On the Generation of (Minimal) Clones Containing Near-Unanimity Operations

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Abstract—For a clone $C$ that contains a near-unanimity operation, one can define $\lambda(C)$ and $\mu(C)$ to be the smallest integers $k_1$ and $k_2$ such that $C$ is generated by its $k_1$-ary part and can be written as the set of polymorphisms of its $k_2$-th graphic, respectively. In this paper, we discuss the meaning of the functions $\lambda$ and $\mu$, elaborate the connection between them, derive some (sharp) bounds, and calculate the functions values for some selected (minimal) clones.

I. INTRODUCTION

Since this paper is submitted as a contribution to the ISMVL session intended to honour Prof. Ivo Rosenberg, let me start the introduction by explaining how the content of this paper is connected to the research of Prof. Rosenberg. During my graduate studies at the Université de Montréal, Prof. Rosenberg was my supervisor. Among other things, he suggested that I examine the generation of clones that contain near-unanimity operations. These examinations led to some results that I included in my M.Sc. thesis [1]. In the present paper, I present some of these (previously unpublished) results as well as some newer results that are motivated by the results of my thesis.

For $d \geq 2$, a $(d+1)$-ary operation $m$ on a set $A$ is called a near-unanimity operation if

$$m(x, y, \ldots, y) \approx \ldots \approx m(y, \ldots, y, x) \approx y.$$ 

Ternary near-unanimity operations are called majority operations. Denote by $\mathcal{M}^d_n$ the set of all clones on $\{0, \ldots, n-1\}$ that contain a $(d+1)$-ary near-unanimity operation. Moreover, for a given clone $C$, denote by $\Gamma^k_C$, the $k$-th graphic of the clone (see the preliminaries for a definition). It is an easy consequence of the Baker-Pixley Theorem [2] (see II.2) that the following numbers are well-defined for all $C \in \mathcal{M}^d_n$, $n \in \mathbb{N}_+$ and $d \geq 2$:

$$\lambda(C) := \min\{k \in \mathbb{N} \mid \text{Clo}(C^{(k)}) = C\},$$
$$\lambda_d(n) := \max\{\lambda(C) \mid C \in \mathcal{M}^d_n\},$$
$$\mu(C) := \min\{k \in \mathbb{N} \mid C = \text{Pol}\Gamma^k_C\},$$
$$\mu_d(n) := \max\{\mu(C) \mid C \in \mathcal{M}^d_n\}.$$ 

To understand these numbers, note that Clo$(C^{(k)})$ is the smallest clone with $k$-ary part $C^{(k)}$ and that Pol$\Gamma^k_C$ is the largest clone with $k$-ary part $C^{(k)}$. Hence, a clone $C$ is uniquely determined by its $\lambda(C)$-ary part and its $\mu(C)$-ary part, but the determinations themselves are done in different, somewhat opposite, ways.

In Section III, we will start by examining the four functions $\lambda$, $\mu$, $\lambda_d$, $\mu_d$. Although, for a single clone $C \in \mathcal{M}^d_n$, we can have $\lambda(C) > \mu(C)$ as well as $\mu(C) < \lambda(C)$, it is easy to show that we have $\lambda_d(n) = \mu_d(n)$ for all $n \in \mathbb{N}_+$, $d \geq 2$ (see Proposition III.1). After establishing these rather simple observations, we will show that the notion of $C$-independent relations provides us with a common bound on $\lambda(C)$ and $\mu(C)$ for a given $C \in \mathcal{M}^d_n$. This is a result from the aforementioned thesis [1], and we outline how this result was used in [3] to determine $\lambda_d(n) (= \mu_d(n))$ for all $d \geq 2$ and large enough $n$.

As an original result of this paper, we will show that the result from [1] can also be used to obtain a (significantly smaller) bound on the restriction of $\lambda$ and $\mu$ to clones containing conservative near-unanimity operations.

In Section IV we will outline the determination of $\mu(C)$ for all minimal majority clones on a three-element set as it was done in [1], and we will use this determination to infer some minor results about the structure of the lattice of clones.

We will finish the paper with some concluding remarks and open problems in Section V.

II. PRELIMINARIES

For the whole paper, let $A$ be a finite set.

Let $O_A := \bigcup_{k \in \mathbb{N}_+} A^k$ be the set of all (non-nullary) finitary operations on $A$. For $F \subseteq O_A$ and $k \in \mathbb{N}_+$, denote by $F^{(k)}$ the set of $k$-ary operations in $F$. A subset $C \subseteq O_A$ is said to be a clone on $A$ if it contains all the projection mappings

$$\pi^k_i : A^k \to A : (x_1, \ldots, x_k) \mapsto x_i \ (1 \leq i \leq k),$$

and is closed with respect to superposition of operations in the following sense: for $f \in O_A^{(k)}$ and $g_1, \ldots, g_k \in O_A^{(l)}$, the $l$-ary operation $f(g_1, \ldots, g_k)$ defined by

$$f(g_1, \ldots, g_k)(x_1, \ldots, x_l) := f(g_1(x_1, \ldots, x_l), \ldots, g_k(x_1, \ldots, x_l))$$

is also in $C$. Given an algebra, the set of its non-nullary term functions is a clone. Conversely, every clone can be realized as the set of term functions of a suitable algebra. Hence, the
clones on a set $A$ represent all possible different behaviours of algebras with carrier set $A$. This is the main motivation behind clone theory. For an overview of clone theory, we refer the reader to [4] and [5].

For every $F \subseteq O_A$ there is a least clone containing $F$. We denote this clone by $\text{Clo}(F)$ and say that $F$ generates $\text{Clo}(F)$. For operations $f_1, \ldots, f_k$, we briefly write $\text{Clo}(f_1, \ldots, f_k)$ instead of $\text{Clo}((f_1, \ldots, f_k))$. It is easy to see that all clones on a given set $A$ form a complete lattice with respect to inclusion. The lattice is countable and completely known for $|A| \leq 2$ [6]. However, for $|A| \geq 3$, there are continuum many clones, and very little is known about the structure of this lattice. One major problem is to determine all minimal clones, that is, the atoms in the lattice of clones. While they are completely described for $|A| \leq 3$ [7], only some partial results are known for $|A| \geq 4$ (see [8] for a survey on minimal clones). Possibly the most important partial result is Rosenberg’s Classification Theorem [9] which yields that each minimal clone falls under one of five classifications (two of which are characterizations). In Section IV, we will look at one of these five types, namely the case in which the clones are generated by majority operations.

Let $R_A := \bigcup_{k \in \mathbb{N}} \mathcal{R}(A^k)$ be the set of all finitary relations on $A$. For $R \subseteq R_A$, let $R^{(k)}$ be the set of $k$-ary relations among $R$. For $a_1, \ldots, a_l \in A$ with $a_i = \left( \begin{smallmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{li} \end{smallmatrix} \right)$ and an $l$-ary operation $f$, set

$$f[a_1, \ldots, a_l] := \left( \begin{array}{c} f(a_{11}, a_{21}, \ldots, a_{1l}) \\ \vdots \\ f(a_{1n}, a_{2n}, \ldots, a_{1n}) \end{array} \right).$$

An $l$-ary operation $f$ is said to preserve $\sigma \in R_A^k$ (or $\sigma$ is said to be invariant under $f$), written $f \triangleright \sigma$, if $\forall f \triangleright \sigma, \partial \in \sigma$ implies $f[\partial_1, \ldots, \partial_l] \in \sigma$. For $F \subseteq O_A$ and $R \subseteq R_A$, define

$$\text{Inv} F := \{ \sigma \in R_A \mid \forall f \triangleright \sigma \}.$$

Furthermore, let $\text{Inv}^{(m)} F := (\text{Inv} F)^{(m)}$ and define $\text{Pol}^{(m)} R$ analogously. For $F \subseteq O_A$, we have $\text{Clo}(F) = \text{PolInv} F$. Thus, a subset $F \subseteq O_A$ is a clone if and only if $F = \text{PolInv} F$ [10]–[13].

For $\sigma \in R_A^k$, we denote by $\Gamma_C(\sigma)$ the smallest relation among $\text{Inv}^{(k)} C$ (with respect to $\subseteq$) that contains $\sigma$ and say that $\Gamma_C(\sigma)$ is $C$-generated by $\sigma$. The following lemma holds:

**Lemma III.1** ([13]). Let $\sigma = \{ \nu_1, \ldots, \nu_k \}$ be a relation on $A$ and let $C \subseteq O_A$ be a clone. Then,

$$\Gamma_C(\sigma) = \{ g[\nu_1, \ldots, \nu_k] \mid g \in C^{(k)} \}.$$

For a clone $C$, a relation $\sigma$ is called $C$-independent if $\nu \notin \Gamma_C(\sigma \setminus \{\nu\})$ for all $\nu \in \sigma$. Note that each relation contains a $C$-independent relation and that each relation $\sigma \in \text{Inv} C$ is $C$-generated by a $C$-independent relation.

Denote by $\chi_k$ the $n^k \times k$ matrix over $A$ whose rows are the elements of $A^k$ listed in lexicographical order. Denoting the columns of $\chi_k$ by $\kappa_1, \ldots, \kappa_k$, we set

$$\Gamma_C := \Gamma_C(\{\kappa_1, \ldots, \kappa_k\}),$$

and we call $\Gamma_C$ the $k$-th graphic of $C$. Its set of polymorphisms $\text{Pol} \Gamma_C$ is the largest clone $C'$ whose $k$-ary part is $C^{(k)}$.

If $C$ contains a $(d+1)$-ary near-unanimity operation, then the Baker-Pixley Theorem yields that the $d$-ary relations among $\text{Inv} C$ already determine $C$, that is, $C = \text{PolInv}^{(d)} C$.

**Theorem II.1** (Baker-Pixley [2]). If $C$ is a clone on $A$ containing a $(d+1)$-ary near-unanimity operation, then $C = \text{PolInv}^{(d)} C$.

The Baker-Pixley theorem has some important consequences. For instance, it implies that clones that contain near-unanimity operations are finitely generated, and it implies that only finitely many clones on $A$ can contain near-unanimity operations of a given arity. Thus, the definitions of $\lambda, \mu, \lambda_d, \mu_d$ are justified.

**III. THE FUNCTIONS $\lambda, \mu, \lambda_d$ AND $\mu_d$**

In this section we will explain the connection between the functions $\lambda, \mu, \lambda_d, \mu_d$, and provide some results concerning their values. Throughout the entire section, let $n \in \mathbb{N}_+$ and fix $A := \{0, \ldots, n-1\}$.

Clearly, for a given clone $C \subseteq M_n^d$, we cannot expect $\lambda(C) = \mu(C)$. In fact, $\lambda(C) \geq \mu(C)$ and $\mu(C) = \lambda(C)$ are both possible and occur regularly. However, as the following proposition shows, we do have $\lambda_d(n) = \mu_d(n)$.

**Proposition III.1.** $\lambda_d(n) = \mu_d(n)$ for all $d \geq 2$.

**Proof:** For $n = 1$, the statement is trivial. For a given $n \geq 2$, it is easy to find clones $C_1, C_2 \subseteq M_n^d$ such that $\lambda(C_1) \geq d+1$ and $\mu(C_2) \geq d+1$. Thus, $\lambda_d(n), \mu_d(n) \geq d+1$ (in fact, the numbers are much greater, as we will see later). Let us assume $\lambda_d(n) > \mu_d(n)$. But then, there exists a clone $C \subseteq M_n^d$ such that $C = \text{Clo}(C^{(\lambda_d(n))}) \subseteq \text{Clo}(C^{(\mu_d(n))})$. Thus, $C' := \text{Clo}(C^{(\mu_d(n))}) \supseteq C^{(d+1)}$ belongs to $M_n^d$ but is properly contained in $C$ and hence not the largest clone with $\mu_d(n)$-ary part $C^{(\mu_d(n))}$. This is contradiction to the maximality of $\mu_d(n)$, so $\lambda_d(n) \leq \mu_d(n)$. By analogous arguments, we obtain $\lambda_d(n) \geq \mu_d(n)$.

To deal with $\mu(C)$, let us note the following lemma. This will also be a key observation for our sketched calculation in Section IV.

**Lemma III.2.** Let $C \subseteq O_A$ be a clone. Then, $\text{Pol} \Gamma_C$ preserves any relation $\sigma \in R_A$ that is $C$-generated by a relation $\sigma' \in R_A$ where $|\sigma'| \leq k$.

**Proof:** Let $\sigma' = \{ \nu_1, \ldots, \nu_k \}$ (note that $\nu_1, \ldots, \nu_k$ do not have to be distinct). Then, by Lemma III.1, we obtain

$$\sigma = \Gamma_C(\sigma') = \{ g[\nu_1, \ldots, \nu_k] \mid g \in C^{(k)} \} = \{ g[\nu_1, \ldots, \nu_k] \mid g \in \text{Pol}^{(k)} \Gamma_C \} = \Gamma_{\text{Pol} \Gamma_C}^{(k)}(\sigma').$$

Thus, $\sigma$ is preserved by $\text{Pol} \Gamma_C$.
Now, for a \((d+1)\)-ary near-unanimity operation \(m \in O_A\), let us introduce the two integers
\[
\omega_m := \max \left\{ |\sigma| \mid \sigma \in R_A^{(d)} \text{ Clo}(m) \text{-independent}, \Gamma_{\text{Clo}(m)}(\sigma) \neq A^d \right\}
\]
and
\[
k_m := \max \{d + 1, \omega_m\}.
\]
Note that in terms of algebras, \(\omega_m\) is the least integer \(k\) such that every proper subuniverse of \((A, m)^d\) has an at most \(k\)-element generating set.

We have prepared everything to derive a common bound on \(\lambda(C)\) and \(\mu(C)\):

**Lemma III.3** ([11]). For a clone \(C \subseteq O_A\) containing a \((d+1)\)-ary near-unanimity operation \(m\), we have

1. \(\text{Clo}(C^{(k_m)}) = C\) (i.e., \(\lambda(C) \leq k_m\)),
2. \(\text{Pol} \Gamma_C^{k_m} = C\) (i.e., \(\mu(C) \leq k_m\)).

**Proof:** 1) For simplicity set \(C' := \text{Clo}(C^{(k_m)})\). \(C' \subseteq C\) is trivial. It remains to show \(C \subseteq C'\), which is equivalent to \(\text{Inv} C' \subseteq \text{Inv} C\). Since \(k_m \geq d + 1\) implies that \(C\) and \(C'\) both contain \(m\), we can apply the Baker-Pixley-Theorem and it suffices to show that any \(d\)-ary relation on \(A\) that is preserved by \(C'\) is also preserved by \(C\). The full \(d\)-ary relation \(A^d\) is trivially preserved by \(C\), so we can assume \(\sigma \neq A^d\). Clearly, \(\sigma\) must be \(C'\)-generated by a \(C'\)-independent relation \(\sigma'\). It is obvious that \(\sigma'\) is also \(\text{Clo}(m)\)-independent and that we have \(\Gamma_{\text{Clo}(m)}(\sigma') \neq A^d\). Thus, we must have \(|\sigma'| \leq k_m\). Hence, there exist elements \(\nu_1, \ldots, \nu_{k_m}\) (not necessarily distinct) such that \(\sigma' = \{\nu_1, \ldots, \nu_{k_m}\}\). By Lemma II.1, this implies

\[
\sigma = \Gamma_C(\sigma') = \{g[\nu_1, \ldots, \nu_{k_m}] \mid g \in C^{(k_m)}\}
\]

and thus \(\sigma \in \text{Inv} C\).

2) Set \(C' := \text{Pol} \Gamma_C^{k_m}\). We have to show \(C' \subseteq C\). By the same arguments as in the first part of the proof, it is sufficient to show that any \(d\)-ary relation \(\sigma \neq A^d\) preserved by \(C'\) is also preserved by \(C'\). Similar as above, \(\sigma\) is \(C'\)-generated by a relation \(\sigma'\) where \(|\sigma'| \leq k_m\). Thus, by Lemma III.2, \(\sigma\) is preserved by \(C'\) and we have shown \(C' = C\). \(\blacksquare\)

For \(d = 2\), an explicit formula for \(\lambda_2(n)\) was given by Harry Lakser in [14]; for \(n \geq 5\), it holds \(\lambda_2(n) = n(n - 2)\). In the same paper, the determination of \(\lambda_d(n)\) for \(d \geq 3\) and sufficiently large \(n\) was stated as an open problem. With the help of Lemma III.3 (which I had used in the thesis only to derive some rather loose upper bounds), I was later able to solve this problem by proving that we have

\[
\max \{\omega_m \mid m \in O_A\text{ near-unanimity operation}\} = (n - 1)^d - 1
\]

whenever \(n\) is large enough, and that, somewhat surprisingly, this is also the exact value of \(\lambda_d(n)\). Let us put some emphasis on this observation: although \(\omega_m\) can be significantly higher than \(\lambda(C)\) (or \(\mu(C)\)) for a \((d+1)\)-ary near-unanimity operation \(m\) and a clone \(C \in \mathcal{M}_d^n\) with \(m \in C\), the maximum of \(\omega_m\) where \(m\) ranges over all \((d+1)\)-ary near-unanimity operations on \(A\) is the same as the maximum of \(\lambda(C)\) (or \(\mu(C)\)) where \(C\) ranges over all clones among \(\mathcal{M}_d^n\).

**Theorem III.4** ([3]). Let \(n, d \geq 2\). Then \(\lambda_d(n) = (n - 1)^d - 1\) for sufficiently large \(n\) (that is, \(n \geq (d - 1)^d + d + 1\)).

**Proof:** The proof of \(\lambda_d(n) \geq (n - 1)^d - 1\) holds for all \(n \geq 3\) and consists of constructing a clone \(C \in \mathcal{M}_d^n\) with \(\lambda(C) = (n - 1)^d - 1\). The other part consists of proving \(k_m \leq (n - 1)^d - 1\) for all \((d+1)\)-ary near-unanimity operations and \(n \geq (d - 1)^d + d + 1\), which is much more difficult and requires some more complicated combinatorial arguments. For details, we refer the reader to [3], a paper whose single purpose is to prove this theorem.

Let us now turn our attention to the case in which a clone \(C \subseteq O_A\) contains a conservative near-unanimity operation. Recall that an operation \(f \in O_A\) is said to be conservative if \(f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}\) for all \(x_1, \ldots, x_k \in A\). Let us denote by \(C_n\) the set of all clones \(C\) on \(\{0, 1, \ldots, n - 1\}\) that contain a conservative \((d+1)\)-ary near-unanimity operation. Similar to Proposition III.1, we can conclude

\[
\max \{\lambda(C) \mid C \in C_n\} = \max \{\mu(C) \mid C \in C_n\}
\]

for all possible choices of \(n\) and \(d\). To obtain a bound on these two numbers, we will use the following lemma:

**Lemma III.5.** Let \(C \subseteq O_A\) be a clone and let \(m \in C\) be a conservative \((d+1)\)-ary near-unanimity operation. Let \(l \in \{1, \ldots, d\}\) and let \(\sigma\) be a \(d\)-ary \(\text{Clo}(m)\)-independent relation on \(A\). Suppose that there exist pairwise distinct \(a_1, \ldots, a_{d+1} \in A\) such that we have

\[
\left( \begin{array}{c}
x_1 \\
\vdots \\
x_l \\
\vdots \\
x_d
\end{array} \right), \left( \begin{array}{c}
x_1 \\
\vdots \\
x_l \\
\vdots \\
x_d
\end{array} \right), \ldots, \left( \begin{array}{c}
x_1 \\
\vdots \\
x_l \\
\vdots \\
x_d
\end{array} \right) \in \sigma
\]

for some \(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_d \in A\). Then,

\[
\left( \begin{array}{c}
y_1 \\
\vdots \\
y_{l-1} \\
m(a_1, \ldots, a_{d+1}) \\
y_{l+1} \\
\vdots \\
y_d
\end{array} \right) \notin \sigma
\]

for all \(y_1, \ldots, y_{l-1}, y_{l+1}, \ldots, y_d \in A\) except if we have \(y_i = x_i\) for all \(i \in \{1, \ldots, l-1, l+1, \ldots, d\}\).

**Proof:** Without loss of generality, let us assume \(l = 1\). Since \(m\) is conservative, we may also assume without loss of generality that we have \(m(a_1, \ldots, a_{d+1}) = a_1\). Suppose that there exists \(y_2, \ldots, y_d \in A\) such that we have \(\left( \begin{array}{c}
y_1 \\
\vdots \\
y_d
\end{array} \right) \in \sigma\). Then,

\[
m \left[ \begin{array}{c}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_d
\end{array} \right], \left( \begin{array}{c}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_d
\end{array} \right), \ldots, \left( \begin{array}{c}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_d
\end{array} \right) = \left( \begin{array}{c}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_d
\end{array} \right) \in \sigma,
\]

where \(\tilde{a}_1, \ldots, \tilde{a}_d\) are arbitrary elements of \(A\).
which implies \( x_i = y_i \) for all \( i \in \{2, \ldots, d\} \) since \( \sigma \) is \( \text{Clo}(m) \)-independent.

Proposition III.6. \( \lambda(C), \mu(C) \leq dn^{d-1} \) for all \( C \in \mathcal{C}_d^n \).

Proof: By Lemma III.3, we can show the claim by proving \( k_m \leq dn^{d-1} \) for all conservative \((d+1)\)-ary near-unanimity operations \( m \). In other words, if we assume that \( \sigma \in R^0_d \) is \( \text{Clo}(m) \)-independent with \( \Gamma_{\text{Clo}(m)}(\sigma) \neq A^d \), then we need to show \( |\sigma| \leq dn^{d-1} \). Let us describe \( \sigma \) as a binary \( n \times \ldots \times n \) matrix \( M \) by setting

\[
M_{i_1 \ldots i_d} := \begin{cases} 1 & \text{if } \left( i_{\delta_i-1} \right) \in \sigma, \\ 0 & \text{otherwise.} \end{cases}
\]

Lemma III.5 translates into this setting as follows: whenever there exist \( l \in \{1, \ldots, d\}, i_1, i_1-1, i_1+1, \ldots, i_d, j_1, \ldots, j_{d+1} \in \{1, \ldots, n\} \) such that \( j_1, \ldots, j_{d+1} \) are pairwise distinct and

\[
M_{i_1 \ldots i_{l-1} j_{l+1} i_{l+2} \ldots i_d} = \ldots = M_{i_1 \ldots i_{l-1} j_{d+1} i_{d+2} \ldots i_d} = 1,
\]

then there exists \( s \in \{1, \ldots, d+1\} \) such that

\[
\sum_{r_1, \ldots, r_{l-1}, r_{l+2}, \ldots, r_d \in \{1, \ldots, n\}} M_{r_1 \ldots r_{l-1} j_{l+1} r_{l+2} \ldots r_d} = 1. \tag{1}
\]

Let us use this property to show

\[
|\sigma| = \sum_{i_1, \ldots, i_d \in \{1, \ldots, n\}} M_{i_1 \ldots i_d} \leq dn^{d-1}.
\]

Whenever there exist \( i_2, \ldots, i_d, j_1, \ldots, j_{d+1} \in \{1, \ldots, n\} \) such that \( j_1, \ldots, j_{d+1} \) are pairwise distinct and

\[
M_{i_1 i_2 \ldots i_d} = \ldots = M_{j_{d+1} i_2 \ldots i_d} = 1,
\]

then let \( s \in \{1, \ldots, d+1\} \) be an integer such that (1) holds for \( l = 1 \), and let us remove all entries \( M_{j_{l+1} r_2 \ldots r_d} (r_2, \ldots, r_d \in \{1, \ldots, n\}) \) from the matrix \( M \). Note that, in each step, only one of the removed \( n^{d-1} \) entries has the value 1. Similarly, whenever there exist \( i_1, i_3, \ldots, i_d, j_1, \ldots, j_{d+1} \in \{1, \ldots, n\} \) such that \( j_1, \ldots, j_{d+1} \) are pairwise distinct and

\[
M_{i_1 i_3 \ldots i_d} = \ldots = M_{i_1 i_3 \ldots i_d} = 1,
\]

then let \( s \in \{1, \ldots, d+1\} \) be an integer such that (1) holds for \( l = 2 \), and let us remove all entries \( M_{r_1 r_3 \ldots r_d} (r_1, r_3, \ldots, r_d \in \{1, \ldots, n\}) \) from \( M \). After removing all such entries, we end up with a \( k_1 \times k_2 \times n \times \ldots \times n \) matrix \( M' \) where

\[
\sum_{i_1, \ldots, i_d \in \{1, \ldots, n\}} M_{i_1 \ldots i_d} = \sum_{i_2 \in \{1, \ldots, k_1\}, i_3 \in \{1, \ldots, k_2\}, i_{d+1} \in \{1, \ldots, n\}} M'_{i_1 \ldots i_d} + (n-k_1) + (n-k_2).
\]

Without loss of generality, let us assume \( k_2 \leq k_1 \). Since we have ensured

\[
\sum_{j \in \{1, \ldots, k_1\}} M'_{j i_2 \ldots i_d} \leq d
\]

for all \( i_2 \in \{1, \ldots, k_2\}, i_3, \ldots, i_d \in \{1, \ldots, n\} \) by removing the entries as described above, we can infer

\[
\sum_{i_2 \in \{1, \ldots, k_2\}, i_3 \in \{1, \ldots, k_2\}, i_{d+1} \in \{1, \ldots, n\}} M'_{i_1 \ldots i_d} \leq dk_2 n^{d-2}.
\]

Thus, we may finish the proof by showing

\[
dk_2 n^{d-2} + (n-k_1) + (n-k_2) \leq dn^{d-1}. \tag{2}
\]

If \( k_1, k_2 = n \), then this is obvious. Hence, let us assume that we have \( k_2 \leq n-1 \) (recall that we assume \( k_2 \leq k_1 \)). For \( d = 2 \), the inequality (2) is true since

\[
2k_2 + (n-k_1) + (n-k_2) = 2n + k_2 - k_1 \leq 2n.
\]

For \( d \geq 3 \), we have

\[
dk_2 n^{d-2} + (n-k_1) + (n-k_2)
\]

\[
\leq d(n-1)n^{d-2} + (n-k_1) + (n-k_2)
\]

\[
\leq d(n-1)n^{d-2} + 2n
\]

\[
= dn^{d-1} - dn^{d-2} + 2n
\]

\[
< dn^{d-1}.
\]

Thus, \( |\sigma| \leq dn^{d-1} \), as required.

For large enough \( n \), this bound is significantly lower than the tight bound on \( \lambda_0(n) \) that is provided by Theorem III.4. However, this does by no means guarantee that \( dn^{d-1} \) is anywhere near a sharp bound on \( \max \{ \lambda(C) | C \in \mathcal{C}_d^n \} \), even if \( n \) is large (note that, for \( n \leq d \), it is obvious that \( dn^{d-1} \) cannot be a tight bound). I, for one, suppose that \( dn^{d-1} \) is not a tight bound even for enormously large \( n \), but I do suspect (with moderate confidence) that \( dn^{d-1} \) is very close to \( \max \{ \lambda(C) | C \in \mathcal{C}_d^n \} \) whenever we have \( n > d \) (see the discussion of Problem 2 in the final section).

IV. THE MINIMAL MAJORITY CLONES ON A THREE-ELEMENT SET

By Rosenberg’s Classification Theorem, one of the five classes of minimal clones consists of majority clones, that is, clones that are generated by majority operations. In this section, we calculate \( \mu(C) \) for all minimal majority clones \( C \subseteq \mathcal{O}_3 \) where \( |A| \leq 3 \). On a two-element set, this is almost trivial: for \( m \) being the (unique) majority operation on a given two-element set, \( \mu(\text{Clo}(m)) = 3 \) follows immediately from Lemma III.3 and the two obvious facts that we have \( \omega_m \leq 3 \) and \( \mu(\text{Clo}(m)) \geq 3 \). Alternatively, we can use the following observation:

Proposition IV.1. If \( C \subseteq \mathcal{O}_3 \) is a minimal majority clone and \( k \geq \max \{3, |A| \} \), then \( C \) is the only minimal clone contained in \( \text{Pol} \Gamma_C \).

Proof: Suppose that \( C_{\text{min}} \subseteq \text{Pol} \Gamma_C \) is a minimal clone. Choose an operation \( f \) that generates \( C_{\text{min}} \) and has minimal arity among all operations that generate \( C_{\text{min}} \). Let \( l \) be this arity. It is a direct consequence of Rosenberg’s Classification.
Theorem [9] that we have \( l \leq \max\{3, |A|\} \leq k \). Hence, we obtain
\[
f \in C(\mu) \subseteq \text{Pol}(\mu) \Gamma_C = C(\mu).
\]
Thus, \( f \in C \), whence \( C = \text{Clo}(f) = C_{\text{min}} \) (note that \( f \in C \) implies that \( f \) generates \( C \) since otherwise \( \text{Clo}(f) \not\subseteq C \) would contradict the minimality of \( C \)).

Consequently, if \( \mu(C) > \max\{3, |A|\} \) for a minimal majority clone \( C \), then there exists a clone \( C' \) that properly contains \( C \) but no other minimal clone. However, in the well-known clone lattice on a two-element set, there is no clone that properly contains the (only) minimal majority clone \( \text{Clo}(m) \) but no other minimal clone. Thus, \( \mu(\text{Clo}(m)) = 3 \).

For \( |A| = 3 \), things are more complicated. If we fix \( A := \{0, 1, 2\} \), then we have (up to isomorphism) three minimal majority clones, each of which is generated by one of the following three majority operations [7] (note that a majority operation is already uniquely determined by its values on the triples of pairwise distinct elements):

\[
<table>
<thead>
<tr>
<th>(x, y, z)</th>
<th>m_0(x, y, z)</th>
<th>m_{44}(x, y, z)</th>
<th>m_{510}(x, y, z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1, 2)</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(1, 2, 0)</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(2, 0, 1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2, 1, 0)</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The naming of the operations is due to [7] but may not be explained here. Perhaps contrary to expectations, the determination of \( \mu(\text{Clo}(m_0)) \), \( \mu(\text{Clo}(m_{44})) \) and \( \mu(\text{Clo}(m_{510})) \) by hand turns out to be rather difficult.

**Theorem IV.2.**
\[\begin{align*}
1) \ & \mu(\text{Clo}(m_0)) = 3, \\
2) \ & \mu(\text{Clo}(m_{44})) = 3, \\
3) \ & \mu(\text{Clo}(m_{510})) = 4.
\end{align*}\]

**Proof:** The proof as presented in [1] is way too long to present it in this paper. Let us therefore simply sketch its idea. Obviously, each of the three numbers \( \mu(\text{Clo}(m_0)) \), \( \mu(\text{Clo}(m_{44})) \) and \( \mu(\text{Clo}(m_{510})) \) must be at least 3 as the binary and unary part of each of these clones is trivial. To show \( \mu(\text{Clo}(m_{510})) > 3 \), one needs to find an operation \( f \) that preserves \( \Gamma_{\text{Clo}(m_{510})}^3 \), but does not preserve a relation \( \sigma \) that is preserved by \( m_{510} \). The following choices do the trick:

\[
f(x_1, x_2, x_3, x_4) = \begin{cases} x_1, & \text{if } x_1 = x_2, \\
x_1, & \text{if } x_1 = x_3, \\
m_{510}(x_2, x_1, x_3), & \text{if } x_1 = x_4, \\
x_2, & \text{if } x_2 = x_3, \\
m_{510}(x_1, x_2, x_3), & \text{if } x_2 = x_4, \\
m_{510}(x_1, x_2, x_3), & \text{if } x_3 = x_4, \\
\end{cases}
\]

\[
\sigma := \left\{ \left( \frac{0}{2}, \frac{2}{0}, \frac{1}{1}, \frac{2}{2} \right) \right\}.
\]

Note that \( f \) is well-defined since, on \( \{0, 1, 2\} \), at least two variables of a quaternary have to coincide, and the cases do not contradict each other if more than two variables coincide. The other parts of the proof require much more work. Each of the three statements \( \mu(\text{Clo}(m_0)) \leq 3, \mu(\text{Clo}(m_{44})) \leq 3 \) and \( \mu(\text{Clo}(m_{510})) \leq 4 \) are shown by contradiction. For instance, we assume \( \text{Pol}(\mu) \Gamma_{\text{Clo}(m)} C \not\subseteq \text{Clo}(m) \). This implies that there exists a nontrivial \( f \in \text{Pol}(\mu) \Gamma_{\text{Clo}(m)} \) that is not generated by \( m_0 \) that is, \( \text{Clo}(f) \not\subseteq \text{Clo}(m_0) \). Since \( f \) is non-trivial, \( \text{Clo}(f) \) must contain a minimal clone. But now, each minimal clone on a three-element set is generated by an at most ternary operation and we have \( \text{Clo}(f) \subseteq \text{Pol}(\mu) \Gamma_{\text{Clo}(m)} = \text{Clo}(m_0) \). Consequently, the minimal clone contained in \( \text{Clo}(f) \) must be \( \text{Clo}(m_0) \). Thus, \( \text{Clo}(m_0) \subseteq \text{Clo}(f) \). This means that \( \text{Clo}(m_0) \) and \( \text{Clo}(f) \) are both majority clones, and we can apply the Baker-Pixley Theorem to obtain that \( \text{Clo}(f) \not\subseteq \text{Clo}(m_0) \) implies \( \text{Inv} \{ m_0 \} \not\subseteq \text{Inv} \{ f \} \). Thus, there exists a binary relation \( \sigma \in R_A \) that is preserved by \( m_0 \) but not by \( f \). This means

\[
f \left[ \left( \frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_k}{y_k} \right) \right] = \left[ \frac{a}{b} \right],
\]

for some \( \left( \frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_k}{y_k} \right) \in \sigma \) and \( \left[ \frac{a}{b} \right] \not\in \sigma \). By using combinatorial arguments and Lemma III.2, one can then reach a contradiction. The proofs of \( \mu(\text{Clo}(m_44)) \geq 3 \) and \( \mu(\text{Clo}(m_{510})) \geq 4 \) start in the same way and also use Lemma III.2, but require different combinatorial arguments. See [1] for details.

As a curiosity, let us note that Lemma III.3 could not be used to substitute any arguments of this proof. In fact, an almost trivial computer calculation or a more elaborate combinatorial analysis yields \( k_{m_0} = 6, k_{m_{44}} = 4 \) and \( k_{m_{510}} = 5 \), which is close to \( \mu(\text{Clo}(m_0)) \) (i.e. \( \{0, 44, 510\} \) in the last two cases, but not quite good enough.

By Proposition IV.1, the last theorem also yields the following information about the lattice of clones on a three-element set:

**Corollary IV.3.** The clone \( \text{Clo}(m_{510}) \) has an upper neighbour in the clone lattice that coincides with \( \text{Clo}(m_{510}) \) on its ternary part and does not contain another minimal clone. In contrast, the other two minimal majority clones \( \text{Clo}(m_0) \) and \( \text{Clo}(m_{44}) \) do not have such an upper neighbour.

**Proof:** It follows from \( \mu(\text{Clo}(m_{510})) = 4 \) that \( C := \text{Pol}(\mu) \Gamma_C \), properly contains \( \text{Clo}(m_{510}) \), and Proposition IV.1 yields that \( C \) does not contain any other minimal clone. By construction, \( C \) and \( \text{Clo}(m_{510}) \) coincide on their ternary part. Moreover, there cannot exist infinitely many clones between \( \text{Clo}(m_{510}) \) and \( C \) since only finitely many clones can contain \( m_{510} \). Thus, the interval \( \{ \text{Clo}(m_{510}), C \} \) is finite, whence it follows that \( \text{Clo}(m_{510}) \) has the desired upper neighbour. The other two minimal majority clones \( \text{Clo}(m_0) \) and \( \text{Clo}(m_{44}) \) cannot have such an upper neighbour in the clone lattice since, according to Theorem IV.2, they are already the largest clones with ternary part \( \text{Clo}(m_0)^{(3)} \) and \( \text{Clo}(m_{44})^{(3)} \), respectively.

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In other words, for $|A| = 3$, there exists a clone $C \subseteq O_A$ that has no lower neighbour in the clone lattice except a minimal majority clone with which it coincides on its ternary part. Since the construction of a minimal majority clone on an $n$-element set can be extended to a minimal majority clone on an arbitrarily larger set, we can conclude that such a clone exists on all sets $A$ with $|A| \geq 3$.

V. Conclusion & Open Problems

In this paper, we discussed the functions $\lambda, \mu, \lambda_d, \mu_d$, gave (sharp) bounds on their values, calculated them for selected minimal clones and explained the connection between them. In the course of this investigation, a few open problems occurred.

Recall that the proof of Theorem III.4 establishes $\lambda_d(n) \geq (n - 1)^d - 1$ for all $n \geq 3$, but $\lambda_d(n) \leq (n - 1)^d - 1$ only for all $n \geq (d - 1)^2 d + d + 1$. Although a restriction to $n$ (depending on $d$) is certainly necessary, this bound seems too big. It is possible to give more complicated but overall similar proofs yielding lower restrictions to $n$ (sometimes formulated as the solution of an optimization-problem). However, even these bounds seem far from being the lowest $N$ such that $\lambda_d(n) = (n - 1)^d - 1$ for all $n \geq N$. This leads to the first open problem:

Problem 1. For $d \geq 2$, find the smallest $N \in \mathbb{N}$ such that $\lambda_d(n) = (n - 1)^d - 1$ for all $n \geq N$.

At the end of Section III, we showed that, for arbitrary $n, d \geq 2$, it holds

$$\max \{\lambda(C) \mid C \in C_n^d\} \leq d n^{d-1}.$$  

However, we did not show that the bound is tight, and in fact, as discussed at the end of Section III, it is very well possible that it is not, even for large $n$. Hence, the following obvious problem arises:

Problem 2. What is $\max \{\lambda(C) \mid C \in C_n^d\}$ for $n$ large enough?

If I had to guess the value of $\max \{\lambda(C) \mid C \in C_n^d\}$ for large enough $n$, then I would put my money on the number $dn^{d-1} - 1$ and would guess that “large enough” means $n > d$. However, I would prefer not to bet much.

A third problem arises from Theorem IV.2, where we have seen that all minimal majority clones are generated by their fourth (but not third) graphic. This raises the paper-concluding question of what happens for minimal majority clones on greater sets:

Problem 3. For $n \in \mathbb{N}_+$, let $\nu(n)$ be the smallest integer $k$ such that each minimal majority clone on an $n$-element set is the set of the polymorphisms of its $k$-th graphic. What is a formula (or a good upper bound) for $\nu(n)$?

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References


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