4k^2G_2(9k^4)^{D-h-t}h^2.
\] (39)
Moreover, in view of (34) and (37)–(39), there exists a number $C_4 := C_4(k)$ such that
\[
\sum_{i=0}^{D} k_i u_i^2 \leq \left( \sum_{i=0}^{h} k_i u_i^2 \right) + \left( \sum_{i=0}^{D-h-t-1} k_{h+i} u_{h+i}^2 \right) + \left( \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \right) 
\leq C_4(9k^4)^{D-h-t}h^5.
\]
In particular, the right-hand inequality of (ii) holds.

We now show that there is a constant $C_3 := C_3(k, a_1)$ for which the left-hand inequality of (ii) holds. Suppose first that $h \geq 17$. By (13) and (14) we have
\[
u_i = 2\omega \sqrt{b_1^{-i}} \cos(\tau + i\varphi) \quad (i = 0, 1, \ldots, h + 1),
\]
where \( \cos\left(\frac{3\pi}{h+1}\right) \leq \cos(\varphi) = \frac{\theta - a_1}{2\sqrt{b_1}} \leq \cos\left(\frac{\pi}{h+1}\right) \). Since \( h \geq 5 \) and \( \frac{2}{\pi}x \leq \sin x \leq x \) \((0 \leq x \leq \frac{\pi}{2})\), it follows that

\[
\frac{2}{h+1} \leq \sin\left(\frac{\pi}{h+1}\right) \leq \sin(\varphi) \leq \sin\left(\frac{3\pi}{h+1}\right) \leq \frac{3}{h+1}\pi.
\]

Moreover, since \( h \geq 5 \), and \( 2\pi x \leq \sin x \leq x \) \((0 \leq x \leq \frac{\pi}{2})\),

it follows that

\[
2 \leq \sin\left(\frac{\pi}{h+1}\right) \leq \sin(\varphi) \leq \sin\left(\frac{3\pi}{h+1}\right) \leq 3\pi.
\]

(40)

Moreover, since \( h \geq 17 \),

\[
u_1 = \frac{\theta - a_1 + \sqrt{3b_1}}{k}.
\]

(41)

Since \( h \geq 2 \), we have \((a_1 + 1)|k \) and \( b_1 > 1 \) by [7, Lemma 5.3.1]. Hence \( a_1\sqrt{b_1} + \sqrt{3b_1} > k \).

Therefore, by (17), (41) and (40), there exists a number \( G = G(k, a_1) > 0 \) (depending only on \( k \) and \( a_1 \)) such that

\[
|\omega| \geq \frac{1}{2\varphi} \left| \sqrt{b_1}u_1 - u_0 \right| \geq \frac{1}{\pi \sin \varphi} \left| \frac{a_1\sqrt{b_1} + \sqrt{3b_1} - k}{k} \right| \geq \frac{h + 1}{3\pi^2} \left| \frac{a_1\sqrt{b_1} + \sqrt{3b_1} - k}{k} \right| \geq \sqrt{Gk}.
\]

(42)

Moreover, by (18), (40) and (42),

\[
\sum_{i=0}^{h} k_i u_i^2 \geq \sum_{i=0}^{h} b_i^2 u_i^2 \geq \omega^2 \left( 2(h+1) - \frac{2}{\sin(\varphi)} \right) \geq Gh^2(h+1) \geq Gh^3.
\]

Now, to complete the proof of (ii), note that if \( h < 17 \) then

\[
\sum_{i=0}^{h} k_i u_i^2 \geq 1 > \left( \frac{1}{17} \right)^3 h^3.
\]

Hence the left-hand inequality of (ii) holds for \( C_3 = C_3(k, a_1) = \min\{1, (\frac{1}{17})^3, G\} \).

(iii) So that we may apply Proposition 4.3, we first show that there exists a positive real number \( \epsilon' \) for which

\[
|\sigma_1 - \theta/k| \geq \epsilon' \max\{|u_0|, \sqrt{b_1}|u_1|\}.
\]

(43)

Moreover, by (40) and (42),

\[
\sum_{i=0}^{h} k_i u_i^2 \geq 1 \geq \left( \frac{1}{17} \right)^3 h^3.
\]

Hence the left-hand inequality of (ii) holds for \( C_3 = C_3(k, a_1) = \min\{1, (\frac{1}{17})^3, G\} \).

(iii) So that we may apply Proposition 4.3, we first show that there exists a positive real number \( \epsilon' \) for which

\[
|\sigma_1 - \theta/k| \geq \epsilon' \max\{|u_0|, \sqrt{b_1}|u_1|\}.
\]

(43)

Note that in the case \( \theta \geq a_1 + 2\sqrt{b_1} + \epsilon \),

\[
0 < \sigma_1 < \frac{1}{\sqrt{b_1}} < \frac{a_1 + 2\sqrt{b_1}}{k} < \frac{\theta}{k}.
\]
and hence
\[ |\sigma_1 - \theta/k| > \frac{a_1 + 2\sqrt{b_1}}{k} - \frac{1}{\sqrt{b_1}} > 0. \]
Now suppose \( \theta \leq a_1 - 2\sqrt{b_1} - \epsilon \). Define a real-valued function \( \phi : [-k, a_1 - 2\sqrt{b_1} - \epsilon] \to \mathbb{R} \) by putting
\[ \phi(\theta) = \sigma_1 - \frac{\theta}{k}. \]
Clearly \( \phi \) is continuous, and \( \phi \) has at most one root, namely \( \theta = -\frac{k}{a_1+1} \). In view of the assumption
\[ |\theta + \frac{k}{a_1 + 1}| \geq \delta, \]
there exists some \( H_1 := H_1(k, a_1, \epsilon, \delta) > 0 \) such that \( |\phi(\theta)| \geq H_1 \) holds for all \( \theta \) with \(-k \leq \theta \leq a_1 - 2\sqrt{b_1} - \epsilon \) and \(|\theta + k/(a_1 + 1)| \geq \delta \).

Therefore, by (26) it follows that (43) holds, since
\[ |\sigma_1 - \theta/k| \geq \epsilon' \sqrt{k} \geq \epsilon' \max\{|u_0|, \sqrt{b_1}|u_1|\}, \]
where \( \epsilon' = \epsilon'(k, a_1, \epsilon, \delta) := \sqrt{k}^{-1} \min\{\frac{a_1 + 2\sqrt{b_1}}{k} - \frac{1}{\sqrt{b_1}}, H_1\} > 0 \).

We now show that there is a number \( C_5 \) for which the left-hand inequality of (iii) holds. Note that there are numbers \( H_2 = H_2(k, a_1, \epsilon) \), \( H_3 = H_3(k, a_1) \) such that \( H_2 \leq b_1 \rho_1^2 \leq H_3 \). Indeed, we can put
\[ H_2 := \left( \epsilon + 2\sqrt{b_1} + \sqrt{\epsilon^2 + 4\epsilon \sqrt{b_1}} \right)^2 / 4b_1, \quad H_3 := \left( k + a_1 + \sqrt{(k + a_1)^2 - 4b_1} \right)^2 / 4b_1. \]
Hence, since \( h + 1 \geq 2 \) and (43) hold, by (23) and (26) there exists a positive number \( H_4 := H_4(k, a_1, \epsilon, \delta) \) so that
\[ \sum_{i=0}^{D} k_i u_i^2 \geq \sum_{i=0}^{h+1} k_i u_i^2 \geq \frac{1}{k} \sum_{i=0}^{h+1} b_i u_i^2 \geq H_4(b_1 \rho_1^2)^{h+1} \geq H_2 H_4(b_1 \rho_1^2)^{h}. \]
Thus the left-hand inequality in (iii) holds with \( C_5 = C_5(k, a_1, \epsilon, \delta) := H_2 H_4 \).

We now show that there is number \( C_6 \) for which the right-hand inequality of (iii) holds. In view of \( k^{-1} k_i \leq b_i^1 (i = 0, 1, \ldots, h + 1) \), (23) and (26), it follows that there is some positive number \( H_5 = H_5(k, a_1, \epsilon) \) so that
\[ \max\{|k_h u_h^2, k_{h+1} u_{h+1}^2| \leq \sum_{i=0}^{h+1} k_i u_i^2 \leq k \sum_{i=0}^{h+1} b_i^1 u_i^2 \leq H_5(b_1 \rho_1^2)^{h+1} \max\{|u_0|, ku_1^2\} \leq k H_3 H_5(b_1 \rho_1^2)^{h}. \]  
(44)
Moreover, by Lemma 5.2 and (44),
\[ \max\{|k_{D-t+i} u_{D-t+i}^2, k_{D-t+i+1} u_{D-t+i+1}^2| \leq (9k^4)^{D-h-t+i} \max\{|k_h u_h^2, k_{h+1} u_{h+1}^2| \} \leq k H_3 H_5(9k^4)^{D-h-t+i} (b_1 \rho_1^2)^{h}. \]  
(45)
and

\[ \sum_{i=0}^{D-h-t-1} k_{h+i} u_{h+i}^2 < (9k^4)^{D-h-t} \max\{k_h u_h^2, k_{h+1} u_{h+1}^2\} \]

\[ \leq k H_3 H_5 (9k^4)^{D-h-t}(b_1 \rho_1^2)^h, \]  \hspace{1cm} (46)

where \(k_{D+1} = u_{D+1} = 0\).

Now, to complete the proof, we consider the sum

\[ \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2. \]

Note that by setting \(\alpha = a_1, \beta = b_1, \gamma = c_1, x_0 = u_0\), and \(x_1 = u_1\) in Proposition 4.3, it follows that there exists a positive constant \(M := M(k, a_1, \epsilon, \delta)\) such that (24) of this proposition holds. Hence, by Lemma 2.1(v), we can consider three cases, namely \(t < M, (t \geq M \text{ and } a_D = 0)\), and \((t \geq M \text{ and } a_D = a_1 + 1)\).

**Case 1**: \(t \leq M - 1\)

By Lemma 5.1(ii) and (45),

\[ k_{D-t+i} u_{D-t+i}^2 \leq \max\{k_{D-t+i} u_{D-t+i}^2, k_{D-t+i+1} u_{D-t+i+1}^2\} \]

\[ \leq k H_3 H_5 (9k^4)^{D-h-t}(b_1 \rho_1^2)^h (9k^4)^i \]

holds for \(i = 0, \ldots, t\), and hence

\[ \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq k H_3 H_5 (9k^4)^{D-h-t}(b_1 \rho_1^2)^h \sum_{i=0}^{t} (9k^4)^i \]

\[ \leq k H_3 H_5 (9k^4)^M (9k^4)^{D-h-t}(b_1 \rho_1^2)^h. \]  \hspace{1cm} (47)

For the next two cases, we let \(v_i := u_{D-i} (i = 0, 1, \ldots, t + 1)\).

**Case 2**: \(t \geq M \text{ with } a_D = 0\)

In this case we have \(v_1 = \frac{\nu}{k} v_0\), and thus there is a number \(\epsilon'' := \epsilon''(k, a_1, \epsilon, \delta) > 0\) so that, by a proof similar to that of (43),

\[ |v_1 - \sigma_1 v_0| \geq \epsilon'' \max\{|v_0|, \sqrt{b_1}|v_1|\}. \]

Hence, since \(t + 1 \geq 2\), setting \((\alpha, \beta, \gamma) = (a_1, b_1, 1)\), we can apply Proposition 4.3 to the numbers \(x_i := v_i, i = 0, 1, \ldots, t + 1\). Therefore, since \(k_{D-i} \leq k k_D b_i^1 (i = 0, 1, \ldots, t + 1)\), by (23), (24) and (45), there exist \(H_6 := H_6(k, a_1, \epsilon) > 0\) and \(H_7 := H_7(k, a_1, \epsilon, \delta) > 0\) such that

\[ \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq \sum_{i=0}^{t+1} k_{D-i} v_i^2 \leq k k_D \sum_{i=0}^{t+1} b_i^1 v_i^2 \]

\[ \leq H_6 (b_1 \rho_1^2)^{t+1} k_D \max\{v_0^2, b_1 v_1^2\} \]

\[ \leq H_7 b_1^2 k_D \max\{v_{t-1}^2, v_t^2\} \]

\[ \leq k^2 H_7 \max\{k_{D-t} u_{D-t}^2, k_{D-t+1} u_{D-t+1}^2\} \]

\[ \leq k^3 H_3 H_5 H_7 (9k^4)^{D-h-t}(b_1 \rho_1^2)^h. \]  \hspace{1cm} (48)

**Case 3**: \(t \geq M \text{ with } a_D = a_1 + 1\)
By (27) and \(a_D = a_1 + 1\),
\[
v_1 = \psi v_0, \tag{49}
\]
where
\[
\psi = \psi(\theta, k, a_1) := \frac{\theta - a_1 - 1}{b_1}.
\]
Note that for all \(\theta \in [-k, k]\) with \(|\theta - a_1| \geq 2\sqrt{b_1} + \epsilon\),
\[
|\psi| - \frac{|\rho_1| + |\sigma_1|}{2} = \frac{|\theta - a_1| - 2}{2b_1} \geq \frac{2\sqrt{b_1} - 2 + \epsilon}{2b_1} := H_8(k, a_1, \epsilon) = H_8.
\]
It is straightforward to check that
\[
\sum_i |\psi| = |\psi| \geq |\rho_1| - |\sigma_1| + H_8 \geq \frac{\sqrt{\epsilon^2 + 4\epsilon \sqrt{b_1}}}{2b_1} + H_8 \tag{50}
\]
and
\[
\max \left\{1, \sqrt{b_1}|\psi| \right\} \leq \max \left\{1, \sqrt{b_1} \left( \frac{2k - b_1}{b_1} \right) \right\} \leq \sqrt{b_1} \left( \frac{2k - b_1}{b_1} \right) \tag{51}
\]
all hold. Therefore by (49)–(51),
\[
|v_1 - v_0\sigma_1| = |\psi - \sigma_1||v_0|
\geq H_9|v_0|
\geq H_9 \sqrt{b_1} \max \left\{1, \sqrt{b_1}|\psi| \right\} |v_0|
= \frac{H_9 \sqrt{b_1}}{2k - b_1} \max \left\{|v_0|, \sqrt{b_1}|v_1| \right\},
\]
where \(H_9 = H_9(k, a_1, \epsilon) := \sqrt{\frac{\epsilon^2 + 4\epsilon \sqrt{b_1}}{2b_1}} + H_8\). Since \(t + 1 \geq 2\), we can thus apply Proposition 4.3 to the numbers \(x_i := v_i (i = 0, 1, \ldots, t + 1)\). Using the same procedure as we used to show that (48) holds, we thus see that
\[
\sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \leq \sum_{i=0}^{t+1} k_{D-i} v_i^2 \leq H (9k^4)^{D-h-t} (b_1 \rho_1^2)^h \tag{52}
\]
holds for some constant \(H = H(k, a_1, \epsilon) > 0\).

Now, to complete the proof of (iii), note that by (44), (46)–(48) and (52), it follows that
\[
\sum_{i=0}^{D} k_i u_i^2 \leq \left( \sum_{i=0}^{h+1} k_i u_i^2 \right) + \left( \sum_{i=0}^{D-h-t-1} k_{h+i} u_{h+i}^2 \right) + \left( \sum_{i=0}^{t} k_{D-t+i} u_{D-t+i}^2 \right) \\
\leq C_6 (9k^4)^{D-h-t} (b_1 \rho_1^2)^h.
\]
We now consider two cases. In particular, by Corollary 2.4, for fixed graphs with valency $H$, hence, by Lemma 2.1(iv), the diameter $D$ of $\Gamma$ is bounded by a number depending only on $k$, $a_1$ and $C$, namely,

$$D \leq h + t + C \leq 2h + C \leq 2H + C.$$ 

In particular, by Corollary 2.4, for fixed $k \geq 3$, $\lambda$ and $C$, there are finitely many distance-regular graphs with valency $k$, intersection number $a_1 = \lambda$ such that the diameter $D$ is at most $h + t + C$. Therefore, since there are only finitely many possibilities for $a_1$ for fixed $k$ (as $0 \leq a_1 \leq k - 2$), the theorem follows.

To this end, let $\Gamma$ be a distance-regular graph with valency $k$, intersection number $a_1$, head $h \geq 3$ and diameter $D$ satisfying $D \leq h + t + C$. Define a quadratic $Q$ by

$$Q(x) := (x - a_1 - 2\sqrt{b_1}) (x - a_1 + 2\sqrt{b_1}).$$

Let $F = F(k, a_1) := \max\{|Q(x)| \mid x \in [-k, k]\}$, noting that $F \geq 4$ as $Q(a_1) = -4b_1$.

Since $h \geq 3$, by Theorem 3.2(ii), $\Gamma$ has an eigenvalue $\theta$ with

$$a_1 + 2\sqrt{b_1} \cos \left(\frac{3}{h + 1}\pi\right) \leq \theta \leq a_1 + 2\sqrt{b_1} \cos \left(\frac{1}{h + 1}\pi\right).$$

We now consider two cases, $|Q(\theta)| \geq F^{-3C-8}$ and $|Q(\theta)| < F^{-3C-8}.$

**Case 1:** $|Q(\theta)| \geq F^{-3C-8}$

Let $G = G(k, a_1, C) := \max\{x \in (a_1, a_1 + 2\sqrt{b_1}) \mid |Q(x)| = F^{-3C-8}\}.$ Since

$$\lim_{h \to \infty} \left(a_1 + 2\sqrt{b_1} \cos \left(\frac{x}{h + 1}\pi\right)\right) = a_1 + 2\sqrt{b_1}$$

holds for any constant $x$ and by $|Q(\theta)| \geq F^{-3C-8},$

$$a_1 + 2\sqrt{b_1} \cos \left(\frac{3}{h + 1}\pi\right) < G.$$ 

Hence

$$h < \frac{3\pi}{\arccos \left(\frac{G - a_1}{2\sqrt{b_1}}\right)} - 1.$$

**Case 2:** $|Q(\theta)| < F^{-3C-8}$

Since $Q$ has integer coefficients and leading coefficient one, and eigenvalue $\theta$ is an algebraic integer with $Q(\theta) \neq 0$, the following product is a positive integer

$$\prod_{\{\theta' \mid \theta' \text{ is an algebraic conjugate of } \theta\}} |Q(\theta')|.$$ 

Note that any algebraic conjugates of $\theta$ are eigenvalues of $\Gamma$ that have the same multiplicity of $\theta$. By Lemma 3.1, there are at most $(3C + 7)$ eigenvalues $\eta$ with $|\eta - a_1| \geq 2\sqrt{b_1}$ (simply delete
the rows and columns of \( L_{2048} \) corresponding to \( \{ (0, 0, k) \} \cup \{(c_i(a, b), a_i(c, a, b), b_i(c, a, b)) \mid (c, a, b) \in \Omega(\Gamma) \} \), where \( i(c, a, b) := \max\{s \mid (c_s, a_s, b_s) = (c, a, b)\} \). Thus, since we are assuming \( |Q(\theta)| < F^{-3C-8} \), there exists an algebraic conjugate \( \theta' \) of \( \theta \) such that both \( |Q(\theta')| > 1 \) and \( |\theta' - a_1| \leq 2\sqrt{b_1} - (a_1 + 2\sqrt{b_1} - H_1) \) hold, where

\[
H_1 = H_1(k, a_1) := \max \left\{ x \in \left( a_1, a_1 + 2\sqrt{b_1} \right) \mid |Q(x)| = 1 \right\}.
\]

By Theorem 5.3(i) and (ii), there exist positive constants \( H_i = H_i(k, a_1) \ (i = 2, 3) \) (depending only on \( k \) and \( a_1 \)) such that

\[
\sum_{i=0}^D k_iu_i(\theta)^2 \leq H_2(9k^4)h
\]

and

\[
\sum_{i=0}^D k_iu_i(\theta)^2 \geq H_3h^3.
\]

Thus, since \( m(\theta) = m(\theta') \), by (8) we have

\[
h \leq \sqrt{\frac{H_2}{H_3}}(9k^4)^C.
\]

Therefore, in view of Cases 1 and 2, \( h \leq H \) holds for the positive number

\[
H = H(k, a_1, C) := \max \left\{ 2, \frac{3\pi}{\arccos \left( \frac{G-a_1}{2\sqrt{b_1}} \right)} - 1, \sqrt{\frac{H_2}{H_3}}(9k^4)^C \right\}.
\]

This completes the proof of Theorem 1.1. ■

7. Proof of Theorem 1.2

In view of Corollary 2.4, there are finitely many distance-regular graphs with fixed valency \( k \) and \( h = 1 \), and so we assume \( h \geq 2 \). Moreover, we also fix \( a_1 \), since for \( k \) fixed there are only finitely many possibilities for this intersection number.

Fix \( 0 \leq \lambda \leq k - 2 \). From now on, we assume that \( \Gamma' \) is a distance-regular graph with valency \( k \geq 3 \), diameter \( D \), intersection number \( a_1 = \lambda \), and head \( h \geq 2 \). In particular, we will also assume \( (a_1 + 1) \mid k \) and \( b_1 \geq 2 \), since both of these relationships hold for such a distance-regular graph.

Now, consider the polynomial

\[
P(x) := \prod_{(c,a,b)\in V_{k,a_1}, c\leq b} \left( x - a - 2\sqrt{bc} \right) \left( x + a - 2\sqrt{bc} \right)
\times \left( x - a + 2\sqrt{bc} \right) \left( x + a + 2\sqrt{bc} \right)
\times \left( x - a - 2a_1 - 2\sqrt{bc} \right) \left( x + a - 2a_1 - 2\sqrt{bc} \right) \left( x - a - 2a_1 + 2\sqrt{bc} \right)
\times \left( x + a - 2a_1 + 2\sqrt{bc} \right) \left( x + \frac{k}{a_1 + 1} \right) \left( x - 2a_1 - \frac{k}{a_1 + 1} \right).
\]
Then it is straightforward to verify that $P$ has the following properties:

- $P$ is a continuous function with $P \neq 0$.
- $P$ has integral coefficients.
- If $(c, a, b) \in V_{k,a_1}$, then $a + 2\sqrt{bc}$ and $a - 2\sqrt{bc}$ are roots of $P$.
- $P$ is symmetric around $a_1$ (i.e., $P(a_1 + x) = P(a_1 - x)$ for $x \in \mathbb{R}$).

Let

$$
\mu := \frac{(a_1 + 2\sqrt{b_1}) + (a_1 + 1 + 2\sqrt{b_1} - 1)}{2}.
$$

Since $(a_1 + 1)/k$ and $b_1 \geq 2$, by (5), it follows that for all $(c, a, b) \in V_{k,a_1}^*$ we have

$$
a_1 + 2\sqrt{b_1} < \mu < a_1 + 1 + 2\sqrt{b_1} - 1 \leq a + 2\sqrt{bc} \leq k.
$$

Let

$$S_{1/2} := \{ x \in [\mu, k] \mid 0 < |P(x)| < 1/2 \}.
$$

Clearly $S_{1/2}$ consists of a collection of disjoint open intervals and $a + 2\sqrt{bc} \notin S_{1/2}$ for all $(c, a, b) \in V_{k,a_1}^*$. Put

$$S_1 := \{ x \in [-k, k] \mid |P(x)| \geq 1 \}.
$$

Then $a_1 \in S_1$. Moreover, since $P$ is continuous and symmetric around $a_1$, it is straightforward to show that there exists a real number $\nu = \nu(k, a_1) > 0$ (depending only on $k$ and $a_1$) such that

$$||x - a_1| - |y - a_1|| \geq \nu$$

holds for all $x \in S_1$ and $y \in S_{1/2}$.

We now prove three claims from which the theorem will easily follow.

**Claim 1.** There exists a positive number $C := C(k, a_1)$ (that depends only on $k$ and $a_1$, and not on $\Gamma$) so that if $D - (h + \tau) > C$ then $\Gamma$ has an eigenvalue $\theta$ in $S_{1/2}$.

**Proof of Claim 1.** Since $b_1 \geq 2$, it follows that $(1, k - 2, 1) \in V_{k,a_1}^*$, and hence $V_{k,a_1}^* \neq \emptyset$. Now suppose $(c, a, b) \in V_{k,a_1}^*$. If $\ell := \ell(c,a,b) \geq 3$ then, by Theorem 3.2(ii), there is an eigenvalue $\theta$ of $\Gamma$ with

$$a + 2\sqrt{bc} \cos \left( \frac{i}{\ell + 1} \pi \right) \leq \theta \leq a + 2\sqrt{bc} \cos \left( \frac{i - 2}{\ell + 1} \pi \right)
$$

for each $i = 3, \ldots, \ell$. Since

$$\lim_{\ell \to \infty} \left( a + 2\sqrt{bc} \cos \left( \frac{3}{\ell + 1} \pi \right) \right) = a + 2\sqrt{bc} = \lim_{\ell \to \infty} \left( a + 2\sqrt{bc} \cos \left( \frac{1}{\ell + 1} \pi \right) \right),$$

there exists a constant $F_{(c,a,b)} \geq 3$, depending only on $k, a_1$ and $(c, a, b)$, so that for all $\ell \geq F_{(c,a,b)},$

$$\left( a + 2\sqrt{bc} \cos \left( \frac{3}{\ell + 1} \pi \right), a + 2\sqrt{bc} \cos \left( \frac{1}{\ell + 1} \pi \right) \right) \subseteq S_{1/2}.$$

Hence, by putting

$$C = C(k, a_1) := \sum_{(c,a,b) \in V_{k,a_1}^*} F_{(c,a,b)},$$

we have

$$D - (h + \tau) > C \implies \theta \in S_{1/2}.$$
Claim 2. If $\Gamma$ has an eigenvalue $\theta \in S_{1/2}$, then $\theta$ has an algebraic conjugate $\theta'$ in $S_1$.

Proof of Claim 2. Since $P$ has integer coefficients and leading coefficient 1, and $\theta$ is an algebraic integer with $P(\theta) \neq 0$ it follows that

$$\prod_{\{\theta'|\theta' \text{ is an algebraic conjugate of } \theta\}} |P(\theta')|$$

is a positive integer. Hence, since $0 < |P(\theta)| < 1/2$, there exists an algebraic conjugate $\theta'$ of $\theta$ with $|P(\theta')| > 1$. ■

Claim 3. Suppose that $\Gamma$ has an eigenvalue in $S_{1/2}$. Then there exist positive numbers $C_1 := C_1(k, a_1), C_2 := C_2(k, a_1)$ (that depend only on $k$ and $a_1$, and not on $\Gamma$) so that $D - (h + t) \leq C_1 h$ then $h \leq C_2$.

Proof of Claim 3. Suppose that $\theta \in S_{1/2}$ is an eigenvalue of $\Gamma$. By Claim 2, $\theta$ has an algebraic conjugate $\theta' \in S_1$. Since $P$ is continuous and $P(-\frac{k}{a_1+1}) = 0$, there exists a positive constant $\epsilon' = \epsilon'(k, a_1)$ so that

$$\left| \theta' + \frac{k}{a_1+1} \right| \geq \epsilon'.$$

As $\theta \geq \mu > a_1 + 2\sqrt{b_1},$

$$\rho_1(\theta) \geq \rho_1(\mu) > 1/\sqrt{b_1}$$

(56)

(where $\rho_1$ is defined as at the beginning of Section 5),

$$|\theta - a_1| - 2\sqrt{b_1} \geq \mu - a_1 - 2\sqrt{b_1} > 0$$

(57)

and

$$\left| \theta + \frac{k}{a_1+1} \right| > a_1 + 2\sqrt{b_1} + \frac{k}{a_1+1} > 0.$$  

(58)

We now consider two cases: either $\theta'$ is contained in the closed interval $[a_1 - 2\sqrt{b_1}, a_1 + 2\sqrt{b_1}]$ or it is not.

Case 1: $\theta' \in [a_1 - 2\sqrt{b_1}, a_1 + 2\sqrt{b_1}]$.

Since $\theta' \in S_1$, and $P$ is continuous and symmetric around $a_1$, there exists a constant $F_1 := F_1(k, a_1)$ with $0 < F_1 < 2\sqrt{b_1}$ so that

$$|\theta' - a_1| \leq F_1 = 2\sqrt{b_1} - \left(2\sqrt{b_1} - F_1\right).$$

(59)

Put $F_2 = F_2(k, a_1) := \rho_1(\mu)\sqrt{b_1}$. Then by (56),

$$\sqrt{b_1}\rho_1(\theta) \geq \rho_1(\mu)\sqrt{b_1} = F_2 > 1.$$  

(60)

Now, let $F_3 := F_3(k, a_1)$ be the number for which $(9k^4)^{F_3} = F_2$ holds, and suppose

$$D - (h + t) \leq F_3 h.$$
Then, by Theorem 5.3(i) and (iii), together with (57)–(60), there exist positive constants \( F_i := F_i(k, a_1) \) \((i = 4, 5)\) such that
\[
\sum_{i=0}^{D} k_i u_i(\theta)^2 \geq F_4 (b_1 \rho_1(\theta)^2)^h \geq F_4 F_2^2 h
\]
and
\[
\sum_{i=0}^{D} k_i u_i(\theta')^2 \leq F_5 (9k^4)^{F_3} h = F_5 F_2^2 h.
\]

Since \( \theta' \) is an eigenvalue of \( \Gamma \) with multiplicity \( m(\theta') = m(\theta) \), it follows that \( F_5^2 h \leq (F_5 / F_4) h \) holds for \( F_2 > 1 \). Hence there exists a positive number \( F = F(k, a_1) \) for which \( h \leq F \).

**Case 2:** \( \theta' \not\in [a_1 - 2\sqrt{b_1}, a_1 + 2\sqrt{b_1}] \).
Since \( \theta' \in S_1 \), and \( P \) is continuous and symmetric around \( a_1 \), there exists a positive number \( G_1 = G_1(k, a_1) \) with
\[
|\theta' - a_1| \geq 2\sqrt{b_1} + G_1. \tag{61}
\]
Let \( v = v(k, a_1) \) be a number for which (54) holds, noting that \( 0 < v < k - a_1 - 2\sqrt{b_1} \), and let \( G_2 = G_2(k, a_1) := \min \left\{ \left| \frac{\rho_1(x + v)}{\rho_1(x)} \right| \left| a_1 + 2\sqrt{b_1} \leq x \leq k - v \right\} \right. \).

Since \( \rho_1 \) is strictly increasing on \([a_1 + 2\sqrt{b_1}, k]\), it follows that \( G_2 > 1 \). Moreover, since \( |\rho_1(a_1 - x)| = |\rho_1(a_1 + x)| \) holds for all \( x \) positive and real,
\[
\frac{|\rho_1(\theta)|}{|\rho_1(\theta')|} \geq G_2 \quad \text{holds for } |\theta - a_1| > |\theta' - a_1| \tag{62}
\]
and
\[
\frac{|\rho_1(\theta')|}{|\rho_1(\theta)|} \geq G_2 \quad \text{holds for } |\theta' - a_1| > |\theta - a_1|. \tag{63}
\]

Now, let \( G_3 = G_3(k, a_1) \) be the number for which \( (9k^4)^{G_3} = G_2 \) holds, and suppose \( D - (h + t) \leq G_3 h \).

Let \( \{\eta, \eta'\} = \{\theta, \theta'\} \) so that \( |\eta - a_1| > |\eta' - a_1| \) holds. Then, by Theorem 5.3(iii) together with (55), (57), (58) and (61)–(63), it follows that there exist positive constants \( G_i = G_i(k, a_1) \) \((i = 4, 5)\) for which
\[
\sum_{i=0}^{D} k_i u_i(\eta)^2 \geq G_4 (b_1 \rho_1(\eta)^2)^h \geq G_4 G_2^2 (b_1 \rho_1(\eta')^2)^h
\]
and
\[
\sum_{i=0}^{D} k_i u_i(\eta')^2 \leq G_5 (9k^4)^{G_3 h} (b_1 \rho_1(\eta')^2)^h = G_5 G_2^h (b_1 \rho_1(\eta')^2)^h.
\]

Therefore \( G_2^h \leq G_5 / G_4 \), where \( G_2 > 1 \), and so there exists a constant \( G = G(k, a_1) > 0 \) for which \( h \leq G \) holds.

In view of these two cases, the proof of Claim 3 follows by putting \( C_1 = C_1(k, a_1) := \min\{F_3, G_3\} \) and \( C_2 = C_2(k, a_1) := \max\{F, G\} \).
We now complete the proof of Theorem 1.2. Suppose first that $\Gamma$ does not have an eigenvalue in $S_{1/2}$. Then by Claim 1 there exists a constant $H_1 = H_1(k, a_1) > 0$, depending only on $k$ and $a_1$, so that $D - (h + t) \leq H_1$. Therefore, by Theorem 1.1, there are finitely many distance-regular graphs with valency $k \geq 3$, diameter $D$, intersection number $a_1 = \lambda$, and head $h \geq 2$ that have no eigenvalue in $S_{1/2}$. 

On the other hand, if $\Gamma$ does have an eigenvalue in $S_{1/2}$, then by Claim 3, there exist positive constants $H_2 := H_2(k, a_1)$ and $H_3 := H_3(k, a_1)$ so that if $D - (h + t) \leq H_2 h$ then $h \leq H_3$. Thus, if $D - (h + t) \leq H_2 h$ then by Lemma 2.1(iv) 

$$D \leq h + t + H_2 h \leq (H_2 + 2) h \leq (H_2 + 2) H_3,$$

and therefore by Corollary 2.4 there are finitely many distance-regular graphs with valency $k \geq 3$, diameter $D$, intersection number $a_1 = \lambda$, and head $h \geq 2$ that have an eigenvalue in $S_{1/2}$. This completes the proof of the theorem. □

Acknowledgements

This work was done when the first author was visiting the Faculty of Mathematics, Graduate School at Kyushu University (Japan). This work was supported by the Korea Research Foundation Grant (KRF-2004-214-C00132).

The second author was supported by the Com²MaC-SRC/ERC program of MOST/KOSEF (grant # R11-1999-054) and by a grant of POSTECH under project number 1RE0601301.

References