Discrete-time windows with minimal RMS bandwidth for given RMS temporal width

Sebastian Starosielec, Daniel Hägele

Arbeitsgruppe Spektroskopie der kondensierten Materie, Ruhr-Universität Bochum, Universitätsstraße 150, D-44780 Bochum, Germany

A R T I C L E   I N F O

Article history:
Received 20 March 2013
Received in revised form 4 February 2014
Accepted 22 March 2014
Available online 29 March 2014

Keywords:
Window functions
Uncertainty principle
Time-bandwidth product
N-point Fourier transform

A B S T R A C T

We derive a family of discrete window functions for the N-point Fourier transform for application in spectral analysis that optimize the root mean square (RMS) frequency width $\sigma_\omega$ for a given temporal RMS width $\sigma_t$. The window family yields as a byproduct the minimum time-bandwidth product $\sigma_t\sigma_\omega$ for given $\sigma_t$ and N. The new windows interpolate for decreasing $\sigma_t$ between the popular Cosine-window and a nearly Gaussian window. The new “confined Gaussian” window function $g^{CG}_k$ (with $k = 0, \ldots, N-1$) is extremely well approximated by $g^{CG}_k \approx G(k) - G(-1/2)(G(k+N) + G(k-N))/(G(-1/2+N) + G(-1/2-N))$ with the Gaussian $G(x) = \exp(-\delta t^2(x - (N-1)/2)^2/4\delta t^2)$, the temporal width $s = \sigma_t$, and time step $\delta t$.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

In this paper we solve the problem of window functions with minimum time-bandwidth product for the N-point Fourier transform. The problem appears in digital spectral analysis whenever time and frequency localization is important. The encoding of audio signals into the MP3 or related formats is probably the most notable example where such requirements need to be met [1]. In general, the application of windows is not limited to audio signals. In fact, many different windows are in use for digital image processing [2] and signal processing their choice being determined by application purposes like sideband suppression, frequency estimation, or peak-area estimation (for a recent review on the use of windows see [3]). Suggestions for windows with special properties appear on a regular basis (e.g. [4,5]).

While Harris mentions already in his famous 1978 review the truncated-Gaussian window family for its close to optimal root mean square (RMS) time-bandwidth product [6], the question of windows with ultimately the smallest time-bandwidth product had not been settled. The subject of an optimal time-bandwidth product is well understood in the case of a Fourier pair $f(t)$ and $F(\omega)$ of continuous functions with $\sigma_t$ and $\sigma_\omega$ being the RMS widths of $|f(t)|^2$ and $|F(\omega)|^2$, respectively. The so-called classical uncertainty principle states that $\sigma_t\sigma_\omega \geq 1/2$ holds for any Fourier pair (proofs can be found in [7] or [8]). It can be shown that any function of Gaussian form $f(t) = A \exp(-\beta t^2)$ with $\beta > 0$ fulfills equality $\sigma_\omega \sigma_t = 1/2$ (see Gabor in [7]). The truncated Gaussian windows enjoy great popularity in signal processing as they are obtained from a Gaussian via discretization and truncation and are therefore believed to be windows with good time-bandwidth product [6]. Here, we will derive new windows with minimum time-bandwidth products for a given $\sigma_t$, surpassing in performance all popular windows including the truncated Gaussian windows. The related problem of uncertainty relations in the case of discrete signals has been tackled in literature before: Ishii derived an uncertainty relation for infinite discrete temporal signals, showing, however, at the same time that no signal exists that can fulfill exact equality for an uncertainty relation with RMS measures [9]. It has been shown that for certain non-RMS measures of the frequency width windows exist that
2. RMS bandwidth and temporal width of finite discrete windows

In order to quantify the spectral width of a window function \( g_k \), we consider the windowed \( N \)-point Fourier transform

\[
a_q = \delta t \sum_{k=0}^{N-1} g_k x_k e^{-j\omega_k t_q}
\]

of the time-discretized signal \( x_k = x(t_k) \) with \( t_k = k \delta t \), \( q = 0, \ldots, N - 1 \), and \( \omega_q = 2\pi q / (N \delta t) \). The coefficients \( a_q \) are related to the continuous Fourier transform

\[
F_x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt
\]

of the original signal \( x(t) \) which is assumed to be band-limited to the interval \([-\omega_N, \omega_N]\) (with \( \omega_N = \pi / \delta t \) being the Nyquist frequency) via a convolution integral

\[
a_q = \frac{1}{2\pi} \int_{-\omega_N}^{\omega_N} F_x(\omega') \tilde{g}(\omega - \omega') \, d\omega',
\]

where

\[
\tilde{g}(\omega) = \delta t \sum_{k=0}^{N-1} g_k e^{-j\omega t_k}
\]

is the so-called window response function [6]. The continuous response function \( \tilde{g}(\omega) \) is therefore the basis for defining measures of the frequency bandwidth via a given discrete window \( g_k \).

In the following, we define the spectral RMS width \( \sigma_w \) and the temporal RMS width \( \sigma_t \) in terms of \( g_k \). The temporal width is analogous to the continuous case defined as

\[
\sigma_t^2 = \langle (t^2) \rangle - \langle t \rangle^2 = \langle (t - \langle t \rangle)^2 \rangle
\]

where the moments of the time

\[
\langle t^p \rangle = \frac{\sum_{k=0}^{N-1} |g_k|^2 (k \delta t)^p}{\sum_{k=0}^{N-1} |g_k|^2}
\]

are given with respect to the absolute square of the window coefficients \( |g_k|^2 \). The right-hand side of (6) can be written as a quadratic form

\[
\langle t^p \rangle = \langle \delta t^p \rangle \langle g^T \rangle^p \langle g \rangle
\]

where \( \langle g \rangle \) is a vector of dimension \( N \) with \( \langle g \rangle_k = g_k \), \( g^H \) denotes the transposed complex conjugate vector, the normalization \( g^H g = 1 \) holds, and

\[
\langle t^p \rangle_{kl} = \begin{cases} k^p & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}
\]

are \( N \times N \) matrices. Similarly,

\[
\sigma_w^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2
\]

is defined with the help of the window response function via

\[
\langle \omega^p \rangle = \frac{\int_{-\omega_N}^{\omega_N} |\tilde{g}(\omega)|^2 \omega^p \, d\omega}{\int_{-\omega_N}^{\omega_N} |\tilde{g}(\omega)|^2 \, d\omega}.
\]

After expressing \( \tilde{g}(\omega) \) in terms of \( g_k \) and evaluation of the integrals we find for the first and second moment

\[
\langle \omega^p \rangle = \langle \delta t \rangle^p \langle g^H \rangle^p \langle g \rangle
\]

with

\[
\langle g^H \rangle^p_{kl} = \begin{cases} \frac{\pi^2}{3} & \text{for } k = l \\ \frac{2(1 - \delta^H (k - l)^2}{(k - l)^2} & \text{for } k \neq l. \end{cases}
\]

Analogous expressions have been given by Monro for the case of wavelets in [12]. The notation in terms of quadratic forms will be used for deriving windows with minimum frequency width \( \sigma_w \) in the next section.

3. Minimum frequency width for given temporal width

In this section, we determine windows \( g_k \) that minimize \( \sigma_w \) for a given temporal width \( \sigma_t \). The following conditions will be imposed on the minimization: (i) The spectral center of \( \tilde{g}(\omega) \) is located at \( \omega = 0 \), i.e. \( \langle \omega \rangle = 0 \) must hold. (ii) The temporal center is located at \( (N - 1)/2 \), i.e. \( \langle t \rangle = (N - 1)/2 \) must hold. (iii) The condition \( \sigma_t^2 = \langle t^2 \rangle^H - \langle (N - 1)/2 \rangle^2 \rangle = \sigma_t^2 \) for the temporal width must hold [compare (5); we set \( \delta t = 1 \) for the numerics]. (iv) The normalization condition \( g^H g = 1 \) must hold. The conditions (i) and (ii) guarantee that a signal does not appear shifted in frequency or time after harmonic analysis. All traditional windows are found to fulfill these conditions.

In order to find the best RMS frequency width \( \sigma_w \), we have to minimize the function \( h(g) = g^H \langle g \rangle^p \langle g \rangle \) under the conditions (i)–(iv). We will do this by first minimizing \( h \) under conditions (iii) and (iv) only. It will turn out that this minimum already fulfills conditions (i) and (ii) which concludes our minimization procedure.

A minimization problem with constraints can be equivalently stated as a minimization problem without constraints using the well known method of Lagrange multipliers (see [13, Chapter three]). An extremum \( g_{\text{ext}} \) of

\[
L(g, \alpha, \lambda) = g^H \langle g \rangle^p \langle g \rangle - \alpha \langle g^H \rangle^p \langle g \rangle - \lambda (g^H g - 1)
\]

are contained in the sum (14) via the so-called Lagrange multipliers \( \alpha \) and \( \lambda \). The derivatives of \( L \) with respect to \( g_k \)
must vanish and give the following necessary conditions for an extremum:

\[
(P^{2} + \alpha T^{1}) - (N - 1)/2)) g = \delta g. \tag{15}
\]

This is a matrix eigenvalue problem for \( \lambda \) with \( N \) solutions \( g^{(n)}(\alpha), \lambda^{(n)}(\alpha) \) that depend on \( \alpha \). The condition \( g^{H} g = 1 \) is fulfilled by choosing normalized eigenvectors \( g^{(n)} \). The minimization problem can now be restated as finding \( \alpha \) and \( n \) which yield a normalized eigenvector \( g^{(n)}(\alpha) \) of \( \mathbf{M}(\alpha) \) with the lowest possible value \( s_{\alpha} \) for the required \( s_{\alpha} \). This greatly simplifies the initial minimization problem which depends on \( N + 2 \) variables to a minimization problem that requires diagonalization of an \( N \times N \)-Matrix while depending on only two variables, \( \alpha \) and \( n \). Fig. 1 gives a full overview over the cases \( N = 16 \) and \( N = 24 \) of the new minimization problem by plotting the time-bandwidth product \( s_{\alpha} \) versus the temporal width \( s_{t} \) for all eigenvectors \( g^{(n)}(\alpha) \) with \( \alpha \) between \(-10^{\pi} \) and \( 10^{\pi} \) (definition of \( \pi \) see Appendix A). The curves are labeled with the index \( n \) of the eigenvectors assorted according to their corresponding eigenvalues starting with \( n = 0 \) for the lowest eigenvalue \( \lambda^{(0)}(\alpha) \). The curves are all continuous and show a moderate structure which allows for the immediate identification of the curve \( n = 0 \) as the one which contains the best time-bandwidth product for all given \( \alpha \). We find for \( N = 16 \) and \( N = 24 \) a time-bandwidth product \( s_{\alpha} \approx 0.5 \) for \( s_{t} \approx 0.15 \). All other curves lie within an area whose outer borders are given by the curve for \( n = 0 \) for lowest \( s_{\alpha} \) and for \( n = N - 1 \) for highest \( s_{\alpha} \). We find the same structure also for \( N = 128 \) and \( N = 1024 \) (not shown). The scaling behavior of \( s_{\alpha} \) and \( \lambda \) strongly suggests that for any even \( N \) (odd case, see below) the windows with the best time-bandwidth product are found for the eigenvector with the lowest eigenvalue (see Appendix A). The numerical minimization procedure can therefore be simplified to finding that value for \( \alpha \) that yields an eigenvector \( g^{(0)}(\alpha) \) with the desired \( s_{\alpha} = \pi \). We note that for odd \( \alpha = 0 \) is possible and consequently \( s_{\alpha} = 0 \). However, windows with \( s_{\alpha} = 0 \) (all \( g_{k} = 0 \) with the exception of \( g_{(N-1)/2} = 1 \)) are without any relevance for applications and are therefore not regarded here.

We conclude the minimization procedure by showing that conditions (i) and (ii) are fulfilled: As the matrix \( \mathbf{M} \) is symmetric with respect to \( (N - 1)/2 \), all eigenvectors are (or can be chosen to be) either symmetric or antisymmetric with respect to that point. All solutions therefore imply \( \xi = (N - 1)/2 \). Moreover, \( \mathbf{M} \) is real and Hermitian implying that all eigenvectors can be chosen to be real guaranteeing \( g = 0 \). This follows for real \( g_{k} \) from \( \hat{g}(\omega) = \mathbf{g}^{*}(\omega) \) (the \( * \) denotes complex conjugate) and therefore \( |\hat{g}(\omega)|^{2} = |\hat{g}(\omega)|^{2} \) [compare Eq. (4)]. Consequently, all eigenvectors including the global minima fulfill conditions (i) and (ii), which concludes our minimization procedure. The value of \( \alpha \) that yields a window \( g^{(0)}(\alpha) \) with the desired \( \pi \) is calculated numerically using MATLAB (see Appendix B, script for calculating Fig. 4).

The global minima \( g_{k} \) for varying \( \alpha \) constitute a new family of window functions that we will call confined Gaussian windows (CGW) as they possess similar to Gaussian functions in the continuous unconfined case a minimal frequency width \( s_{\alpha} \) for given \( \alpha \), however, now for the case of finite discrete-time signals. Fig. 2(a) shows a series of confined Gaussian windows for increasing \( s_{\alpha} \) and \( N = 1024 \) (compare MATLAB script in Appendix B). The confined Gaussian windows appear to interpolate between the Cosine-window for the largest displayed \( s_{t} \) (found for \( \alpha = 0 \)) and a nearly Gaussian shaped window for decreasing \( s_{\alpha} \). The corresponding frequency response function \( |\hat{g}(\omega)|^{2} \) shows an improving side lobe suppression for decreasing \( s_{\alpha} \) [Fig. 2(b)]. It is instructive to compare our approach to the continuous unconfined case (see Appendix A). Gabor derived eigenfunctions with minimal time-bandwidth products fully analogous to our approach and found \( s_{\alpha} = n + 1/2 \) where \( n = 0, 1, 2, \ldots \) are the indices of the eigenfunctions with ascending eigenvalues [7]. The optimal eigenfunction for \( n = 0 \) was shown to be a Gaussian. In comparison, Fig. 3(a) shows the time-bandwidth product in dependence on \( s_{\alpha} \) for the eigenvectors \( n = 0, 1, 2 \) that were obtained from (15) for \( N = 1024 \). Gabor’s result is recovered for small \( s_{\alpha} \). We find for the lowest eigenvalue an almost perfectly Gaussian shaped window function \( g \) (the window shapes are indicated above the curves). The time-bandwidth products exceed Gabor’s values as \( s_{\alpha} \) is increased and the finite temporal interval requires the window functions to strongly deviate from the Gaussian shape in order to vanish outside the borders. Windows with additional nodes and inferior time-bandwidth product are found for all \( n \geq 1 \). The star-symbol on the curves indicates the case

![Fig. 1. The time-bandwidth products versus the temporal widths of all eigenvectors of (15) for varying parameter \( \alpha \) for the cases \( N = 16 \) and \( N = 24 \). The optimal time-bandwidth products \( s_{\alpha} \approx 0.5 \) are obtained for \( n = 0 \), i.e. the eigenvector with the lowest eigenvalue for a given \( \alpha \) (black line at the bottom).](image-url)
These considerations support our finding that the lowest value eigenvector of (15) is the only candidate for yielding a window with lowest bandwidths for given temporal width (further arguments based on scaling behavior are given in Appendix A).

4. Comparison with traditional windows

Next, we compare the confined Gaussian windows with a number of frequently used windows. In Fig. 3(b), we plot the time-bandwidth product $\sigma_\omega \sigma_t$ versus $\sigma_t$ for the new confined Gaussian window family (lowest bold line) and several other well known window functions for $N=1024$ (see [6,14] for their definitions). The dotted horizontal line at $\sigma_\omega \sigma_t = 1/2$ indicates the classical time–frequency uncertainty limit for the case of continuous functions. The new confined Gaussian window functions exhibit the best possible time–frequency uncertainty for all windows and get close to the 0.5 limit for $\sigma_t \lesssim 0.13N\delta t$. While the often used truncated-Gaussian window exhibits for $\sigma_t \lesssim 0.10N\delta t$ the best time–frequency uncertainty of all traditional windows, it performs still worse than our confined Gaussian window and falls behind most other windows for $\sigma_t \gtrsim 0.11N\delta t$. The Cosine-window, which is known to possess the best absolute frequency width $\sigma_\omega$ [8], almost exactly coincides with the confined Gaussian window for $\sigma_t \approx 0.181N\delta t$ and $\sigma_\omega \sigma_t \approx 0.568$.

Fig. 3(c) shows that the celebrated high-performance windows (Blackman, de la Valle Poussin, Bohmann, Blackman-Harris, Nuttall, Blackman-Nuttall) are all within 2% of the optimal time-bandwidth value set by the confined Gaussian window. This proves that the shapes of these windows must be very close to that of the confined Gaussian as they are all very close to fulfilling the minimization criterion.

Comparing directly the popular Blackman window with a confined Gaussian window with the same $\sigma_\omega$ we find $\alpha = 0$. These considerations support our finding that the lowest value eigenvector of (15) is the only candidate for yielding a window with lowest bandwidths for given temporal width (further arguments based on scaling behavior are given in Appendix A).
approximately the same height of the first side-lobe and a slightly better temporal width for the confined Gaussian window (not shown). The Nuttall window exhibits for the same $\sigma_\alpha$, a better side-lobe suppression compared to the confined Gaussian windows, however, on the expense of a larger temporal width. Note that the confined Gaussian window family is the only family that interpolates between the Cosine-window and the high performance windows.

5. Approximate confined Gaussian window

It is highly desirable to have an explicit approximate expression available for a straightforward use of the confined Gaussian windows to avoid the tedious minimization procedure from above. While we were not able to directly derive such an approximation, we found after some guessing the expression

$$g_k^{(\alpha C)}(k) \approx G(k) - G(-1/2) \frac{G(k+N) + G(k-N)}{G(-1/2+N) + G(-1/2-N)} \quad (16)$$

with the Gaussian function $G(x) = \exp\left(-\frac{\delta t^2}{2}((N-1)/2)^2/4\delta t^2\right)$ to be an excellent approximation of $g_k^{(G)}$. Our guessing was lead by the requirements that a Gaussian function has to be recovered for $\alpha \to 0$ and that the amplitude of the window function has to approach zero at the interval borders. The approximated confined Gaussian window (ACGW) is a Gaussian of width $s \approx \sigma_t$ whose tails are compensated by two shifted negative Gaussians located outside the window. Unlike in the case of the traditional truncated Gaussian window, the ACGW coefficients vanish exactly at $k = -\frac{1}{2}$ and $k = N/2 - \frac{1}{2}$, i.e. at half a sampling point outside the window. Fig. 2(a), (b) shows that the ACGWs and their response functions are almost indistinguishable from the exact case. The thin lines in Fig. 3(b), (c) demonstrate the performance of the approximate confined window functions for $N=1024$. The curve almost perfectly matches the line of the exact confined Gaussian windows. The simple functional form of the ACGW allows for a straightforward use of this window with the immediate benefit of an almost minimal time-bandwidth product. The Cosine window is best approximated for $s = N\delta t / \sqrt{8}$ (i.e. when the second derivative of $g_k^{(\alpha C)}$ vanishes for $k = -1/2$ and $k = N - 1/2$ as required for the Cosine window).

Fig. 4 displays the window parameters $\alpha$ and $s$ of the confined Gaussian and approximate confined Gaussian window in dependence of the window width $\sigma_t$, for $N=1024$ (the MATLAB code for generating the plot is given in Appendix B). The scaling behavior of $\alpha$, $s$, and $\sigma_t$ is regarded by plotting $\sigma_t$ and $s$ in units of $N\delta t$ and $\alpha$ in units of $\pi$ (see Appendix A). The relation $\sigma_t \approx \sigma_t$ for the Gaussian limit is correct for $\sigma_t \leq 0.14 N \delta t$. Fig. 4 provides all information for picking the correct window parameters $\alpha$ and $n$ for a desired temporal width $\sigma_t$. The plot is not only valid for $N=1024$ but also in very good approximation for any $N \geq 16$ as the scaling behavior of $\alpha$, $s$, and $\sigma_t$ with $N$ has been regarded.

6. Conclusion

We have introduced the confined Gaussian window family which possesses the minimum RMS frequency bandwidth for a given RMS temporal width. The optimal window functions are extremely well approximated by a compact analytical expression making the ACGW readily available for applications. The new windows provide the best possible time–frequency localization that can be obtained in the context of finite discrete Fourier transforms for a given temporal width. This property suggests their use in all applications where up to now truncated Gaussian windows had been in use. The confined Gaussian windows should also find application in time–frequency analysis in general and in the context of audio coding by replacing the frequently used Hann and Kaiser-Bessel windows.

Appendix A. Continuous limit and scaling

Continuous limit: The problem of finding windows $g_k$ with minimum time-bandwidth product can in the limit $N \to \infty$ be formulated in terms of a continuous function $g(t)$ on the interval $[0, T]$ via

$$\sqrt{\frac{N}{T}} g_k = g(t) \quad (A.1)$$

where $N \delta t = T = \text{const.}$ and the normalization $\int_0^T |g(t)|^2 \, dt = 1$ is supposed to hold. The factor $\sqrt{N/T}$ guarantees that in the case of a given $g(t)$ the normalization $\sum_{k=0}^{N-1} g_k g_k^* \approx 1$ holds independent of $N$. The expression for $\sigma_t^2$ [see (5) and (6)] transforms in the continuous limit into

$$\sigma_t^2 = \int_0^T g(t) g^*(t) (t - \langle t \rangle)^2 \, dt \quad (A.2)$$

The window response function [Eq. (4)] assumes the form

$$\hat{g}(\omega) = \frac{1}{\sqrt{T}} \int_0^T g(t) e^{-j \omega t} \, dt$$

$$= \sqrt{\frac{T}{\pi}} \int_{-\infty}^{\infty} g(t) e^{-j \omega t} \, dt \quad (A.3)$$

where the integration interval in the second line expanded to $\pm \infty$ since $g(t)$ is zero outside the interval
The expression for $\sigma^2_w$ [Eq. (10)] transforms into

$$\sigma^2_w = \int_{-\infty}^{\infty} g(\omega) G(\omega) \varphi^2(\omega) d\omega,$$

(A.4)

where $\hat{g}(\omega) \propto G(\omega)$ with $G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt / \sqrt{2\pi}$ being the Fourier partner of $g(t)$. Thus, for $N \to \infty$ and $T \to \infty$ the problem of a minimal frequency width for a given temporal width treated in Section 3 transforms into the problem of finding a Fourier pair of functions $g(t)$ and $G(\omega)$ with minimal temporal product. The continuous unconfined version of the problem has the well known solutions of $g(t)$ being Gaussian functions [7].

Scaling behavior: The spectral and temporal width of $g_k$ for a given $g(t)$ scale with $N$ like $\alpha \propto N^{\alpha}$ and $\sigma_w \propto 1/(N^{\alpha})$ which can be easily deduced from (A.1) and (A.4). Consequently, the time bandwidth product $\alpha \sigma_w$ does not depend on $N$ nor on $dt$ (apart from approximation errors due to coarse graining). The scaling $\alpha \propto (N^{\alpha})^{-1}$ and $\lambda \propto (N^{\alpha})^{-1}$ can be deduced from the eigenvalue equation (15): Suppose that a solution of (15) is found with $\sigma_1 = X_0 N^{\alpha}$ and $\sigma_w = Y_0 N^{\alpha}$ with $X > 0$ and $Y > 0$ being scaling factors, then we obtain the relation

$$\left( \frac{Y}{N^{\alpha}} \right)^2 + \alpha (X_0 N^{\alpha})^2 = \lambda \left( \frac{\lambda}{\lambda_0} \right)$$

where the scaling behavior of $\alpha$ and $\lambda$. It is convenient to define a unit $\pi = (N^{\alpha}/10)^{-1}$ for $\alpha$ that has been made use of in Section 3 and Fig. 4. The values of $\sigma_\pi$ correspond to windows $g_k^{(\pi)}(x)$ with a temporal width of $\sigma_\pi \approx N^{\alpha}/10$.

The scaling behavior has important implications for the derivation of confined Gaussian windows (Section 3). Suppose $g_k^{(\pi)}$ is a solution of the eigenvalue equation (15) for some $N$ with a smooth corresponding function $g(t)$, then $g(t)$ can be used to construct other approximate solutions $g_k^{(\pi)}$ of (15) for different values of $N$ with scaled parameters $\alpha^{(\pi)}, \lambda^{(\pi)},$ and $\sigma_\pi \sigma_w = \sigma_\pi \sigma_w$. The scaling behavior guarantees that the order of the eigenvalues $\lambda^{(\pi)} \leq \lambda^{(\pi)} \leq \ldots$ is maintained. This implies that if for a given $N$ windows with optimal time-bandwidth product are found for the index $n$ (e.g. $n = 0$ yields the best window for $N = 24$, see Fig. 1), then for all other $N$ optimal time-bandwidth products can only be found for that value of $n$ (i.e. $n = 0$), at least as long as their $g_k$‘s are well approximated by $g(t)$. This explains our finding that optimal windows are only found for $n = 0$. The scaling behavior also explains why the curves with low $n = 0, 1, 2, \ldots$ in Fig. 1 are almost identical for $N = 16$ and $N = 24$. For large $n$ the curves deviate as the corresponding eigenvectors exhibit more structure and can no longer be approximated by a smooth common $g(t)$ for different $N$.

Appendix B. MATLAB code

MATLAB code for calculating data shown in Figs. 1 and 4

```matlab
function init(N)
% define alphabar, and matrices T and P
% global N T P alphabar
% T=zeros(N,N); P=zeros(N,N);
% for k=1:N
%    T(k,k)=(k-(N-1)/2)/2;
%    for l=1:N
%        if k==1
%            T(k,l)=2*(1-(k-1)/(k-1))^2;
%        else
%            T(k,l)=pi^2/3;
%        end
%    end
%    alphabar=1/(N/4)^4;
% end

function [f11]=helperCGW(anorm)
% determine eigenvectors of M(alpha)
% n = 1 yields CGW
% global N P T
% opts.maxit=10000;
% if(n==N)
%    [g,lambda]=eigs(P+alpha*T, n, 'sa', opts);
% else
%    [g,lambda]=eig(P+alpha*T);
% end
% sigmat=sqrt(diag(g'*g)/norm(g));
% sigma_w=sqrt(diag(g'*g)/norm(g));

function [f11]=helperACGW(anorm)
% determine eigenvectors of M(alpha)
% global N target
% [~ =~ sigma_t]=ACGW(n,abar); % script Figure 1
% global N T P target
% for k=1:N
%    sigma_t=CGW(anorm,abar,1);
%    plot(sigma_t');
% end

function [f11]=helperACGW(anorm)
% determine eigenvectors of M(alpha)
% global N target
% [~ =~ sigma_t]=ACGW(n,abar); % script Figure 1
% global N T P target
% for k=1:N
%    sigma_t=CGW(anorm,abar,1);
%    plot(sigma_t');
% end
```

% Script Figure 1 for sig_t = 0.1 N dt
% global N T P target_stnorm; init(1024);
% target_stnorm=0.1;
% [anorm, aval]=fzero(@helperCGW, 0.1/target_stnorm); CGWg, CGWsigma_t=CGW(anorm,abar,1);
% plot((0:N-1)/N, CGWg*sign(CGWg(512))); % Script Figure 4
global N abar target_stnorm;
init(1024);
fprintf(1, 'ACGW sigma_t/N\%ts/N\%tCGW sigma_t/
N\%ta/abar\n');
for target_stnorm=(0.1:0.01:0.2)
[snorm, sval]=fzero(@helperACGW, target_stnorm);
[anorm, aval]=fzero(@helperCGW,0.1 / ...
target_stnorm);
[ACGWg, ACGWsigma_w, ACGWsigma_t]=ACGW(snorm\nN);
[CGWg, CGWsigma_w, CGWsigma_t]=CGW(anorm\nabar,1);
fprintf(1, '%e\t%e\t%e\t%e\t%e\t%e\n', ...ACGWsigma_t / N, snorm, sval, ...
CGWsigma_t / N, anorm, aval);
end

References