Self-Stabilizing Mutual Exclusion
Under Arbitrary Scheduler*

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A self-stabilizing algorithm, regardless of the initial system state, converges in finite time to a set of states that satisfy a legitimacy predicate. The mutual exclusion problem is fundamental in distributed computing, since it allows processors competing to access a shared resource to be able to synchronize and get exclusive access to the resource (i.e. execute their critical section). It is well known that providing self-stabilization in general uniform networks (e.g. anonymous rings of arbitrary size) can only be probabilistic. However, all existing uniform probabilistic self-stabilizing mutual exclusion algorithms designed to work under an unfair distributed scheduler (that may choose processors to execute their code in an arbitrary manner) suffer from the following common drawback: once stabilized, there exists no upper bound on time between two successive executions of the critical section at a given processor. In this paper, we present the first self-stabilizing algorithm that guarantees such a bound (\(O(n^3)\), where \(n\) is the network size) while working using an unfair distributed scheduler. Our algorithm works in an anonymous unidirectional ring of any size and has a polynomial expected stabilization time.

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1. INTRODUCTION

Mutual exclusion. Mutual exclusion is a fundamental problem in the area of distributed computing. Consider a distributed system of \(n\) processors. Every processor, from time to time, may need to execute a critical section in which exactly one processor is allowed to use some shared resource. A distributed system solving the mutual exclusion problem must guarantee the following two properties:

(i) **Mutual exclusion**: Exactly one processor is allowed to execute its critical section at any time.
(ii) **Fairness**: Every processor must be able to execute its critical section infinitely often.

Self-stabilization. The concept of self-stabilization was first introduced by Edsger W. Dijkstra in 1974 [1]. It is now considered to be the most general technique to design a system to tolerate arbitrary transient faults. A self-stabilizing system guarantees that starting from an arbitrary state, the system converges to a legal configuration in a finite number of steps, and remains in a legal state until another fault occurs (see also [2]).

In the context of computer networks, resuming correct behavior after a fault occurs can be very costly [3]—the whole network may have to be shut down and globally reset in a good initial state. While this approach is feasible for small networks, it is far from practical in large networks such as the Internet. Self-stabilization provides a way to recover from faults without the cost and inconvenience of a generalized human intervention: after a fault is diagnosed, one simply has to remove, repair or reinitialize the faulty components, and the system, by itself, will return to a good global state within a relatively short time span.

Scheduler. All components (processors and communication links) of distributed systems may not share the same speed assumptions (i.e. one processor may execute its code speedily, while many others are very slow). The scheduler is a way to model such different behaviors. A scheduler chooses processors to execute their code at a given time. If the scheduler is given more freedom (power) to make its selections, then the task of designing an algorithm to cope with this scheduler becomes more challenging. In other words, the scheduler works as an adversary against the algorithm ([4, 5]). The synchronous scheduler is one of the simplest (or weakest) schedulers—in every computation step, all processors are allowed to execute their code in lock-step. This scheduler models systems where all processors run (almost) at the same speed. The \(k\)-bounded scheduler may choose enabled processors in such a way that the ratio of speeds between any two processors is at most \(k\). This scheduler models a situation where one processor is at most \(k\) times faster than another. So, the \(k\)-bounded scheduler is a stronger adversary

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than the synchronous one. The arbitrary scheduler represents the strongest possible adversary. It can simply and arbitrarily choose any enabled processors.

**Related work.** Dijkstra’s [1] three self-stabilizing mutual exclusion algorithms are deterministic and non-uniform (in such an algorithm, some processors are distinguished in the sense that they are allowed to execute a program that is different from that of the other processors). In [6], Burns and Pachl presented a deterministic algorithm for uniform unidirectional rings of prime size and proved that no deterministic solution exists for rings of composite size.

Several papers investigated the mutual exclusion problem in the probabilistic (or randomized) setting. Randomization was used to reduce the space in [7, 8], and to deal with anonymous networks in [9, 10]. However, a common problem in all these probabilistic algorithms is that once stabilized, there is no upper bound on the time between two entries into the critical section at a particular processor. In other words, although the expected time between two critical section executions is bounded, there exist computations in which a particular processor may not get the token infinitely often. We refer to these kinds of algorithms as weak probabilistic stabilizing algorithms. Kakugawa and Yamashita [11] presented a probabilistic uniform self-stabilizing algorithm on uniform rings that does guarantee an upper bound between two critical section entries. We call this class of algorithms strong probabilistic stabilizing algorithms. However, the algorithm of [11] works only under the central scheduler (which allows exactly one enabled processor at any time). All previously known algorithms solving the mutual exclusion problem ensure fairness using one of the two well-known methods: (i) by choosing an ad hoc scheduler (e.g. the fair scheduler in [7] or randomized central scheduler in [8]) and (ii) by requiring that the correctness of the system is probabilistic (as in [9] and [10]). The open question in [11] was to design a strong probabilistic stabilizing algorithm that solves the mutual exclusion problem under an unfair distributed scheduler.

**Our contributions.** We answer the open question of [11] and provide a strong probabilistic stabilizing algorithm for the mutual exclusion problem in an anonymous unidirectional ring of any size running under an unfair distributed scheduler. (The distributed scheduler selects an arbitrary non-empty subset of enabled processors in a computation step at any time.) We describe the probabilistic self-stabilizing systems in Section 3. We start with a strong probabilistic algorithm that works under a synchronous scheduler—all processors are activated simultaneously. This first algorithm is derived from the space-optimal weak probabilistic algorithm of [9]. Then, we transform it to a strong probabilistic stabilizing algorithm to work under a k-bounded scheduler (that bounds the ratio of relative speeds of executions of any two processors to k). Finally, we use the composition technique described in [4] to stabilize the algorithm under an unfair distributed scheduler (Section 4). We show that the maximum expected stabilization time is $O(n^3)$ under the unfair and $k$-bounded scheduler, and $O(n^2)$ under the synchronous scheduler. After stabilization, the upper bound between two occurrences of the privilege at a given processor is $O(n^3)$ under the unfair scheduler, $O(kn)$ under the $k$-bounded scheduler and $O(n)$ for the synchronous scheduler.

**Outline.** In Section 2, we present the underlying model for our algorithms. We also define properties related to self-stabilization in the context of probabilistic systems. We present three algorithms and their correctness proofs in Section 4. The complexity results of all algorithms are provided in Section 5. Concluding remarks are made in Section 6.

## 2. MODEL

**Distributed systems.** We model a distributed system $S = (G, \tau, I)$ as a transition system where $G$ is the set of system configurations, $\tau$ is a transition function from $G$ to $G$ and $I$ is the set of initial configurations. A probabilistic distributed system is a distributed system where a probabilistic distribution is defined on the transition function of the system.

We consider unidirectional ring networks where the processors maintain two types of variables: local variables and field variables. Each processor, $P_i$, has two neighbors denoted by $left_i$ (the counterclockwise neighbor of $P_i$) and $right_i$ (the clockwise neighbor of $P_i$). The local variables of $P_i$ cannot be accessed by any of its neighbors, whereas the field variables of $P_i$ are part of the shared register which is used to communicate with $P_i$’s right neighbor. A processor can write only into its own shared register and can read only from the shared registers owned by its left neighbor or itself. The state of a processor is defined by the values of its local and field variables. A processor may change its state by executing its local algorithm (defined below). A configuration of a distributed system is an instance of the state of its processors.

The algorithm executed by each processor is described by a finite set of guarded actions of the form $(guard) \rightarrow (statement)$. Each guard of processor $P_i$ is a Boolean expression involving $P_i$’s variables and $left_i$’s field variables. A processor $P_i$ is enabled in configuration $c$ if at least one of the guards of the program of $P_i$ is true in $c$. Let $c$ be a configuration and $CH$ be a subset of enabled processors in $c$. We denote by $(c, CH)$ the set of configurations that are reachable from $c$ if every processor in $CH$ executes an action starting from $c$. A computation step is a tuple $(c, CH, c')$, where $c' \in (c, CH)$. Note that all configurations $\in (c, CH)$ are reachable from $c$ by executing exactly one computation step. In a probabilistic distributed system, every computation step is associated with a probabilistic value (the sum of the probabilities of the computation steps determined by $(c, CH)$ is 1). A computation of a distributed system is a maximal sequence of computation steps. A history of a computation is a finite prefix of the computation. A history of length $n$ (denoted by $h_n$) starting with the initial configuration $c_0$ can
be defined recursively as follows:

\[ h_n = \begin{cases} (c_0, CH_0, c_1) & \text{if } n = 1 \\ \{h_{n-1}(c_{n-1}, CH_{n-1}, c_n)\} & \text{otherwise} \end{cases} \]

The probabilistic value of a history is the product of the probabilities of all the computation steps in the history. If \( h_n \) is a history such that

\[ h_n = [(c_0, CH_0, c_1) \cdots (c_{n-1}, CH_{n-1}, c_n)] \]

then we use the following notations: the length of the history \( h_n \) (equal to \( n \)) is denoted as \( \text{length}(h_n) \), the last configuration in \( h_n \) (which is \( c_n \)) is represented by \( \text{last}(h_n) \) and the first configuration in \( h_n \) (which is \( c_0 \)) is referred to as \( \text{first}(h_n) \) (first can also be used for an infinite computation). A computation fragment is a finite sequence of computation steps. Let \( h \) be a history, \( x \) be a computation fragment such that \( \text{first}(x) = \text{last}(h) \) and \( e \) be a computation such that \( \text{first}(e) = \text{last}(h) \). Then \( [hx] \) denotes a history corresponding to the computation steps in \( h \) and \( x \), and \( (he) \) denotes a computation containing the steps in \( h \) and \( e \).

3. PROBABILISTIC SYSTEMS

In this section, we give an outline of the probabilistic model used in the rest of the paper. A detailed description of this model is available in [4].

**Scheduler.** A scheduler is a predicate over the system computations. In a computation, a transition \((c_i, c_{i+1})\) occurs due to the execution of a nonempty subset of the enabled processors in configuration \( c_i \). In every computation step, this subset is chosen by the scheduler. The interaction between a scheduler and the distributed system generates some special structures called strategies. The scheduler strategy definition is based on the tree of computations (all the computations having the same initial configuration).

Let \( c \) be a system configuration and \( S \) a distributed system. The tree representing all computations in \( S \) starting from the configuration \( c \) is the tree rooted at \( c \) and is denoted as \( \text{Tree}(S, c) \). Let \( n_1 \) be a configuration in \( \text{Tree}(S, c) \). A branch originating from \( n_1 \) represents the set of all \( \text{Tree}(S, c) \) computations starting in \( n_1 \) with the same first transition. The degree of \( n_1 \) is the number of branches rooted in \( n_1 \).

**Definition 3.1. (Strategy)** Let \( S \) be a distributed system, \( D \) be a scheduler and \( c \) be a configuration in \( S \). We define a strategy w.r.t. \( D \) as the set of computations represented by the tree obtained by pruning \( \text{Tree}(S, c) \) such that the degree of any processor is at most 1 and any execution in the pruned tree satisfies \( D \).

The following definition introduces the notion of cone of execution which can be seen intuitively as a branch in a strategy (see Figure 1).

**Definition 3.2. (Cone)** Let \( s \) be a strategy of a scheduler \( D \). A cone \( C_h(s) \) corresponding to a history \( h \) is defined as the set of all possible computations under \( D \) which create the same history \( h \).

For example, Figure 1 presents a cone \( C_h(s) \) whose history \( h \) is equal to \([c_0, CH_0, c^1_0]\).

The probabilistic value of a cone \( C_h(s) \) is the probabilistic value of the history \( h \) (i.e. the product of the probabilities of all computation steps in \( h \)).

**Definition 3.3. (Subcone)** A cone \( C_{h'}(s) \) is called a subcone of \( C_h(s) \) if and only if \( h' = [hx] \), where \( x \) is a computation fragment.

For example, Figure 1 represents a cone \( C_{h'}(s) \) whose history \( h' \) is equal to

\[ [(c_0, CH_0, c^1_0) \cdots (c^1_{n-1}, CH_{n-1}, c^1_n)] \]
Since \( h' = [h x] \) (with \( x \) equal to \( (e_1^1, CH_1, e_2^1) \ldots (e_n^1, CH_{n-1}, e_n^1) \)), \( C'_h(s) \) is a subcone of \( C_h(s) \).

Let \( S \) be a system, \( D \) a scheduler and \( s \) a strategy of \( D \). The set of computations under \( D \) that reach a configuration \( c' \) satisfying predicate \( P \) (denoted as \( c' \models P \)) is denoted as \( \mathcal{E}P \), and its associated probabilistic value is represented by \( \Pr(\mathcal{E}P) \). We call a predicate \( P \) a closed predicate if the following is true: if \( P \) holds in configuration \( c \), then \( P \) also holds in any configuration reachable from \( c \).

**Probabilistic self-stabilizing systems.** A probabilistic self-stabilizing system is a probabilistic distributed system satisfying two important properties: probabilistic convergence (the probability of the system to converge to a configuration satisfying a legitimacy predicate is 1) and correctness (once the system is in a configuration satisfying a legitimacy predicate, it satisfies the system specification). In this context, the correctness comes in two variants: weak correctness (the system correctness is only probabilistic) and strong correctness (the system correctness is certain).

**Definition 3.4. (Strong probabilistic stabilization)** A system \( S \) is strong self-stabilizing under scheduler \( D \) for a specification \( SP \) if and only if there exists a closed legitimacy predicate \( L \) such that in any strategy \( s \) of \( S \) under \( D \), the two following conditions hold:

(i) The probability of the set of computations under \( D \), starting from \( c \), reaching a configuration \( c' \), such that \( c' \) satisfies \( L \), is 1 (probabilistic convergence).

(Formally, \( \forall s, \Pr(\mathcal{EL}) = 1 \).)

(ii) All computations, starting from a configuration \( c' \) such that \( c' \) satisfies \( L \), satisfy \( SP \) (strong correctness).

(Formally, \( \forall s, \forall e, e' \in s, e = (he') : last(h) \models L \Rightarrow e' \models SP. \))

**Convergence of probabilistic stabilizing systems.** We borrow a result of [4] to prove the probabilistic convergence of the algorithms presented in this paper. This result is built upon some previous work on probabilistic automata [12, 13, 14, 15] and provides a complete framework for the verification of self-stabilizing probabilistic algorithms. We need to introduce a few terms before we are ready to present this result. First, we explain a key property called local convergence, denoted by \( LC \). Informally, the \( LC \) property characterizes a probabilistic self-stabilizing system in the following way: the system reaches a configuration which satisfies a particular predicate, in a bounded number of computation steps with a positive probability.

**Definition 3.5. (Local convergence)** Let \( s \) be a strategy, and \( P_1 \) and \( P_2 \) be two predicates on configurations, where \( P_1 \) is a closed predicate. Let \( \delta \) be a positive number \( \in \{0, 1\} \) and \( N \) is a positive integer. Let \( C_h(s) \) be a cone with last(h) \( \models P_1 \) and let \( M \) denote the set of subcones \( C_h(s) \) of \( C_h(s) \) such that last(h') \( \models P_2 \) and length(h') \( - \) length(h) \( \leq N \). Then \( C_h(s) \) satisfies the local convergence property denoted as \( LC(P_1, P_2, \delta, N) \) if and only if \( \Pr(\bigcup_{C_h(s) \in M} C_h(s)) \geq \delta \).

**Field variables:**
- \( t_p \in [0, \text{md}(n) - 1] \) (the privilege.)
- Variables:
  - \( \text{go}_\text{ahead}_p \in \{ \text{pass}, \text{wait} \} \)
  - \( \text{rand}_\text{bool}_p \) holds a random value in \{1, 0\}. Each value has a probability of 1/2.

**Predicate:**
- \( \text{Privilege}(p) \equiv t_p - t_{p'} \neq 1 \text{ mod mnd}(n) \)

**Macro:**
- \( \text{Pass}_\text{privilege}(p) : t_p := (t_p + 1) \text{ mod mnd}(n) \)

**Actions:**
- \( A_1 : \text{Privilege}(p) \land \text{go}_\text{ahead}_p = \text{wait} \rightarrow \text{if (rand}_\text{bool}_p = 1) \text{ then go}_\text{ahead}_p = \text{pass}; \text{else Pass}_\text{privilege}(p); \)
- \( A_2 : \text{Privilege}(p) \land \text{go}_\text{ahead}_p = \text{pass} \rightarrow \text{Pass}_\text{privilege}(p); \text{if (rand}_\text{bool}_p = 0) \text{ then go}_\text{ahead}_p = \text{wait}; \)

**Algorithm 4.1.** Mutual exclusion under a synchronous scheduler (for \( p \)).

Now, if in strategy \( s \), there exists \( \delta > 0 \) and \( N > 1 \) such that any cone \( C_h(s) \) with last(h) \( \models P_1 \) satisfies \( LC(P_1, P_2, \delta, N) \), then the result of [4] states that the probability of the set of computations under \( D \) reaching configurations satisfying \( P_1 \land P_2 \) is 1. Formally:

**Theorem 3.1. ([4])** Let \( s \) be a strategy. Let \( P_1 \) and \( P_2 \) be closed predicates on configurations such that \( \Pr(\mathcal{E}P_{1x}) = 1 \). If \( \exists \delta > 0 \) and \( \forall N > 1 \) such that any cone \( C_h(s) \) with last(h) \( \models P_1 \) satisfies \( LC(P_1, P_2, \delta, N) \), then \( \Pr(\mathcal{E}P_{11x}) = 1 \), where \( P_{12} = P_1 \land P_2 \).

**4. Strong probabilistic stabilizing mutual exclusion**

In this section, we present three solutions to the mutual exclusion problem for three different schedulers: synchronous scheduler (Section 4.1), \( k \)-bounded scheduler (Section 4.2) and unfair scheduler (Section 4.3).

**Specification of the mutual exclusion problem.** We specify the mutual exclusion problem (\( SP_{ME} \)) as follows: there is exactly one privilege in the system at any time and every processor obtains the privilege infinitely often.

**4.1. Synchronous scheduler**

The algorithm for mutual exclusion under the synchronous scheduler is presented as Algorithm 4.1.

In Algorithm 4.1, every processor \( p \) in the system has a field variable \( t_p \). A processor is privileged if and only if the difference between \( t_p \) and \( t_{p'} \) (the \( t_p \) variable of its left neighbor) is not 1. It was proven in [9] that if operations on \( t_p \) variables are made always modulo \( \text{md}(n) \), where \( n \) is the minimum non-divisor of \( n \). For example, \( \text{md}(5) = 2 \).
number of processors in the ring, then at least one privilege is always present in the ring.

A privileged processor tosses a coin to decide whether it wants to pass the privilege or not. If it decides to keep the privilege, it can do so for one more computation step. The purpose of the go_ahead_p variable is to prevent processor p from keeping the token too long. This allows the algorithm to achieve an upper bound on the service time (the time between two consecutive grants given to processor p to enter its critical section).

We define the legitimacy predicate $L_{ME}$ as follows:

$$L_{ME} \equiv \text{There exists exactly one privilege.}$$

Correctness proof. In this subsection, we will show that every processor enjoys the privilege infinitely often.

**Lemma 4.1.** Starting from any legitimate configuration, each node is privileged once in at most every $2 \times n$ computation steps, where $n$ is the size of the ring.

**Proof.** Let $c$ be a legitimate configuration. Let $p_1$ be the processor holding the privilege in $c$. If the value of $p_1$’s local variable go_ahead equals wait, then $p_1$ executes Action $A_1$. A coin is tossed with two possible outcomes:

(i) $p_1$ changes go_ahead to pass, and in the next computation step, Action $A_2$ is executed, making $p_1$ pass the privilege to its right neighbor.

(ii) $p_1$ passes the privilege immediately to its right neighbor.

Therefore, within at most two computation steps, the privilege is passed from $p_1$ to $p_2$. Since the size of the ring is $n$, $p_1$ gets privileged again within $2 \times n$ computational steps.

**Corollary 4.1.** Starting from any legitimate configuration, the privilege circulates infinitely often in the ring.

Proof of convergence. In order to prove the probabilistic convergence of Algorithm 4.1, we first show that for any strategy $s$, all cones of $s$ satisfy the local convergence property. Then, by Theorem 3.1, the probabilistic convergence of the system will be established.

**Notation 4.1.** Let $\text{Priv}(c)$ denote the number of privileged processors in configuration $c$.

**Lemma 4.2.** Let $s$ be a strategy of Algorithm 4.1 under a synchronous scheduler starting from configuration $c$. There exist $\delta > 0$ and $N \geq 1$ such that any cone of $s$ satisfies

$$LC(\text{true, } L_{ME}, \delta, N)$$

**Proof.** The proof is done using the following two steps:

(i) In strategy $s$, there exists a cone $C_{h1}(s)$, such that configuration last($h1$) satisfies the property:

$$\text{Priv}(\text{last}(h1)) \leq \text{Priv}(c) - 1$$

(ii) Repeat the argument of (1) by considering $C_{h1}(s)$ as the current strategy, until the number of privileges becomes equal to 1.

**Proposition 4.1.** Let $s$ be a strategy. There exists $\delta_1 > 0$, $N_1 \geq 1$ and a cone $C_{h1}(s)$ such that

$$\Pr(C_{h1}(s)) \geq \delta_1$$

$$\text{length}(h1) \leq N_1$$

and

$$\Pr(\text{last}(h1)) \leq \Pr(c) - 1$$

**Proof.** Assume that $\Pr(c_0) = m$. We number the privileges $(t_i)_{i=1,m}$ clockwise such that

$$\text{dist}(t_m, t_1) = \min_{1 \leq i \leq m-1} \text{dist}(t_i, t_{i+1})$$

where $\text{dist}(p, q)$ is the distance from $p$ to $q$ measured following the clockwise direction. Let $d_1$ be the distance in $c_0$ between $t_1$ and $t_{i+1}$. We want to prove that the probability that the distance between the privileges $t_m$ and $t_{i+1}$ decreases, is strictly positive. Intuitively, this means that there is a positive probability that the ‘speed’ of privilege $t_m$ increases as it approaches $t_1$. We consider privileges $t_m, t_1$ and $t_2$, and calculate the probability of $t_2$ leaving a go_ahead variable equal to wait, and that of $t_1$ leaving a go_ahead variable equal to pass. The worst initial configuration for our scenario is when every processor $p$ between $t_m$ and $t_1$ has go_ahead, $t_1$ to reach the processor which held $t_1$ in $c_0$, (2) $t_1$ reached the processor which held $t_2$ in $c_0$ and (3) all the processors visited by $t_1$ set their variable go_ahead to pass and all processors visited by all other privileges set their variable go_ahead to wait. With the above three conditions satisfied, we establish that:

$$\delta_0 = \delta_1 \times \delta_2 \times \delta_0^3$$

where

(a) $\delta_1$ is the probability for $t_m$ to reach the processor which held $t_1$ in $c_0$.

$$\delta_1 \geq \left(\frac{1}{4}\right)^{d_m}$$

(b) $\delta_2$ is the probability for $t_1$ to reach the processor which held $t_2$ in $c_0$ while all processors visited by $t_1$ set their go_ahead variable to pass.

$$\delta_2 \geq \left(\frac{1}{4}\right)^{d_1}$$
stabilizing for
and

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In Algorithm 4.2, the variable $k$ is controlled to include values between 0 and $2\theta + 1$ (where $\theta$ is a parameter of the algorithm). When $go_{\text{ahead}}^p = (2k + 1)$, the processor must pass the privilege and randomly change its value. Otherwise, the processor strictly increases its $go_{\text{ahead}}^p$ value. All the values between 0 and $2k$ represent $\text{wait}$ states and $2k + 1$ represents a $\text{pass}$ state.

**THEOREM 4.1.** Algorithm 4.1 is strong probabilistic stabilizing for $\text{SP}_{\text{ME}}$ under a synchronous scheduler.

### 4.2. $k$-Bounded scheduler

In Algorithm 4.2, the variable $go_{\text{ahead}}^p$ is now extended to include values between 0 and $2\theta + 1$ (where $\theta$ is a parameter of the algorithm). When $go_{\text{ahead}}^p = (2\theta + 1)$, the processor must pass the privilege and randomly change its value. Otherwise, the processor strictly increases its $go_{\text{ahead}}^p$ value. All the values between 0 and $2\theta$ represent $\text{wait}$ states and $2\theta + 1$ represents a $\text{pass}$ state.

**LEMMA 4.3.** (Strong correctness) Starting from any legitimate configuration, each processor is privileged within $(2\theta + 2) \times n$ computation steps, where $n$ is the size of the ring.

**Proof.** The worst scenario in terms of a privilege being held at processor $p$ is as follows: $p$ has its $go_{\text{ahead}}^p = 0$ and every time $p$ is chosen, $\text{rand}_\text{bool}^p = 0$. In the worst case, the privilege remains at $p$ for $(2\theta + 2)$ steps (for $2\theta + 1$ steps, $go_{\text{ahead}}^p$ is incremented, and finally, in step $2\theta + 2$, the privilege is passed).

**LEMMA 4.4.** (Probabilistic convergence) Let $s$ be a strategy of Algorithm 4.2 under a $k$-bounded scheduler starting in configuration $c$. There exists $\delta > 0$ and $N \geq 1$ such that every cone of $s$ satisfies $\text{LC}(true, \text{L}_{\text{ME}}, \delta, N)$.

**Proof.** In the following, $\text{dist}(p, q)$ denotes the distance between processors $p$ and $q$, which is equal to the number of processors between $p$ and $q$ in the clockwise direction, plus 1. Assume that in $c$, there are $m$ privileges $t_1, \ldots, t_m$, where

$$\text{dist}(t_m, t_1) = \min_{1 \leq i \leq m-1} (\text{dist}(t_i, t_{i+1}))$$

Let $d_i$ be the distance in $c$ between $t_i$ and $t_{i+1}$. In the worst case, every processor $p$ between $t_m$ and $t_1$ is in the ‘worst’ $\text{wait}$ state ($go_{\text{ahead}} = 0$), and every processor $p'$ between $t_1$ and $t_2$ is in the $\text{pass}$ state ($go_{\text{ahead}}^p = 2k + 1$). We now consider the following two cases:

(i) We calculate the probability $\delta_0$ to obtain a cone $C_{h_0}(s)$, where last($h_0$) satisfies the following three properties:

1. $t_m$ reached the processor which held $t_1$ in $c$.
2. $t_1$ reached the processor which held $t_2$ in $c$.
3. All the processors visited by $t_1$ set their variable $go_{\text{ahead}}$ to $2k + 1$, and the processors visited by $t_m$ set their variable $go_{\text{ahead}}$ to 0. Then in last($h_0$),

$$\text{dist}(t_m, t_1) \leq (2k + 2) \times d_m,$$

$$\text{length}(h_0) \leq \text{max}(d_1, (2k + 2)d_m),$$

and

$$\delta_0 = \delta_0^1 \times \delta_0^2 \times \delta_0^3,$$

where

(a) $\delta_0^1$ represents the probability for $t_m$ to reach the processor which had $t_1$ in $c$

$$\delta_0^1 \geq \left(\frac{1}{2}\right)^{(2k+2)d_m}$$

(b) $\delta_0^2$ represents the probability for $t_1$ to reach the processor which had $t_2$ in $c$, while all processors visited by $t_1$ set their $go_{\text{ahead}}$ variable to $2k + 1$

$$\delta_0^2 \geq \left(\frac{1}{2} \times \frac{1}{2k + 2}\right)^{d_1}$$
We calculate the probability \( \delta_3 \):

Field:

\[ t_p \in [0, mnd(n) - 1] \text{ (the privilege) } \]

Variables:

\( \text{rand\_bool}_p \) holds any value in \([0, 1]\). Each value has a probability of \(1/2\).

\( \text{go\_ahead}_p \) holds any integer value in \([0..(2k + 1)]\).

Predicate:

\[ \text{Privilege}(p) \equiv t_p - t_{ip} \neq 1 \text{ mod } mnd(n) \]

Macro:

\[ \text{Pass\_privilege}(p) : t_p := (t_p + 1) \text{ mod } mnd(n) \]

Actions:

\[ A_1 :: \text{Privilege}(p) \land \text{go\_ahead}_p \neq (2k + 1) \rightarrow \]

if \( \text{rand\_bool}_p = 1 \) then \( \text{go\_ahead}_p := (2k + 1) \) else \( \text{go\_ahead}_p := \text{go\_ahead}_p + 1 \);

\[ A_2 :: \text{Privilege}(p) \land \text{go\_ahead}_p = (2k + 1) \rightarrow \]

\[ \text{Pass\_privilege}(p); \text{ if } \text{rand\_bool}_p = 0 \text{ then } \text{go\_ahead}_p := \text{random}(0..2k + 1); \]

**ALGORITHM 4.2.** Mutual exclusion under a \( k \)-bounded scheduler (for \( p \)).

(c) \( \delta_3 \) represents the probability that all processors visited by all other privileges set their \( \text{go\_ahead} \)
variable to 0

\[ \delta_3 \geq \left( \frac{1}{4} \times \frac{1}{2k + 2} \right) (m-2) \max((2k+2)d_m,d_l) \]

Thus,

\[ \delta_0 \geq \left( \frac{1}{2} \right)^{\max((2k+2)d_m,d_l)m} \]

\[ \times \left( \frac{1}{2k + 2} \right)^{d_l+(m-2) \max((2k+2)d_m,d_l)} \]

\[ \geq \left( \frac{1}{2} \times \frac{1}{2k + 2} \right)^{\max((2k+2)d_m,d_l)m} \]

(ii) We calculate the probability \( \delta_1 \) that cone \( C_{h1}(s) \) has a subcone \( C_{hm+1}(s) \) such that

\[ \text{Priv}(\text{last}(h1)) \leq \text{Priv}(c) - 1 \]

In order to keep tokens \( t_m \) and \( t_1 \) as far as possible from each other, the \( k \)-bounded scheduler chooses the processors holding \( t_1 \) as often as possible. Note that in order to keep the tokens \( t_m \) and \( t_1 \) apart, the \( k \)-bounded scheduler chooses the processors holding \( t_1 \). By the scheduler definition, anytime the processor holding \( t_1 \) is chosen \( k \) times, every other processor is chosen at least once. Nevertheless, after \( 2k + 2 \) choices of the processor holding \( t_1 \), the distance between \( t_m \) and \( t_1 \) is decreased by 1 (i.e. \( t_1 \) moves one step forward, while \( t_m \) moves two steps forward). Then, after at most \((2k + 2)^2d_m \) choices of \( t_1 \), tokens \( t_m \) and \( t_1 \) merge. Hence, the probability that tokens \( t \) and \( t_1 \) merge and conditions (2) and (3) be verified is

\[ \delta_1 \geq \delta_0 \left( \frac{1}{2} \right)^{2k d_m(2k+2)} \left( \frac{1}{2} \times \frac{1}{2k + 2} \right)^{2d_m(2k+2)} \]

The length of the history \( h1 \) is

\[ \text{length}(h1) \leq \text{length}(h0) + d_m \times (2k + 2)^2 \]

\[ \leq \max(d_l, (2k + 2) \times d_m) + d_m \times (2k + 2)^2 \]

and

\[ \delta_1 \geq \left( \frac{1}{2} \right)^{2(2k+2)n(n+2(k+1))} \left( \frac{1}{2k + 2} \right)^{2(2k+2)n(n+1)} \]

Reapplying (2) for the cone \( C_{h1}(s) \), there is a positive probability to obtain a subcone \( C_{h2}(s) \) such that \( \text{Priv}(\text{last}(h2)) \leq m - 2 \), and then a positive probability \( \delta \) to obtain a subcone \( C_{hm-1}(s) \), where the number of privileges is 1. We now have

\[ \delta \geq \left( \frac{1}{2} \right)^{2(2k+2)n^2(n+2(k+1))} \left( \frac{1}{2k + 2} \right)^{2(2k+2)n^2(n+1)} \]

and

\[ \text{length}(hm - 1) \leq (2k + 2)^2 \times n^2 \]

\[ \square \]

From Lemmas 4.3 and 4.4 and Theorem 3.1, we can claim the following result.

**THEOREM 4.2.** Algorithm 4.2 is strong probabilistic stabilizing for \( \text{SP\_ME} \).

### 4.3. Unfair scheduler

In this section, we extend Algorithm 4.2 for an unfair scheduler. The idea of crossover composition (introduced in [4]) is used to implement this extension. The crossover composition can be seen as a black box with two algorithms as input (denoted by \( A \) and \( B \)) and one algorithm as output (denoted by \( O \)). The composition goal is to improve Algorithm \( A \) with nice properties of Algorithm \( B \). The output Algorithm \( O \) solves the same problem as \( A \), but benefits from the properties of Algorithm \( B \).
Fields:
\[ t_p \in [0, m\text{nd}(n) - 1] \text{ (the privilege)} \]
\[ ft_p \in [0, m\text{nd}(n) - 1] \text{ (the fair privilege)} \]

Variables:
\[ \text{rand\_bool}_p \text{ holds any value in } [0, 1]. \] Each value has a probability of 1/2.
\[ \text{go\_ahead}_p \text{ holds any integer value in } [0..(2k+1)]. \]

Predicates:
\[ \text{Privilege}(p) \equiv t_p - ft_p \not\equiv 1 \text{ mod } m\text{nd}(n) \]
\[ \text{Fair\_privilege}(p) \equiv ft_p - t_p \not\equiv 1 \text{ mod } m\text{nd}(n) \]

Macros:
\[ \text{Pass\_privilege}(p) : t_p := (t_p + 1) \text{ mod } m\text{nd}(n) \]
\[ \text{Pass\_Fair\_privilege}(p) : ft_p := (ft_p + 1) \text{ mod } m\text{nd}(n) \]

Actions:
\[
\text{A}_1::\text{Privilege}(p) \land \text{go\_ahead}_p \neq (2k+1) \land \text{Fair\_privilege}(p) \rightarrow
\quad \text{Pass\_Fair\_privilege}(p) \]
\[
\text{if } (\text{rand\_bool}_p = 1) \text{ then go\_ahead}_p = (2k+1) \text{ else go\_ahead}++; \]
\[
\text{A}_2::\text{Privilege}(p) \land \text{go\_ahead}_p = 2k+1 \land \text{Fair\_privilege}(p) \rightarrow
\quad \text{Pass\_Fair\_privilege}(p) \land \text{Pass\_privilege}(p); \]
\[
\text{if } (\text{rand\_bool}_p = 0) \text{ then go\_ahead}_p = \text{random(0..2k+1)}; \]
\[
\text{A}_3::\neg\text{Privilege}(p) \land \text{Fair\_privilege}(p) \rightarrow
\quad \text{Pass\_Fair\_privilege}(p) \]

ALGORITHM 4.3. Mutual exclusion under an unfair scheduler (for \( p \)) with \( k = n - 1. \)

5. COMPLEXITY

5.1. Space complexity

The minimum non-divisor of \( n \) is \( O(\log(n)) \) [16]. Therefore, Algorithm 4.3 needs \( O(\log(n-1) + 2 \times \log(\log(n))) \) bits per processor. Algorithms 4.1 and 4.2 use \( O(\log(\log(n))) \) and \( O(\log(k) + \log(\log(n))) \) bits per processor respectively. Moreover, since it has been shown in [17] that \( m\text{nd}(n) \) is constant on average (over all values of \( n \)), it follows that on average, the space complexity of Algorithm 4.3 is \( \log(n) \), and that the space complexity of Algorithms 4.1 and 4.2 is \( O(1) \) and \( O(\log(k)) \) respectively.

5.2. Time complexity

In this section, we first provide the stabilization time for all three algorithms (i.e. the time to recover from a fault), and then the propagation delay of the token once the system is stabilized (i.e. the service time needed to grant a processor to enter its critical section).

5.2.1. Stabilization time

As the convergence of our algorithms is probabilistic, we can only guarantee maximum expected stabilization time. In the literature (e.g. [5]), the maximum expected stabilization time is expressed in terms of rounds, where a round is a minimal computation fragment during which every processor executes one action.

From Lemmas 4.1 and 4.3, a round is \( O(n) \) computation steps under the synchronous scheduler, and \( O(nk) \) computation steps using the \( k \)-bounded and unfair scheduler.

ALGORITHM 4.1. We consider the behavior of \( m \) pairs of successive tokens during one round. The probability that no
two consecutive tokens merge in this round is less than
\[ p = \left( \frac{1}{2} \right)^{m \times n} \]

Thus, the probability that at least one pair of consecutive tokens merges is more than \( q = 1 - p \). Then, the expected number of rounds before the system reaches a configuration with \( m - 1 \) tokens is \( E[m, m - 1] < \frac{2}{q} \), i.e.
\[ E[m, m - 1] < \frac{2^{m \times n}}{2^{2m \times n} - 1} \]

The maximum expected number of rounds (let us denote it by \( T_{4.1} \)) to converge from an arbitrary configuration (where the number of privileges is \( m \)) to a legitimate configuration (where the number of privileges is \( 1 \)) is given by the formula
\[ T_{4.1} \leq \sum_{i=2}^{m} \frac{2^{i \times n}}{2^{2i \times n} - 1} \leq m + 2 \]

If \( T_{4.1} \) is \( O(m) \) rounds and \( m \leq n \), then \( T_{4.1} \) is \( O(n^2) \) computation steps.

**Algorithm 4.2** where \( k = n - 1 \) and Algorithm 4.3. We now calculate the maximum expected number of rounds for Algorithm 4.2 (where \( k = n - 1 \)) to stabilize starting from the worst possible configuration (with \( m \) tokens).

First, we find an upper bound on the expected number of rounds needed to reach a configuration where the number of tokens is one less than that in the starting configuration. We consider the behavior of \( m \) pairs of successive tokens during one round. The probability that no two consecutive tokens merge is less than
\[ p = \left( \frac{1}{2} \times \frac{1}{(2k + 2)} \right)^{m \times n} \]

Thus, the probability that at least one pair of consecutive tokens merges is more than \( q = 1 - p \). Then, the expected number of rounds before the system reaches a configuration with \( m - 1 \) tokens is \( E[m, m - 1] < (1/q) \), i.e.
\[ E[m, m - 1] < \frac{2^{(2k + 2) \times m \times n}}{2^{(2k + 2)^{m 	imes n} - 1}} \]

The maximum expected number of rounds \( T_{4.2} \) before stabilization of Algorithm 4.2 (where \( k = n - 1 \)) from a configuration with \( m \) privileges to a configuration with 1 privilege is given by the formula
\[ T_{4.2} \leq \sum_{i=2}^{m} E[i, i - 1] = \sum_{i=2}^{m} \frac{2(2n - 1)^{i \times n}}{2(2n - 1)^{i \times n} - 1} \leq m \]

Since \( T_{4.2} \) is \( O(m) \) rounds and \( m \leq n \), \( T_{4.2} \) is \( O(n^3) \) computation steps. A processor executing Algorithm 4.3 executes a rule of Algorithm 4.2 if and only if it holds a fair token. For the time complexity analysis, the worst number of fair tokens is 1. Hence, the bound provided for Algorithm 4.2 holds for Algorithm 4.3 too. Therefore, its time complexity is \( O(n^3) \).

### 5.2.2 Propagation delay

Once stabilized, in the worst case, the upper bound between two appearances of a privilege at the same processor \( p \) in Algorithms 4.1, 4.2 and 4.3 is \( 2 \times n \), \( (2k + 2) \times n \) and \( n^3 \) respectively. The average delays are \( (3 \times n)/2 \), \((2k + 3) \times n)/2 \) and \((n^3(n + 1))/2 \) respectively.

### 6. CONCLUSION

We presented a solution to the open problem of having a strong probabilistic self-stabilizing mutual exclusion algorithm under an unfair distributed scheduler. Once the system is stabilized, a processor only waits for a bounded (polynomial) amount of time. Bounding the coin tossing as presented in this paper can be applied to several other probabilistic algorithms (e.g. [16, 18]) to provide a bound of the service time. The service time provided by our solutions is \((2k + 2) \times n \) (Algorithm 4.2) and \( n^3 \) (Algorithm 4.3) respectively. The average service times are \((2k + 3) \times n)/2 \) and \((n^3(n + 1))/2 \) for the two algorithms.

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### REFERENCES


