A GENERAL DUALITY THEORY FOR CLONES

SEBASTIAN KERKHOFF
Department of Mathematics, TU Dresden
01069 Dresden, Germany
Sebastian.Kerkhoff@tu-dresden.de

Received 24 March 2012
Accepted 14 November 2012
Published 20 March 2013

Communicated by K. A. Kearnes

Inspired by work of Mašulović, we outline a general duality theory for clones that will allow us to dualize any given clone, together with its relational counterpart and the relationship between them. Afterwards, we put the approach to work and illustrate it by producing some specific results for concrete examples as well as some general results that come from studying the duals of clones in a rather abstract fashion.

Keywords: Clones; duality theory; dual equivalences; Galois connections; coclones; clones of dual operations; centralizer clones; distributive lattices; Lawvere theories.

Mathematics Subject Classification: 06D50 (03C05, 06A15)

1. Introduction

The principle of duality is “a very pervasive and important concept in (modern) mathematics” [10] and “an important general theme that has manifestations in almost every area of mathematics” [9]. However, for clones, a general duality theory seems to be non-existent. The usual approach is to consider a clone as the set of term functions of a suitable algebra and then try to dualize this algebra, which may, or may not, be possible. Another approach was introduced by Mašulović in [17], where clones are dualized by treating them as sets of homomorphisms in a quasivariety that is then understood and dualized as a category. Although new results were obtained by using this duality, the approach is somewhat limited. It only works for a tiny fraction of clones (the centralizer clones on finite sets), and it does not give us any information about what happens to the Galois connection Pol–Inv, that is set up between operations and relations and is arguably the single most effective item in a clone theorist’s toolbox (it is nicknamed the “most basic Galois connection in algebra” in [18]).
The aim of this paper is to extend the theory from [17] such that both of these drawbacks are overcome. In other words, we will build a duality theory that is general enough to work for any clone and also dualizes all parts of the aforementioned Galois connection Pol–Inv. To do so, we will follow the author’s Ph.D. thesis [12], which carries the same name as this paper. In fact, most (but not all) of this paper’s content is taken from this thesis, and the main purpose of this paper can be seen as a presentation of the thesis’ core results in a concise form.

After the preliminaries, the actual work starts in Sec. 3 where we will build our framework by introducing the notion of clones over objects in arbitrary categories, essentially treating clones as images of models of Lawvere theories. As it was shown in [12, 15], the Galois connection Pol–Inv can also be generalized into this framework so that it is constituted between operations and what we will introduce as generalized relations. The obtained category-theoretic setting allows us to apply the duality principle to all of these notions. In this way, we obtain the notion of clones of dual operations (generalizing coclones as introduced in [4]) and a Galois theory for dual operations and what we call dual relations.

In Sec. 4, we use the constructed framework to introduce clone dualities. More precisely, we will explain how dual equivalences can be used to dualize clones of operations in a category $\mathcal{A}$ to clones of dual operations in a dually equivalent category $\mathcal{X}$. We will also explain the connection between these clone dualities and the generalized Galois theory. At the end of the section, the reader is hopefully convinced that a general duality theory for clones is indeed obtained, justifying the name of this paper.

In practice, of course, one wants to dualize a clone of operations to a clone of dual operations that is as easy as possible. In particular, one is interested in dualizing clones over (finite) sets to clones of dual operations in concrete and easily approachable categories. In Sec. 5, we discuss to what extend this can be achieved.

Finally, in Sec. 6, we put all the techniques to work and illustrate the approach with several examples. We show how clone dualities can be used to produce general and abstract results as well as specific and concrete ones, where the latter will be exemplified by the study of clones over distributive lattices.

We will conclude with some remarks and suggestions for further research in the final Sec. 7.

2. Preliminaries

In the preliminaries, we will introduce all the ingredients that we need to construct clone dualities. We will recall some basic notions from category theory, present some facts about topological structures and introduce clones in the classical sense.

2.1. Category theory

We assume that the reader is familiar with the rudimentary basics of category theory. By that, we mean that the reader should be familiar with the definitions...
of categories, functors, natural transformations, products and coproducts. In this
section, we only introduce our notation and the terminology of duality. For a
category $C$, we write $A \in C$ to indicate that the object $A$ is in the category $C$ and
we denote the set of morphisms from an object $A$ to an object $B$ in $C$ by $C(A,B)$.
For $A \in C$, we write $A^n$ to mean the $n$th power of $A$ (provided it exists) and we
denote the associated projection morphisms by $\pi_i^n : A^n \to A$ ($i \in \{1,\ldots,n\}$).
For morphisms $f_1,\ldots,f_n : B \to A$, we denote by $\langle f_1,\ldots,f_n \rangle : B \to A^n$ the
tupling of $f_1,\ldots,f_n$. Dually, for an object $X \in C$, we denote by $n \cdot X$ the $n$th
copower of $X$ (provided it exists) and by $\iota^n_i : X \to n \cdot X$ ($i \in \{1,\ldots,n\}$) the
associated injection morphisms. For morphisms $g_1,\ldots,g_n : X \to Y$, we denote by
$[g_1,\ldots,g_n] : n \cdot X \to Y$ the catupling of $g_1,\ldots,g_n$.

If an object $A \in C$ has finite powers, then we can use the following contravariant
functor from $\mathbb{N}_+$ (understood as a category by treating $n \in \mathbb{N}_+$ as the set $\{1,\ldots,n\}$
and taking all set-functions as morphisms) to $C$:

$$A^{-} : \mathbb{N}_+ \to C,$$

$$n \mapsto A^n,$$

$$\varphi \in \mathbb{N}_+(n,m) \mapsto A^{\varphi} := \langle \pi_{\varphi(1)}^m,\ldots,\pi_{\varphi(n)}^m \rangle : A^m \to A^n.$$  

Similarly, if an object $X \in C$ has finite copowers, then we can use the covariant
functor

$$(-) \cdot X : \mathbb{N}_+ \to C,$$

$$n \mapsto n \cdot X,$$

$$\varphi \in \mathbb{N}_+(n,m) \mapsto \varphi \cdot X := [\iota_{\varphi(1)}^m,\ldots,\iota_{\varphi(n)}^m] : n \cdot X \to m \cdot X.$$  

Note that these functors can be used to write identities in categories. For instance,
if we take some $A \in \text{Set}$, $f : A^n \to A$ and define

$$\varphi : n \to n : i \mapsto \begin{cases} 1 & \text{if } i \in \{1,2\}, \\ i & \text{otherwise}, \end{cases}$$

then $f = A^{\varphi} \circ f$ is equivalent to the identity $f(x_1,\ldots,x_n) \approx f(x_1,x_1,x_3,\ldots,x_n)$.

A dual equivalence between two categories $\mathcal{A}$ and $\mathcal{X}$ is a quadruple $(D,E,e,\epsilon)$,
where $D : \mathcal{A} \to \mathcal{X}$ and $E : \mathcal{X} \to \mathcal{A}$ are contravariant functors (i.e. functors that
reverse the direction of the morphisms) and $e : \text{id}_{\mathcal{A}} \to ED$ and $\epsilon : \text{id}_{\mathcal{X}} \to DE$ are
natural isomorphisms. The notion “dual equivalence” is justified since $D$ and $E$
are full, faithful and preserve all purely category-theoretic properties, except that
they reverse the order of the morphisms. For instance, monomorphisms become
epimorphisms and products become coproducts. In particular, we have $A^n \in \mathcal{A}$ if
and only if $n \cdot D(A) \in \mathcal{X}$.

A concrete category is a category $C$ that is equipped with a faithful functor
$U : C \to \text{Set}$. This functor allows one to think of the objects in $C$ as sets (possibly
with some additional structure) and of the morphisms as mappings between these
sets. In this paper we will be less formal and whenever we speak of a concrete
category $C$ we will omit the faithful functor and directly assume that all the objects in $C$ are sets, possibly with some additional structure, and all the morphisms are set-functions. Clearly, this goes without loss of generality.

2.2. (Topological) structures

In this subsection, we will recall the notion of a structure, and we will remind the reader of a famous duality between two categories of structures that will be used as an example throughout the paper.

A structure is a set that might be equipped with total operations, partial operations and relations. By abuse of notation, for a structure $X$, we will sometimes also write $X$ when we actually refer to the underlying set of $X$. For instance, we may write $x \in X$ to indicate that $x$ is in the carrier set of the structure $X$, and we may refer to its cardinality by $|X|$.

A homomorphism between two structures is a mapping that commutes with the operations and preserves the relations and the domains of the partial operations. For a homomorphism $h : X \to Y$, we denote by $h[X]$ the image of $X$ under $h$. If a structure is also equipped with a topology, then it is said to be a topological structure and the homomorphisms are required to be continuous. For a class of (topological) structures $K$, we denote by $I(K)$, $S(K)$, $P(K)$ the classes of isomorphic copies, non-empty substructures and products of structures from $K$, respectively. Moreover, we denote by $S^0(K)$ the class of all, possibly empty, topologically closed substructures of structures from $K$, and by $P^+(K)$ the class of all products over non-empty index sets of structures from $K$. Recall that quasi-varieties are ISP-closed classes of structures with the same type. In the following, we will sometimes speak of categories of certain classes of (topological) structures. When we do so, we mean the category that contains these structures as objects and the (continuous) homomorphisms between them as morphisms.

Since we will need it in the remainder of the paper, let us quickly recall the well-known result that the category of finite distributive lattices is dually equivalent to the category of finite bounded posets [2, 25, 29]. To understand the Priestley duality well enough for our purposes, we need to recall the definition of a prime filter: a prime filter of a lattice $A = \langle A, \lor, \land \rangle$ is a lattice-filter $F \subseteq A$ such that we have $a \in F$ or $b \in F$ whenever $a \lor b \in F$. Here, we will also consider the empty and the full subset as prime filters and denote the set of all prime filters of $A$ by $\text{Spec}(A)$. For the content of this paper, we do not need to know the whole quadruple.

---

<sup>a</sup>The reason for this is to avoid confusion when we refer to coproducts or products of a structure. If we always used the convention of using bold letters for structures and normal letters for their underlying carrier set, then the carrier set of $X^n$ would be denoted by $X^n$. However, the carrier set of $X^n$ is not necessarily the $n$th Cartesian power of the set $X$, so confusion would arise. The other (and formally correct) option would be to use the forgetful functor, but this would be at the expense of notational simplicity.

<sup>b</sup>The Priestley duality is constituted between the category of distributive lattices and the category of bounded Priestley spaces. If it is restricted to the finite part of both categories (as it is done here), we obtain a duality that was already described by Birkhoff in [2].
that constitutes the dual equivalence between the category \( A \) of finite distributive lattices and the category \( X \) of finite bounded posets. It is enough to understand the contravariant functor \( D: A \to X \) that is given as follows:

\[
D: A \to X,
A \mapsto \langle \text{Spec}(A), \emptyset, A, \subseteq \rangle,
\]

\[
f : A \to B \mapsto D(f): D(B) \to D(A) : x \mapsto f^{-1}[x].
\]

2.3. Clones (on sets)

Until the end of this subsection, let \( A \) be a non-empty (but not necessarily finite) set. For \( n \in \mathbb{N}_+ \), denote by \( O_A^{(n)} \) the set of all \( n \)-ary operations over \( A \) and set \( O_A := \bigcup_{n \in \mathbb{N}_+} O_A^{(n)} \). Note that \( O_A \) does not contain nullary operations (see the discussion in Sec. 7 of this paper).

The \( i \)th variable of an \( n \)-ary operation \( f \) is said to be nonessential if

\[
f(x_1, \ldots, x_n) \approx f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n).
\]

A variable is called essential if it is not nonessential. Moreover, an operation is said to be essentially \( k \)-ary if it has exactly \( k \) essential variables.

A subset \( C \subseteq O_A \) is called a clone (or clone of operations) if it contains all the projection mappings

\[
\pi^n_i : A^n \to A : (x_1, \ldots, x_n) \mapsto x_i
\]

and is closed with respect to superposition of operations in the following sense: for an \( n \)-ary operation \( f \in C \) and \( k \)-ary operations \( f_1, \ldots, f_n \in C \), the \( k \)-ary operation \( f(f_1, \ldots, f_n) \), defined by setting

\[
f(f_1, \ldots, f_n)(x_1, \ldots, x_k) := f(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k)),
\]

is also in \( C \). Given an algebra, the set of its non-nullary term functions is a clone. Conversely, every clone can be realized as the set of term functions of a suitable algebra. Hence, the clones on \( A \) represent all possible different behaviors of algebras with carrier set \( A \). This is the main motivation behind clone theory.

For each \( F \subseteq O_A \), there is a least clone containing \( F \). We denote this clone by \( \text{Clo}(F) \), and we say that \( F \) generates \( \text{Clo}(F) \). It is easy to see that the clones over \( A \) form a lattice, which we will denote by \( \mathcal{L}_A \). On a two-element set, there are countably many clones, and the lattice was completely described by Post in [24]. However, for \( |A| \geq 3 \), there are continuum many clones, and a full description of these lattices seems to be hopeless, even for \( |A| = 3 \). For more details on clone theory, we refer to [22, 30].

We will now see that there is a correspondence between clones of operations and certain sets of relations.
Denote by $R_A^{(n)}$ the set of all $n$-ary relations on $A$ and set $R_A := \bigcup_{n \in \mathbb{N}} R_A^{(n)}$.

**Definition 1.** An operation $f \in O_A^{(n)}$ is said to preserve a relation $\sigma \in R_A^{(k)}$ if

$\begin{pmatrix}
\nu_{11} \\
\nu_{12} \\
\vdots \\
\nu_{1k}
\end{pmatrix}, \ldots, \begin{pmatrix}
\nu_{n1} \\
\nu_{n2} \\
\vdots \\
\nu_{nk}
\end{pmatrix} \in \sigma \Rightarrow \begin{pmatrix}
f(\nu_{11}, \nu_{21}, \ldots, \nu_{n1}) \\
f(\nu_{12}, \nu_{22}, \ldots, \nu_{n2}) \\
\vdots \\
f(\nu_{1k}, \nu_{2k}, \ldots, \nu_{nk})
\end{pmatrix} \in \sigma.$

For $F \subseteq O_A$ and $R \subseteq R_A$, define

$\text{Inv } F := \{ \sigma \in R_A | \forall f \in F : f \triangleright \sigma \}$,

$\text{Pol } R := \{ f \in O_A | \forall \sigma \in R : f \triangleright \sigma \}$.

In terms of algebras, a $k$-ary relation $\sigma$ belongs to $\text{Inv } F$ if and only if $\sigma$ forms a subalgebra of $(A, F)^k$.

Obviously, $\text{Pol–Inv}$ is a Galois connection between operations and relations. For $A$ being finite, the Galois closed classes were characterized in [3, 8]: they are precisely the clones of operations and clones of relations, that is, sets of relations on $A$ that contain all trivial relations (that is, all diagonal relations and the empty relation $\emptyset$), are closed under direct (Cartesian) products, under intersection of (any family of) relations of the same arity, under permutation of coordinates and under projections onto a set of coordinates. Thus, for a finite set $A$, the clone lattice $\mathcal{L}_A$ is dually isomorphic to the lattice of clones of relations. In the case $|A| = \infty$, it was shown in [20, 21] that the Galois closed classes of $\text{Pol–Inv}$ are local closures of the clones of operations and local closures of the clones of relations, whereby the lattices formed by these locally closed sets are the ones that are dually isomorphic. The scenario in which the relations (but not the operations) are allowed to be of infinite arity was studied in [26].

### 3. Clones in Categories

#### 3.1. Clones of operations

Since duality theory is a notion from the field of category theory, we first need to lift the notion of a clone to categories. A first attempt to do so was done by Lawvere in his doctoral thesis [16]. A **Lawvere theory** (Lawvere himself called them **algebraic theories**) is a small category consisting of a family of objects $(t_i)_{i \in \mathbb{N}}$ such that $t_i$ is the $i$th power of $t_1$. This notion is the category-theoretic counterpart of an abstract clone [32]. Just like abstract clones can have concrete realizations, a Lawvere theory can have models in other categories. More precisely, a **model** of a Lawvere theory $\mathcal{L}$ in a category $\mathcal{C}$ is a product-preserving functor $M : \mathcal{L} \to \mathcal{C}$. We refer to [11, 18] for more details about Lawvere theories and their connection to clones as studied in universal algebra.
What we are about to define as a clone over an object $A$ in a category $C$ follows [17] and is precisely the set of morphisms $\{M(f) \mid f \in \mathcal{L}(t_n, t_1), n \in \mathbb{N}_+\}$ where $M$ is a model of a Lawvere theory $\mathcal{L}$ with $M(t_1) = A$. Note that $\mathcal{L}(t_0, t_1)$ is excluded from this definition. This is done because clone theory is usually pursued without nullary operations (see the discussion at the end of this paper).

From now on until the end of this section let $A$ be an object in a category $C$ such that all finite non-empty powers of $A$ also exist in $C$. In other words, we require \( \{A^n \mid n \in \mathbb{N}_+\} \subseteq C \).

Definition 2. Let $n \in \mathbb{N}_+$. A morphism $f : A^n \to A$ is called an $n$-ary operation over $A$. Denote by $O_A^{(n)}$ the set of all $n$-ary operations over $A$, define $O_A := \bigcup_{n \in \mathbb{N}_+} O_A^{(n)}$ and, for $F \subseteq O_A$, set $F^{(n)} := F \cap O_A^{(n)}$.

Definition 3. A subset $C \subseteq O_A$ is called a clone of operations, written $C \leq O_A$, if $C$ contains all the projection morphisms $\pi_i^n : A^n \to A$ and, for $f \in C^{(n)}$ and $f_1, \ldots, f_n \in C^{(k)}$, the superposition $f \circ (f_1, \ldots, f_n)$ is also in $C$.

If $C$ is the category of sets, then this definition coincides with the usual notion of a clone. It is easy to verify that the clones over an object $A$ form a complete lattice with respect to inclusion. We call this lattice the lattice of clones over $A$, and we denote it by $\mathcal{L}_A$. The top element of $\mathcal{L}_A$ is the full clone $O_A$, and the bottom element is the clone that contains only the projection morphisms.

Since clones are closed under arbitrary intersection, we can define the closure operator $\text{Clo}$ that assigns to each subset $F \subseteq O_A$ the least clone of operations over $A$ that contains $F$. It is called the clone generated by $F$. For a single operation $f$, we write $\text{Clo}(f)$ to mean $\text{Clo}(\{f\})$.

Example 4.

(i) If $C = \text{Set}$, then $O_A$ is the full clone on the set $A$ and $\mathcal{L}_A$ is the usual clone lattice.

(ii) If $C$ is a variety (or a quasivariety) of algebras, then $O_A$ is the centralizer clone of the algebra $A$ and $\mathcal{L}_A$ is the lattice of subclones of $O_A$. Centralizer clones are of particular interest in universal algebra (see [18], for instance).

(iii) If $C$ is the category of topological spaces and $A \in C$, then $O_A$ is the clone of the topological space $A$ as investigated by Taylor in [31].

(iv) For each clone $C$ on a finite set $A$, we obtain $C = O_A$ if we define $A$ to be a relational structure $(A, R)$ in a variety (or quasivariety) of relational structures such that $C$ is the set of polymorphisms of $R$ (i.e. the set of operations that preserve each $\sigma \in R$). Such a set of relations $R$ can always be found. In this case, $\mathcal{L}_A$ is the lattice of subclones of $C$.

These examples show that one can investigate clones over sets by treating them as clones over objects in (abstract or concrete) categories different from $\text{Set}$. 
We can lift every notion from clone theory to our setting as long as we can write it in purely category-theoretic terms. For instance, we can write all kinds of identities. Example, we can define essential variables of an operation as follows.

**Definition 5.** For \( n \in \mathbb{N}_+ \) and \( i \in \{1, \ldots, n\} \), the \( i \)th variable of an operation \( f \in O_A^{(n)} \) is said to be **nonessential** if

\[
T \circ A^{n+1} = T \circ A^{\psi^{n+1}_i},
\]

where

\[
\psi^{n+1}_i : n \to n + 1 : j \mapsto \begin{cases} 
  n + 1 & \text{if } j = i, \\
  j & \text{if } j \neq i.
\end{cases}
\]

A variable is called **essential** if it is not nonessential. Moreover, we say that an operation is **essentially \( k \)-ary** if it has exactly \( k \) essential variables.

Without using the functor \( A(-) \) introduced in the preliminaries, the equation defining the nonessentiality of the \( i \)th variable of \( f \) reads as follows:

\[
f \circ \langle \pi_1^{n+1}, \ldots, \pi_n^{n+1} \rangle = f \circ \langle \pi_1^{n+1}, \ldots, \pi_{i-1}^{n+1}, \pi_i^{n+1}, \pi_{i+1}^{n+1}, \ldots, \pi_{n+1}^{n+1} \rangle.
\]

This definition coincides with the usual definition of (non-)essential variables as presented at the beginning of Sec. 2.3 whenever the latter is applicable (that is, if the powers of \( A \) are Cartesian powers and the morphisms are set-functions).

As we have learned in the preliminaries, clones over sets have a relational counterpart where the connection is given by the Galois connection \( \text{Pol} \dashv \text{Inv} \) that is induced by the notion of preserving. This raises the question of whether we can also generalize this powerful tool into our more general framework. It was shown in \([12, 15]\) that the answer is a positive one.

To understand the idea of the more general approach, note that one can interpret relations in the usual sense as sets of mappings. If we do so, we can say that \( \sigma \) is a \( k \)-ary relation on the set \( A \) if \( \sigma \) is a subset of \( \text{Set}(\{1, \ldots, k\}, A) \). Thus, a relation on \( A \) is nothing but a set of morphisms from the object \( \{1, \ldots, k\} \) to the object \( A \) in the category of sets, i.e. it is a subset of \( \text{Set}(\{1, \ldots, k\}, A) \). This is precisely the view on relations that we will now use to generalize relations on sets to relations on objects: analogously to defining \( k \)-ary relations on the set \( A \) to be sets of mappings from \( \{1, \ldots, k\} \) to \( A \), we will define a relation of type \( B \in \mathcal{C} \) on the object \( A \) to be a set of morphisms from the object \( B \) to the object \( A \).

**Definition 6.** Let \( B \in \mathcal{C} \). A relation of type \( B \) on \( A \) is a subset of \( \mathcal{C}(B, A) \). Denote the class of all relations of type \( B \) on \( A \) by \( R_A^{(B)} \).

We will now define the notion of invariant relations on \( A \) by generalizing the usual notion of invariant relations that we have presented in the preliminaries. If we interpret the relation \( \sigma \) as a subset of \( \text{Set}(\{1, \ldots, k\}, A) \), then we can rephrase the condition of preserving by using the tupling:

\[
f \triangleright \sigma \iff f \circ \langle r_1, \ldots, r_n \rangle \in \sigma \quad \text{for all } r_1, \ldots, r_n \in \sigma.
\]
Since this notion of preserving relies on purely category-theoretic properties, we can lift it to other categories.

**Definition 7.** Let \( \sigma \) be a relation of type \( B \) on \( A \) and let \( f \in O_A \). Say that \( \sigma \) is invariant for \( f \) or that \( f \) preserves \( \sigma \), written \( f \triangleright \sigma \), if
\[
f \circ \langle r_1, \ldots, r_n \rangle \in \sigma
\]
whenever \( r_1, \ldots, r_n \in \sigma \). Furthermore, a set of operations \( F \subseteq O_A \) is said to preserve \( \sigma \), written \( F \triangleright \sigma \), if every \( f \in F \) preserves \( \sigma \).

Clearly, for \( C \) being the category of sets and \( B = \{1, \ldots, k\} \), this notion coincides with the usual notion of \( f \) preserving a \( k \)-ary relation. To define clones of relations on \( A \), we need to introduce the notion of a typeclass.

**Definition 8.** A typeclass is a non-empty subclass \( T \subseteq C \) in which any two different objects are non-isomorphic.

In other words, a typeclass is a non-empty subclass of a skeleton. For a given typeclass \( T \), we set
\[
R_T^A := \bigcup_{B \in T} R_{\sigma}^{(B)}(A)
\]
and we say that \( R_T^A \) is the class of relations of the typeclass \( T \) on \( A \). For a class of relations \( R \subseteq R_T^A \) and \( B \in T \), we write \( R^{(B)} \) to mean \( R \cap R_T^{(B)} \).

**Definition 9.** A class \( R \subseteq R_T^A \) is called a clone of relations of the typeclass \( T \) on \( A \), written \( R \leq R_T^A \), if
\begin{enumerate}
\item \( \emptyset \in R \),
\item \( R \) is closed under general superposition, that is, the following holds: let \( I \) be an index class, let \( \sigma_i \in R_{\sigma}^{(B_i)} \) \( (i \in I) \) and let \( \varphi : B \to C \) and \( \varphi_i : B_i \to C \) be morphisms where \( C \in C \) and \( B \in T \). Then, the relation
\[
\bigwedge_{(\varphi_i) \in I} \sigma_i := \{ r \circ \varphi \mid \forall \sigma_i, r \in C(C, A) \}
\]
belongs to \( R \).
\end{enumerate}

**Example 10.** Let \( C \) be the category of sets and let \( A \in C \). If we choose \( T \) to be the set of all finite positive cardinal numbers, then our notion of a clone of relations coincides with the usual notion of a clone of finitary relations. If we choose \( T \) to be the set of all positive cardinal numbers, then our notion coincides with the usual notion of a clone of (possibly infinitary) relations [26].

For a given typeclass \( T \), it is obvious that \( R_T^A \) is a clone of relations. Furthermore, it is easy to see that the intersection of clones of relations is again a clone of relations. Thus, we can define the operator \( \text{Clo}^T : \Psi(R_T^A) \to \Psi(R_T^A) \) that maps each \( R \subseteq R_T^A \)
to the least clone of relations that contains $R$. We call this clone the clone of relations generated by $R$. Moreover, it follows that the clones of relations of the typeclass $T$ on $A$ form a complete lattice with respect to inclusion. We denote this lattice by $\mathcal{L}_A^T$ and call it the lattice of clones of relations of the typeclass $T$ on $A$. Clearly, $R_A^T$ is the greatest clone of relations on $A$, whereas the least clone of relations on $A$ is $\text{Clo}^T(\emptyset)$. In the scenarios from Example 10 (i.e. the universal algebra case with finitary or infinitary relations), the latter is precisely the set of all diagonal relations.

**Definition 11.** Let $T$ be a typeclass. We define the two operators $\text{Inv}^T_A : \mathcal{P}(O_A) \to \mathcal{P}(R_A^T)$ and $\text{Pol}_A : \mathcal{P}(R_A^T) \to \mathcal{P}(O_A)$ as follows: for $F \subseteq O_A$ and $R \subseteq R_A^T$, set

$$\text{Inv}^T_A F := \{ \sigma \in R_A^T \mid \forall f \in F : f \triangleright \sigma \},$$

$$\text{Pol}_A R := \{ f \in O_A \mid \forall \sigma \in R : f \triangleright \sigma \}.$$

For $B \subseteq C$ and $n \in \mathbb{N}_+$, we use the following notation:

$$\text{Inv}^{(B)}_A F := \{ \sigma \in R_A^{(B)} \mid \forall f \in F : f \triangleright \sigma \},$$

$$\text{Pol}^{(n)}_A R := \text{Pol}_A R \cap O_A^{(n)}.$$

Note that $\text{Pol}_A R$ and $\text{Inv}^{(B)}_A F$ are always sets, while $\text{Inv}^T_A F$ can be a proper class. Hence, the operators $\text{Pol}_A$ and $\text{Inv}^T_A$ constitute a Galois connection between the subsets of $O_A$ and the subclasses of $R_A^T$.

For $C = \mathbb{S}et$, it follows directly from the observations in Example 10 that $\text{Pol}_A \text{–} \text{Inv}^T_A$ coincides with $\text{Pol} \text{–} \text{Inv}$ if we choose $T$ to be the set of all positive finite cardinal numbers and that it coincides with the Galois connection from [26] if we choose $T$ to be the class of all positive cardinal numbers.

**Example 12.**

(i) Let $\sigma := \{ \text{id}_A \}$. Then, $\text{Pol}_A\{\sigma\}$ is the set of all idempotent operations over $A$. That is, $\text{Pol}_A\{\sigma\} = \{ f \in O_A \mid f \circ \{ \text{id}_A, \ldots, \text{id}_A \} = \text{id}_A \}$.

(ii) Let $C \subseteq O_A$. Note that $C^{(n)}$ is a relation of type $A^n$. Now, $\text{Pol}_A C^{(n)}$ is the largest clone $C'$ that agrees with $C$ on its $n$-ary part, i.e. $\text{Pol}_A^{(n)} C^{(n)} = C^{(n)}$.

(iii) If two operations $f, f'$ are essentially the same (i.e. after eliminating all non-essential variables, the two operations arise from each other by a suitable permutation of variables), then $\text{Inv}_A^{(n)}(f) = \text{Inv}_A^{(n)}(f')$.

(iv) Let $C$ be the category of finite distributive lattices. Let $B$ be the (up to isomorphism) unique two-element distributive lattice $\langle \{0, 1\}, \lor, \land \rangle$. Let $\sigma$ be the relation that contains each morphism $r : B \to A$ with $r(0) \neq r(1)$. Now, we have $f \in \text{Pol}_A\{\sigma\}$ if and only if $f$ is strictly monotone, that is, $f(a_1, \ldots, a_n) > f(b_1, \ldots, b_n)$ whenever $a_i > b_i$ for all $i \in \{1, \ldots, n\}$ (where the order relation $> \in$ is obtained from the lattice in the obvious way).
Let \( F \subseteq O_A \), \( R \subseteq R^T_A \), \( s \geq 1 \) and let \( C \in C \). We define the following local closure operators:

\[
\begin{align*}
\text{C-Loc } F &:= \{ f \in O^{(n)}_A | n \geq 1, \forall r_1, \ldots, r_n \in C(A) : \times \exists f' \in F : f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle \}, \\
s-LOC^T R &:= \{ \sigma \in R^T_A | \forall B \subseteq \sigma, |B| \leq s : \exists \sigma' \in R : B \subseteq \sigma' \subseteq \sigma \}.
\end{align*}
\]

Furthermore, let

\[
\text{Loc}^T F := \bigcap_{C \in T} \text{C-Loc } F
\]

and

\[
\text{LOC}^T R := \bigcap_{s \in \mathbb{N}_+} s-\text{LOC}^T R.
\]

In other words, \( \text{C-Loc } F \) is the set of all operations \( f \in O_A \) such that, for all tuplings \( \langle r_1, \ldots, r_n \rangle \) of morphism from \( C \) to \( A \), there exists an operation \( f' \in F \) such that \( f \) and \( f' \) cannot be distinguished if they are applied after \( \langle r_1, \ldots, r_n \rangle \). Moreover, \( s-\text{LOC}^T R \) is the class of all relations \( \sigma \in R^T_A \) such that, for every \( B \subseteq \sigma \) with at most \( s \) elements, there exists a member \( \sigma' \) of \( R \) that agrees with \( \sigma \) on \( B \) and is contained in \( \sigma \).

Before we present the main result of the Galois theory, let us note that there is an order relation \( \leq \) on the typeclass that has some interesting properties and is defined as follows: we write \( C_1 \leq C_2 \) if there is an epimorphism from \( C_2 \) to \( C_1 \).

**Proposition 14 ([12, 15]).** For \( F \subseteq O_A \), \( B, C \in T \) and \( B \leq C \), we have

(i) \( B\text{-Loc } F \subseteq C\text{-Loc } F \),

(ii) \( \text{Pol}_A \text{Inv}_A^{(C)} F \subseteq \text{Pol}_A \text{Inv}_A^{(B)} F \).

Note that this also generalizes an observation for the classical case, since, for two sets \( B \) and \( C \), we have \( B \leq C \) if and only if \( |B| \leq |C| \).

As the main result of the generalized Galois theory, one can show \( \text{Loc}^T \text{Clo}(F) = \text{Pol}_A \text{Inv}_A^{(s)} F \) and \( \text{C-Loc } \text{Clo}(F) = \text{Pol}_A \text{Inv}_A^{(C)} F \) for every \( F \subseteq O_A \) and each \( C \in T \). Similarly, one has \( \text{LOC}^T \text{Clo}^T(R) = \text{Inv}_A^{(s)} \text{Pol}_A R \) and \( s-\text{LOC}^T \text{Clo}^T(R) = \text{Inv}_A^{(s)} \text{Pol}_A F^* \) for every \( R \subseteq R^T_A \) and \( s \geq 1 \). These equations allow us to characterize those subsets \( F \subseteq O_A \) and those subclasses \( R \subseteq R^T_A \) which can be represented as \( \text{Pol}_A R' \) and \( \text{Inv}_A^{(s)} F^* \) for some \( R' \subseteq R^T_A \) and \( F^* \subseteq O_A \), respectively.
Corollary 15 ([12, 15]). For $F \subseteq O_A$, the following are equivalent:

(a) $F \leq O_A$ (i.e. $F = \text{Cl}(F)$) and $\text{Loc}^\uparrow F = F$.
(b) $F = \text{Pol}_A \text{Inv}_A F$.
(c) $\exists R \subseteq R_A^A : F = \text{Pol}_A R$.

Corollary 16 ([12, 15]). For $R \subseteq R_A^A$, the following are equivalent:

(a) $R \leq R_A^T$ (i.e. $R = \text{Cl}(R)$) and $\text{LOC}^\uparrow R = R$.
(b) $R = \text{Inv}_A \text{Pol}_A R$.
(c) $\exists F \subseteq O_A : R = \text{Inv}_A^T F$.

Denoting by $\text{Loc}^\uparrow \mathcal{L}_A$ the lattice of locally closed clones of operations over $A$ and by $\text{LOC}^\uparrow \mathcal{L}_A^* A$ the lattice of locally closed clones of relations on $A$, we obtain that $\text{Loc}^\uparrow \mathcal{L}_A$ and $\text{LOC}^\uparrow \mathcal{L}_A^{* T}$ are dually isomorphic via $\text{Inv}_A^T$.

Of course, this raises the obvious question of when clones of operations and clones of relations are already locally closed. The following proposition gives an answer to this question.

Lemma 17 ([12, 15]).

(i) We have $\text{LOC}^\uparrow R = R$ for all $R \subseteq R_A^T$ if and only if $C(B, A)$ is finite for all $B \in T$.
(ii) If one of the following two conditions hold, then we have $\text{Loc}^\uparrow C = C$ for all $C \leq O_A$:

- For each $k \in \mathbb{N}_+$, there exists $n \geq k$ such that $A^n \leq B$ for some $B \in T$.
- Each $f \in O_A$ is essentially at most $n$-ary and $A^n \leq B$ for some $B \in T$.

Note that statement (ii) implies that we can always choose $T$ such that we obtain $\text{Loc}^\uparrow F = F$ for all $F \subseteq O_A$ (i.e. $\text{Loc}^\uparrow$ becomes obsolete). However, we cannot necessarily choose $T$ such that we have $\text{LOC}^\uparrow R = R$ for all $R \subseteq R_A^T$. This somewhat unsymmetrical behavior could be avoided by allowing operations of infinite arity, that is, we had to define $O_A$ to be the class of morphisms from any non-empty power of $A$ to $A$.

The following examples show that the proposition from above does indeed generalize an observation for the local closure operators in the universal algebra case.

Example 18. If $\mathcal{C} = \text{Set}$ and $T$ is the set of all finite positive cardinal numbers, then Lemma 17 establishes that we have $\text{LOC}^\uparrow R = R$ for all $R \subseteq R_A^T$ if and only if $A$ is a finite set. Furthermore, the lemma also yields that $A$ being a finite set implies $\text{Loc}^\uparrow F = F$ for all $F \subseteq O_A$. An easy proof shows that the other direction is also true. Thus, both local closure operators can be dismissed if and only if $A$ is a finite set.
3.2. Clones of dual operations

Having written operations, relations, their clones, the local closure operators and the notion of preserving in purely category-theoretic terms, we can dualize all these notions. For the remainder of this section, let $X$ be an object in $C$ such that all non-empty finite copowers of $X$ exist in $C$.

**Definition 19.** Let $n \in \mathbb{N}_+$. An $n$-ary dual operation over $X$ (or cooperation over $X$) is a morphism from $X$ to $n \cdot X$. Denote by $\mathcal{O}_X^{(n)}$ the set of all $n$-ary dual operations over $X$, define $\mathcal{O}_X := \bigcup_{n \in \mathbb{N}_+} \mathcal{O}_X^{(n)}$ and, for a set of dual operations $G \subseteq \mathcal{O}_X$, set $G^{(n)} := G \cap \mathcal{O}_X^{(n)}$.

**Definition 20.** A subset $C \subseteq \mathcal{O}_X$ is called a clone of dual operations (or coclone), written $C \leq \mathcal{O}_X$, if it contains all the injection morphisms and, for $g \in C^{(n)}$ and $g_1, \ldots, g_n \in C^{(k)}$, the superposition $[g_1, \ldots, g_n] \circ g$ is also in $C$.

If $X$ is a set in the category of sets, then a clone of dual operations over $X$ is a coclone as introduced in [4].

**Definition 21.** For $n \in \mathbb{N}_+$ and $i \in \{1, \ldots, n\}$, the $i$th variable of a dual operation $g \in \mathcal{O}_X^{(n)}$ is said to be nonessential if

$$\subseteq_{n+1}^n X \circ g = \psi_{i+1}^n \cdot X \circ g.$$

A variable is called essential if it is not nonessential. Moreover, we say that an operation is essentially $k$-ary if it has exactly $k$ essential variables.

Again, clones of dual operations form a complete lattice, which we will denote by $\mathcal{L}_X$ and call the lattice of clones of dual operations over $X$.

Analogously to the closure operator Clo on sets of operations, we can define $\overline{\text{Clo}}$ for a set of dual operations $G \subseteq \mathcal{O}_X$, we denote by $\overline{\text{Clo}}(G)$ the least clone of dual operations that contains $G$. Again, for a single dual operation, we write $\overline{\text{Clo}}(g)$ instead of $\overline{\text{Clo}}\{g\}$.

We can also dualize the Galois theory from the last section to obtain a general Galois theory for dual operations and something that we will introduce as dual relations.

**Definition 22.** Let $Y \in C$. A dual relation of type $Y$ on $X$ is a subset of $C(X, Y)$. Denote the class of all dual relations of type $Y$ on $X$ by $\mathcal{R}_X^{(Y)}$. Moreover,

$$\mathcal{R}_X := \bigcup_{Y \in \mathcal{T}} \mathcal{R}_X^{(Y)}$$

is called the class of dual relations of the typeclass $\mathcal{T}$ on $X$. For a class of dual relations $R \subseteq \mathcal{R}_X$, let $R^{(Y)} := R \cap \mathcal{R}_X^{(Y)}$. 
Every dual relation of type \( \{1, \ldots, k\} \) on a set \( X \in \mathcal{S}et \) is a \( k \)-ary correlation as introduced in [23].

We will now dualize the remaining notions of the last subsection.

**Definition 23.** Let \( \sigma \) be a dual relation of type \( Y \) on \( X \), and let \( g \) be an \( n \)-ary dual operation over \( X \). We say that \( \sigma \) is invariant for \( g \) or that \( g \) preserves \( \sigma \), written \( g \triangleright \sigma \), if \( [r_1, \ldots, r_n] \circ g \in \sigma \) whenever \( r_1, \ldots, r_n \in \sigma \). Furthermore, we say that a set of dual operations \( G \subseteq \mathcal{O}_X \) preserves \( \sigma \), written \( G \triangleright \sigma \), if every \( g \in G \) preserves \( \sigma \).

**Definition 24.** A class \( R \subseteq \overline{\mathcal{R}}_X^Y \) is called a clone of dual relations of the typeclass \( T \) on \( X \), written \( R \leq \overline{\mathcal{R}}_X^T \), if

(i) \( 0 \in R \),

(ii) \( R \) is closed under general superposition, that is, the following holds: let \( I \) be an index class, let \( \sigma_i \in R(Y_i) \) \( (i \in I) \) and let \( \varphi : Z \to Y \) and \( \varphi_i : Z \to Y_i \) be morphisms where \( Z \in \mathcal{C} \) and \( Y_i \in T \). Then, the dual relation \( \bigwedge_{(\varphi_i)_{i \in I}} (\sigma_i)_{i \in I} \in \overline{\mathcal{R}}_X^Y \) defined by

\[
\bigwedge_{(\varphi_i)_{i \in I}} (\sigma_i)_{i \in I} := \{ \varphi \circ r \mid \forall i \in I : \varphi_i \circ r \in \sigma_i, r \in \mathcal{C}(X, Z) \}
\]

belongs to \( R \).

Again, \( \overline{\mathcal{R}}_X^T \) is a clone of dual relations and the intersection of clones of dual relations is a clone of dual relations. Hence, for \( R \subseteq \overline{\mathcal{R}}_X^T \), there exists a least clone of dual relations that contains \( R \). We call it the clone of dual relations generated by \( R \) and denote it by \( \mathcal{C}lo^\triangleright (R) \). It follows that clones of dual relations also form a complete lattice with respect to inclusion, which we will denote by \( \mathcal{R}_X^T \) and call the lattice of clones of dual relations of the typeclass \( T \) on \( X \).

**Example 25.** If \( \mathcal{C} \) is the category of sets and we choose \( T \) to be the set of all finite positive cardinal numbers, then the notion of clones of dual relations and that of clones of correlations as introduced in [23] coincide in \( \mathcal{C} \).

**Definition 26.** We define the two operators \( \overline{\mathcal{I}}_{\mathcal{O}_X} : \mathcal{P}(\mathcal{O}_X) \to \mathcal{P}(\overline{\mathcal{R}}_X^T) \) and \( \overline{\mathcal{P}}_{\mathcal{O}_X} : \mathcal{P}(\overline{\mathcal{R}}_X^T) \to \mathcal{P}(\mathcal{O}_X) \) as follows: for \( G \subseteq \mathcal{O}_X \) and \( R \subseteq \overline{\mathcal{R}}_X^T \), set

\[
\overline{\mathcal{I}}_{\mathcal{O}_X} G := \{ \sigma \in \overline{\mathcal{R}}_X^T \mid \forall g \in G : g \triangleright \sigma \},
\]

\[
\overline{\mathcal{P}}_{\mathcal{O}_X} R := \{ g \in \mathcal{O}_X \mid \forall \sigma \in R : g \triangleright \sigma \}.
\]

For \( Y \in \mathcal{C} \) and \( n \in \mathbb{N}_+ \), we use the following notation:

\[
\overline{\mathcal{I}}_{\mathcal{O}_X}(Y) := \{ \sigma \in \overline{\mathcal{R}}_X^Y \mid \forall g \in G : g \triangleright \sigma \},
\]

\[
\overline{\mathcal{P}}_{\mathcal{O}_X}(n) R := \overline{\mathcal{P}}_{\mathcal{O}_X} R \cap \mathcal{O}_X^n.
\]
Note that, for \( \mathcal{C} = \text{Set} \) and \( \mathbb{T} = \{ \{1, \ldots, k\} \mid k \in \mathbb{N}_+ \} \), it follows directly from the observations in Example 25 that the Galois Connection \( \overline{\text{Pol}}_{\mathbf{X}} - \overline{\text{Inv}}_{\mathbf{X}} \) coincides with the Galois connection \( c_{\text{Pol}} - c_{\text{Inv}} \) that is presented in [23].

**Example 27.** The following examples are dualized version of the examples presented in Example 12:

(i) Let \( \sigma := \{ \text{id}_X \} \). Then, \( \overline{\text{Pol}}_{\mathbf{X}} \{ \sigma \} \) is the set of all dual idempotent operations over \( X \). That is, \( \overline{\text{Pol}}_{\mathbf{X}} \{ \sigma \} = \{ g \in \overline{\text{G}}_{\mathbf{X}} \mid [\text{id}_X, \ldots, \text{id}_X] \circ g = \text{id}_X \} \).

(ii) Let \( C \leq \overline{\text{G}}_{\mathbf{X}} \). Note that \( C^{(n)} \) is a dual relation of type \( n \cdot \mathbf{X} \). Now, \( \overline{\text{Pol}}_{\mathbf{X}} C^{(n)} \) is the largest clone \( C' \) that agrees with \( C \) on its \( n \)-ary part, i.e. \( \overline{\text{Pol}}_{\mathbf{X}} C^{(n)} = C^{(n)} \).

(iii) If two dual operations \( g, g' \) are essentially the same, then \( \overline{\text{Inv}}_{\mathbf{X}} \{ g \} = \overline{\text{Inv}}_{\mathbf{X}} \{ g' \} \).

(iv) Recall that, by the Priestley duality, the category of finite distributive lattices is dually equivalent to the category of finite bounded posets and that the lattice of locally closed clones of dual relations on \( X \) coincides with the lattice of locally closed clones of dual relations on \( X \).

The only thing left to dualize are the local closure operators.

**Definition 28.** Let \( G \subseteq \overline{\text{G}}_{\mathbf{X}} \), \( R \subseteq \overline{\text{G}}_{\mathbf{X}}^{\mathbb{T}} \), \( s \geq 1 \) and let \( Z \in \mathcal{C} \). We define the following local closure operators:

\[
\overline{Z-\text{Loc}}_{\mathbf{X}} G := \{ g \in \overline{\text{G}}_{\mathbf{X}}^{(n)} \mid n \geq 1, \forall r_1, \ldots, r_n \in C(\mathbf{X}, Z) : \times \exists g' \in G : [r_1, \ldots, r_n] \circ g = [r_1, \ldots, r_n] \circ g' \},
\]

\[
\overline{s-\text{LOC}}_{\mathbf{X}}^{\mathbb{T}} R := \{ \sigma \in \overline{\text{G}}_{\mathbf{X}} \mid \forall B \subseteq \sigma', |B| \leq s : \exists \sigma' \in R : B \subseteq \sigma' \subseteq \sigma \}.
\]

Furthermore, let

\[
\overline{\text{Loc}}_{\mathbf{X}}^{\mathbb{T}} G := \bigcap_{Z \in \mathbb{T}} \overline{Z-\text{Loc}}_{\mathbf{X}} G
\]

and

\[
\overline{\text{LOC}}_{\mathbf{X}}^{\mathbb{T}} R := \bigcap_{s \in \mathbb{N}_+} \overline{s-\text{LOC}}_{\mathbf{X}}^{\mathbb{T}} R.
\]

Denote by \( \overline{\text{Loc}}_{\mathbf{X}}^{\mathbb{T}} \) and \( \overline{\text{LOC}}_{\mathbf{X}}^{\mathbb{T}} \) the lattice of locally closed clones of dual operations over \( X \) and the lattice of locally closed clones of dual relations on \( X \), respectively.

We have dualized every definition of the last section. Thus, each proposition, lemma and theorem from the last subsection holds in its dualized version. In particular, we immediately obtain our main result.
Corollary 29. For $G \subseteq \overline{O}_X$, the following are equivalent:

(a) $G \leq \overline{O}_X$ (i.e. $G = \overline{\text{clo}}(G)$) and $\overline{\text{Loc}}^T G = G$.
(b) $G = \overline{\text{Pol}}_X \overline{\text{Inv}}_X G$.
(c) $\exists R \subseteq \overline{R}_X : G = \overline{\text{Pol}}_X R$.

Corollary 30. For $R \subseteq \overline{R}_X$, the following are equivalent:

(a) $R \leq \overline{R}_X$ (i.e. $R = \overline{\text{clo}}^T(R)$) and $\overline{\text{Loc}}^T_R R = R$.
(b) $R = \overline{\text{Inv}}_X \overline{\text{Pol}}_X R$.
(c) $\exists G \subseteq \overline{O}_X : R = \overline{\text{Inv}}_X G$.

Thus, $\overline{\text{Loc}}^T \overline{O}_X$ and $\overline{\text{Loc}}^T \overline{R}_X$ are dually isomorphic via $\overline{\text{Inv}}^T_X$, and the local closure operators can be dismissed in the cases dual to those that are presented in the last subsection.

4. From Dual Equivalences to Clone Dualities

In the last section, we have constructed a framework in which reversing all morphisms in a category $C$ carries a clone of operations to a clone of dual operations and a clone of relations to a clone of dual relations. Thus, every clone (operational and relational) in $C$ is a dual clone in $C^{op}$. Although this allows us to treat clones and their duals as essentially the same thing (which might be nice), dualizing the clones in this way is hardly helpful as it is basically just a change of notations. What one really wants is the possibility to dualize clones from a category $\mathcal{A}$ into any dually equivalent category $\mathcal{X}$. Of course, this should be done in a way such that the duality somehow corresponds with the generalized Galois theory.

In this section, we will explain how this can be done. Eventually, we will end up with a framework where we can move back and forth between clones of operations, clones of relations, clones of dual operations and clones of dual relations in their respective categories, enabling us with the possibility to move a problem from one place to another, looking for the spot where it is the easiest to solve.

For the whole section, let $\langle D, E, e, \epsilon \rangle$ be a dual equivalence between two categories $\mathcal{A}$ and $\mathcal{X}$, and let $\mathcal{A} \in \mathcal{A}$ such that all finite non-empty powers of $\mathcal{A}$ are also in $\mathcal{A}$. Set $\mathcal{X} := D(\mathcal{A})$. Since $\mathcal{A}$ and $\mathcal{X}$ are dually equivalent, $\mathcal{X}$ contains all finite non-empty copowers of $\mathcal{X}$. The functor $D$ carries $\mathcal{A}$ to $\mathcal{X}$ and reverses the direction of the morphisms, so wishful thinking suggests that it should map a morphism $f \in O_\mathcal{A}$ to a morphism in $O_{\mathcal{X}}$. Unfortunately, $D$ only maps $f$ to a morphism from $\mathcal{X}$ to $D(\mathcal{A}^n)$ and the latter is only isomorphic and not necessarily equal to $n \cdot \mathcal{X}$. However, we can easily get around this technical problem by finding a family of

\footnote{Of course, we could avoid the trouble by defining $n \cdot \mathcal{X} := D(\mathcal{A}^n)$ for all $n \in \mathbb{N}_+$. But then, the copowers of $\mathcal{X}$ might not be canonical and they would depend on the choice of the dual equivalence. One usually wants to avoid both.}
In fact, \(\eta_n\) must have a unique morphism such that the following diagram commutes for all \(n\) isomorphism (\(\eta_n\)) \(\in\mathbb{N}_+\) such that \(f \mapsto \eta_{\text{ar}(f)} \circ D(f)\) becomes a clone isomorphism from \(O_A\) to \(\overline{O}_X\) (recall that \(\text{ar}(f)\) denotes the arity of \(f\)).

**Lemma 31.** There exists a unique family of isomorphisms

\[
(\eta_n : D(A^n) \rightarrow n \cdot X)_{n \in \mathbb{N}_+}
\]

such that the following diagram commutes for all \(n \in \mathbb{N}_+\) and \(i \in \{1, \ldots, n\}\):

\[
\begin{array}{ccc}
n \cdot X & \xrightarrow{\eta_n} & D(A^n) \\
D(\pi_n) & & D(\pi_n) \\
X = D(A) & \xleftarrow{\iota_n} & \\
\end{array}
\]

In fact, \(\eta_n = [D(\pi_n^1), \ldots, D(\pi_n^n)]^{-1}\).

**Proof.** Let \(n \in \mathbb{N}_+\). Since \(A\) and \(X\) are dually equivalent, products in \(A\) are taken to coproducts in \(X\). Hence, \(D(A^n)\), together with the associated morphisms \(D(\pi_n^1), \ldots, D(\pi_n^n)\), fulfills the condition of being a coproduct in \(X\). Therefore, we must have a unique morphism \(\eta_n : D(A^n) \rightarrow n \cdot X\) such that the above diagram commutes for all \(i \in \{1, \ldots, n\}\). It remains to show that \([D(\pi_n^1), \ldots, D(\pi_n^n)]\) is the inverse of \(\eta_n\). On the one hand, we have

\[
\eta_n \circ [D(\pi_n^1), \ldots, D(\pi_n^n)] = [\eta_n \circ D(\pi_n^1), \ldots, \eta_n \circ D(\pi_n^n)] = [\iota_n^1, \ldots, \iota_n^n] = \text{id}_{n \cdot X}.
\]

On the other hand, we have

\[
[D(\pi_n^1), \ldots, D(\pi_n^n)] \circ \eta_n \circ D(\pi_n^i) = [D(\pi_n^1), \ldots, D(\pi_n^n)] \circ \iota_n^i = D(\pi_n^i) = \text{id}_{D(A^n)} \circ D(\pi_n^i)
\]

for all \(i \in \{1, \ldots, n\}\). Since \(D(A^n)\), together with the morphisms \(D(\pi_n^1), \ldots, D(\pi_n^n)\), fulfills the condition to be a coproduct, there exists a unique morphism \(v\) such that \(v \circ D(\pi_n^i) = D(\pi_n^i)\) for all \(i \in \{1, \ldots, n\}\). But now, the uniqueness of \(v\) yields the desired equation \([D(\pi_n^1), \ldots, D(\pi_n^n)] \circ \eta_n = \text{id}_{D(A^n)}\).

With the help of \(\eta_n\), we can now define the clone duality.

**Definition 32.** The mapping \((-)^\circ : O_A \rightarrow \overline{O}_X\), defined by setting

\[
f^\circ := \eta_{\text{ar}(f)} \circ D(f)
\]

is called the clone duality with respect to \(D\). For \(F \subseteq O_A\), set \(F^\circ := \{f^\circ \mid f \in F\}\).

It remains to show that the mapping \((-)^\circ : O_A \rightarrow \overline{O}_X\) has the properties that its name suggests. That is, we need to show that \((-)^\circ\) is a clone isomorphism and that it is uniquely determined by the functor \(D\).
Lemma 33. Let $\mathcal{B} \in \mathfrak{A}$, $k, n \in \mathbb{N}_+$ and let $f_1, \ldots, f_n : \mathcal{B} \to \mathfrak{A}$. Then,

(i) $(-)^\varnothing : O_\mathfrak{A} \to \overline{O}_\mathbf{X}$ is a bijection.

(ii) $D((f_1, \ldots, f_n)) = [D(f_1), \ldots, D(f_n)] \circ \eta_n$.

(iii) $(\pi_n)^\varnothing = i_n^\varnothing$ and $(f \circ (f_1, \ldots, f_n))^\varnothing = [f_1^\varnothing, \ldots, f_n^\varnothing] \circ f^\varnothing$ for all $f_1, \ldots, f_n \in O_\mathfrak{A}(\mathcal{K})$, whence it follows that $C$ is a clone of operations over $\mathfrak{A}$ if and only if $C^\varnothing$ is a clone of dual operations over $\mathbf{X}$.

(iv) $(-)^\varnothing$ is the only mapping from $O_\mathfrak{A}$ to $\overline{O}_\mathbf{X}$ satisfying the properties (i)–(iii).

Proof. (i) Since $(\mathcal{D}, E, e, \varepsilon)$ is a dual equivalence, $D$ is both full and faithful. Furthermore, we just saw that $\eta_n$ is an isomorphism for all $n \in \mathbb{N}_+$. Thus, $f \mapsto \eta_{\mathfrak{A}(f)} \circ D(f)$ is bijective on $O_\mathfrak{A}$.

(ii) By Lemma 31, we have

$$D(f_i) = D(\pi_n^\varnothing \circ (f_1, \ldots, f_n)) = D((f_1, \ldots, f_n)) \circ D(\pi_n) = D((f_1, \ldots, f_n)) \circ \eta_n^{-1} \circ i_n$$

for all $i \in \{1, \ldots, n\}$. Thus,

$$[D(f_1), \ldots, D(f_n)] = [D((f_1, \ldots, f_n)) \circ \eta_n^{-1} \circ i_n, \ldots, D((f_1, \ldots, f_n)) \circ \eta_n^{-1} \circ i_n]$$

$$= D((f_1, \ldots, f_n)) \circ \eta_n^{-1} \circ [i_1^\varnothing, \ldots, i_n^\varnothing]$$

$$= D((f_1, \ldots, f_n)) \circ \eta_n^{-1}.$$

(iii) By Lemma 31, we have $(\pi_n)^\varnothing = \eta_n \circ D(\pi_n) = i_n^\varnothing$. For the second part, according to (ii), we get

$$(f \circ (f_1, \ldots, f_n))^\varnothing = \eta_k \circ D(f \circ (f_1, \ldots, f_n))$$

$$= \eta_k \circ D((f_1, \ldots, f_n)) \circ D(f)$$

$$= \eta_k \circ [D(f_1), \ldots, D(f_n)] \circ \eta_n \circ D(f)$$

$$= [\eta_k \circ D(f_1), \ldots, \eta_k \circ D(f_n)] \circ \eta_n \circ D(f)$$

$$= [f_1^\varnothing, \ldots, f_n^\varnothing] \circ f^\varnothing.$$

(iv) Follows from the uniqueness of $(\eta_n)_{n \in \mathbb{N}_+}$.

Now, the following theorem is an immediate consequence.

Theorem 34. $L_\mathfrak{A} \cong \overline{L}_\mathbf{X}$, where an isomorphism between $L_\mathfrak{A}$ and $\overline{L}_\mathbf{X}$ is given by $C \mapsto C^\varnothing$.

Thus, $(-)^\varnothing$ has all the properties that the name ‘clone duality’ suggests. In fact, a (purely category-theoretic) statement holds for a clone of operations $C \leq O_\mathfrak{A}$ if and only if the dualized statement holds for the clone of dual operations $C^\varnothing \leq \overline{O}_\mathbf{X}$.

For instance, it is an obvious consequence of Lemma 33 that an identity holds in $C$ if and only if its dualized version holds in $C^\varnothing$.

Proposition 35. Let $f \in O_\mathfrak{A}(\mathcal{K})$, $h \in O_\mathfrak{A}(\mathcal{I})$. For functions $\varphi : \mathcal{K} \to \mathfrak{A}$, $\varphi' : \mathfrak{I} \to \mathfrak{A}$, we have

$$f \circ A^{\varphi} = h \circ A^{\varphi'} \iff \varphi \cdot X \circ f^\varnothing = \varphi' \cdot X \circ h^\varnothing.$$
Note that the duality provides us with a new technique to examine clones: Instead of trying to solve a problem for a clone of operations $C$, we can solve the dualized problem for $C^\partial$, which might be easier. Indeed, being able to do this is quite possibly the main benefit of our theory.

After dualizing clones of operations over $A$ to clones of dual operations over $X$, one might want to dualize them back, requiring the inverse of $(-)^\partial$. The following proposition shows that $(-)^\partial - 1$ can be calculated with the functor $E$ and the natural isomorphism $e$.

**Proposition 36.** The inverse of $(-)^\partial$ is given by

$$g \mapsto e^{-1}_A \circ E(\eta^{-1}_{ar(g)} \circ g) \circ e_{A^{ar(g)}}.$$  

**Proof.** Let $f \in O^{(n)}_A$ such that $g = f^\partial$. Then, $g = \eta_n \circ D(f)$. Hence, we obtain $E(\eta^{-1}_n \circ g) = ED(f)$. Since $e : \text{id}_A \rightarrow ED$ is a natural isomorphism, we have $ED(f) = e_A \circ f \circ e_A^\partial$. Thus, we get $E(\eta^{-1}_n \circ g) = e_A \circ f \circ e_A^\partial$, whence it follows $f = e^{-1}_A \circ E(\eta^{-1}_n \circ g) \circ e_{A^{ar(g)}}$. □

Now that we know how to dualize clones of operations over $A$ to clones of dual operations over $X$, the question arises how this duality corresponds with the general Galois theory of (dual) operations and (dual) relations. For any choice of typeclasses $T \subseteq A$ and $T' \subseteq X$, we have a Galois connection $\text{Pol}_A \rightarrow \text{Inv}^T_A$ between the subsets of $O_A$ and the subclasses of $R^T_A$, and a Galois connection $\text{Pol}_X \rightarrow \text{Inv}^{T'}_X$ between the subsets of $\overline{O}_X$ and the subclasses of $\overline{R}^{T'}_X$. Recall that $\mathcal{L}_A^T$ and $\overline{\mathcal{L}}^T_X$ denote the lattice of clones of relations on $A$ and the lattice of clones of dual relations on $X$, respectively. Figure 1 summarizes the situation. Note that the mappings in the diagram commute only for the locally closed clones of the four lattices.

To complete this diagram, we need to find a mapping from $R^T_X$ to $\overline{R}^{T'}_X$ that induces a clone-isomorphism from $\mathcal{L}_A^T$ to $\overline{\mathcal{L}}^{T'}_X$ and commutes with the other mappings in the diagram for the locally closed clones. To achieve this, it is obvious that the typeclasses $T$ and $T'$ must correspond in a certain way. In fact, we need to require that $T'$ is the image of $T$ under the functor $D$ up to isomorphism (and equivalently, $T$ to be the image of $T'$ under the functor $E$ up to isomorphism).

---

**Fig. 1.** How can the diagram be completed?
Definition 37. Let \( C \) and \( C' \) be categories and let \( F : C \to C' \) be a full and faithful functor. Say that two typeclasses \( T \subseteq C \) and \( T' \subseteq C' \) are equivalent under \( F \) if there exists a family of isomorphisms \( (\psi_B)_{B \in T} \) in \( C \) such that we have

\[
T' = \{ (\psi_B \circ F)(B) \mid B \in T \}.
\]

In other words, \( T \) and \( T' \) are equivalent under \( F \) if and only if each \( Y \in T' \) is isomorphic to \( F(B) \) for some \( B \in T \) and, for each \( B \in T \), we have some \( Y \in T' \) such that \( Y \cong F(B) \). Note that, for a dual equivalence \( \langle D, E, e, \epsilon \rangle \), two typeclasses \( T \) and \( T' \) are equivalent under \( D \) if and only if they are equivalent under \( E \). Of course, the easiest (but not always best) way to ensure the equivalence of \( T \) and \( T' \) is to set \( T' := D(T) \).

Example 38.

(i) Two skeletons \( T \subseteq A \) and \( T' \subseteq X \) are always equivalent under \( D \). In fact, if \( T \) is a skeleton of \( A \), then \( T \) and \( T' \) are equivalent if and only if \( T' \) is a skeleton of \( X \).

(ii) Let \( A \) be the category of finite distributive lattices, let \( X \) be the category of finite bounded partially ordered sets, and let \( \langle D, E, e, \epsilon \rangle \) be the Priestley duality that we have recalled in Sec. 2.2. The two typeclasses

\[
T := \{ \langle \{0, 1\}, \lor, \land \rangle^n \mid n \in \mathbb{N}_+ \},
\]

\[
T' := \{ \langle \{\emptyset, \{1\}, \ldots, \{n\}, \emptyset, \mathbb{N}, \subseteq \rangle \mid n \in \mathbb{N}_+ \}
\]

are equivalent under \( D \).

We will now define the mapping that will eventually complete the diagram in Fig. 1.

Definition 39. Let \( T \subseteq A \) and \( T' \subseteq X \) be two typeclasses that are equivalent under \( D \), and let \( (\psi_B)_{B \in T} \) be a corresponding (fixed) family of isomorphisms. We set

\[
\sigma^{\partial^*} := \{ \psi_B \circ D(r) \mid r \in \sigma \}.
\]

We will now spend the next few pages to prove that \((-)^\partial^* \) is indeed the mapping we are looking for.

Lemma 40. \( R^{\partial^*} \) is a clone of dual relations on \( X \) if and only if \( R \) is a clone of relations on \( A \).

Proof. For each \( r \in \sigma \subseteq R_A^{(B)} \), \( D(r) \) is a morphism from \( X \) to \( D(B) \). Hence, \( \psi_B \circ D(r) \) is a morphism from \( X \) to some \( Y \in T' \), and so it follows \( \sigma^{\partial^*} \subseteq R_X^{(Y)} \). Thus, \((-)^{\partial^*} \) is well-defined. Since \( D \) is full and faithful and \( \psi_B \) is an isomorphism, it follows that the mapping \( \psi_B \circ D(-) : R_A^{(B)} \to R_X^{(Y)} \) is bijective for each \( B \in T \).
Since the typeclasses are equivalent under $D$, this implies that $(-)^{\sigma^*} : R^X_A \rightarrow \overline{R}^X_A$ is bijective.

Let us show that $R^{\sigma^*}$ is a clone of dual relations if and only if $R$ is a clone of relations. Obviously, we have $R^{\sigma^*} = [\overline{R}_X^A]^\sigma = \overline{R}^X_{\overline{A}^\sigma}$. Since the mappings $\varphi : B \rightarrow C$ and $\varphi_i : B_i \rightarrow C$ ($i \in I$) are morphisms where $C \in \mathcal{A}$ and $B \in T$. We will show

$$\left( \bigwedge_{(\varphi_i) \in I} (\sigma_i)_{i \in I} \right)^{\sigma^*} = \bigwedge_{(\varphi_i) \in I} (\sigma_i^{\sigma^*})_{i \in I},$$

where

$$\varphi' := \psi_B \circ D(\varphi), \quad \varphi'_i := \psi_{B_i} \circ D(\varphi_i) \quad (i \in I).$$

We have

$$\left( \bigwedge_{(\varphi_i) \in I} (\sigma_i)_{i \in I} \right)^{\sigma^*} = \{ \psi_B \circ D(r \circ \varphi) | \forall i \in I : r \circ \varphi_i \in \sigma_i, r \in \mathcal{A}(C, A) \}$$

$$= \{ \psi_B \circ D(r \circ \varphi) | \forall i \in I : \psi_{B_i} \circ D(r \circ \varphi_i) \in \sigma_i^{\sigma^*}, r \in \mathcal{A}(C, A) \}$$

$$= \{ \varphi' \circ D(r) | \forall i \in I : \varphi'_i \circ D(r) \in \sigma_i^{\sigma^*}, r \in \mathcal{A}(C, A) \}$$

$$= \{ \varphi' \circ s | \forall i \in I : \varphi'_i \circ s \in \sigma_i^{\sigma^*}, s \in X(X, D(C)) \}$$

$$= \bigwedge_{(\varphi_i) \in I} (\sigma_i^{\sigma^*})_{i \in I}.$$
Hence,
\[
f \circ \langle r_1, \ldots, r_n \rangle \in \sigma \iff (f \circ \langle r_1, \ldots, r_n \rangle)^\sigma \in \sigma^\sigma
\]
\[
\iff \psi_B \circ D(f \circ \langle r_1, \ldots, r_n \rangle) \in \sigma^\sigma
\]
\[
\iff \psi_B \circ D((r_1, \ldots, r_n)) \circ D(f) \in \sigma^\sigma
\]
\[
\iff \psi_B \circ [D(r_1), \ldots, D(r_n)] \circ \eta_n \circ D(f) \in \sigma^\sigma
\]
\[
\iff \psi_B \circ [D(r_1), \ldots, D(r_n)] \circ f^0 \in \sigma^\sigma
\]
\[
\iff \psi_B \circ D(r_1), \ldots, \psi_B \circ D(r_n) \circ f^0 \in \sigma^\sigma.
\]
The claim follows since \( \psi_B \circ D(-) : \sigma \to \sigma^\sigma : r \mapsto \psi_B \circ D(r) \) is bijective.

Let us now turn our attention to the local closure operators.

**Lemma 42.** Let \( C \in \mathbb{T}, Z \in \mathbb{T}' \) such that \( D(C) \cong Z \). Then, for all \( F \subseteq O_A \) and \( R \subseteq R_A \), we have

(i) \( Z_{-\text{Loc}} F^\sigma = (C_{-\text{Loc}} F)^\sigma \),

(ii) \( \text{Loc}^{\sigma^\sigma} F^\sigma = (\text{Loc}^\sigma F)^\sigma \),

(iii) \( s_{-\text{Loc}}^C R^\sigma = (s_{-\text{Loc}}^C R)^\sigma \),

(iv) \( \text{Loc}^{\sigma^\sigma} R^\sigma = (\text{Loc}^\sigma R)^\sigma \).

**Proof.**

(i) Let \( f \in O_A^{(n)} \) and let \( \psi_C \) be the isomorphism from Definition 37. We need to show that \( f^0 \in Z_{-\text{Loc}} F^\sigma \) is equivalent to \( f \in C_{-\text{Loc}} F \). Before we do so, let us note the following equivalences for each \( f' \in F \):

\[
[\psi_C \circ D(r_1), \ldots, \psi_C \circ D(r_n)] \circ f^0 = [\psi_C \circ D(r_1), \ldots, \psi_C \circ D(r_n)] \circ f^0
\]
\[
\iff [D(r_1), \ldots, D(r_n)] \circ f^0 = [D(r_1), \ldots, D(r_n)] \circ f^0
\]
\[
\iff [D(r_1), \ldots, D(r_n)] \circ \eta_n \circ D(f) = [D(r_1), \ldots, D(r_n)] \circ \eta_n \circ D(f')
\]
\[
\iff D([r_1, \ldots, r_n]) \circ D(f) = D([r_1, \ldots, r_n]) \circ D(f')
\]
\[
\iff D(f \circ \langle r_1, \ldots, r_n \rangle) = D(f' \circ \langle r_1, \ldots, r_n \rangle)
\]
\[
\iff f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle.
\]
Recall that \( f^0 \in Z_{-\text{Loc}} F^\sigma \) is equivalent to the following statement: For all \( s_1, \ldots, s_n \in \mathcal{A}(X, Z) \), there exists \( g' \in F^\sigma \) such that:

\[
[s_1, \ldots, s_n] \circ f^0 = [s_1, \ldots, s_n] \circ g'.
\]

This, in turn, is equivalent to the following statement: For all \( r_1, \ldots, r_n \in \mathcal{A}(C, A) \), there exists \( f' \in F \) such that

\[
[\psi_C \circ D(r_1), \ldots, \psi_C \circ D(r_n)] \circ f^0 = [\psi_C \circ D(r_1), \ldots, \psi_C \circ D(r_n)] \circ f^0.
\]

But now, we have seen above that this equation holds if and only if

\[
f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle,
\]
The mapping Theorem 43.

Proof. (i) Let $f^\partial \in \overline{Z}\cdot \text{Loc} F^\partial$ is equivalent to $f \in \text{C-Loc} F$.

(ii) Since $T$ and $T'$ are equivalent under $D$, (i) yields that, for each $C \in T$, there exists $Z \in T'$ such that $\overline{Z}\cdot \text{Loc} F^\partial = (\text{C-Loc} F)^\partial$. Thus,

$$\text{Loc}^T F^\partial = \bigcap_{Z \in T'} \overline{Z}\cdot \text{Loc} F^\partial = \bigcap_{C \in T} (\text{C-Loc} F)^\partial \overset{(*)}{=} \left( \bigcap_{C \in T} \text{C-Loc} F \right)^\partial = (\text{Loc}^T F)^\partial,$$

where ($*$) follows from the bijectivity of $(-)^\partial$.

(iii) Let $\varrho \in R^\partial_A$ and let $\sigma \in R$ such that $\sigma^\partial = \varrho$. We have

$$\varrho \in \overline{\text{LOC}}^T R^\partial \Leftrightarrow \forall B \subseteq \varrho, |B| \leq s : \exists \varrho' \in R^\partial : B \subseteq \varrho' \subseteq \varrho$$

$$\Leftrightarrow \forall B \subseteq \sigma^\partial, |B| \leq s : \exists \sigma' \in R : B \subseteq \sigma' \subseteq \sigma$$

where the last but one step follows from the obvious fact that $(-)^\partial$ preserves the cardinality of a relation.

(iv) Follows from (iii) in the same way that (ii) follows from (i). \qed

We are ready to prove the desired theorem.

Theorem 43. The mapping $(-)^\partial : R^\partial_A \to R^\partial_X$ induces a clone-isomorphism from $\mathcal{L}^T_A$ to $\mathcal{L}^T_X$ with the following properties:

(i) $\overline{\text{Pol}}_X R^\partial = (\text{Pol}_A R)^\partial$ for all $R \subseteq R^\partial_A$.

(ii) $\text{Inv}^T_X F^\partial = (\text{Inv}^T_A F)^\partial$ for all $F \subseteq O_A$.

(iii) $\text{Inv}^T_X (\text{Pol}_A R)^\partial = R^\partial$ for all locally closed $R \subseteq R^\partial_A$.

(iv) $\overline{\text{Pol}}_X (\text{Inv}^T_A F)^\partial = F^\partial$ for all locally closed $F \subseteq O_A$.

(v) The following diagram commutes:

Proof. (i) Let $g \in \overline{\text{Pol}}_X R^\partial$, i.e. $g \supseteq R^\partial$. Choose $f \in O_A$ such that $g = f^\partial$. By Lemma 41, $f^\partial \supseteq R^\partial$ is equivalent to $f \supseteq R$. Thus, $g \in \overline{\text{Pol}}_X R^\partial$ is equivalent to $g \in (\text{Pol}_A R)^\partial$.

(ii) Analogous to (i).
Corollary 44. Let \( n \in \mathbb{N}_+ \), \( B \in T \) and let \( Y \in T \) such that \( D(B) \equiv Y \). Then,

(i) \( \overline{\text{Inv}}_X(Y) F^\partial = (\text{Inv}_A^{(B)} F)^\partial^\ast \) for all \( F \subseteq O_A \).

(ii) \( \overline{\text{Pol}}_X R^\partial = (\text{Pol}_A^{(n)} R)^\partial \) for all \( R \subseteq R_A^n \).

Proof. (i) We have \( R_A^{(B)} = R_X^{(Y)} \). Together with Theorem 43, we obtain

\[
\overline{\text{Inv}}_X(Y) F^\partial = \overline{\text{Inv}}_X(Y) \cap \overline{\text{Inv}}_X F^\partial = (\text{Inv}_A^{(B)} F)^\partial^\ast = (\text{Inv}_A^{(B)} F)^\partial^\ast.
\]

(ii) Follows in the same way (note that \((-)^\partial\) preserves the arity).

The following corollary is an easy consequence.

The diagram from the last theorem summarizes our results and is the core of this paper. Within this diagram, we can move freely between clones of operations, clones of relation, clones of dual operations and clones of dual relations.

Let us now look at the following example.

Example 45. Let \( A \) be the category of finite distributive lattices and let \( X \) be the category of finite bounded posets. Moreover, let \( A \in A \), \( (D, E, c, e) \) be the Priestley duality that we have recalled in the preliminaries, and let \( X := D(A) = \langle \text{Spec}(A), \emptyset, A, \subseteq \rangle \) be the dual of \( A \). The copowers of finite bounded posets are usually defined as follows: if we denote by \( X' \) the underlying set of \( X \) without the two constants \( 0^X \) and \( 1^X \) and by \( n \cdot X' \) the \( n \)th copower of \( X' \) in the category of sets (that is, \( n \cdot X' = \{ \langle i, x \rangle \mid i \in \{1, \ldots, n\}, x \in X' \} \)), then we have

\[
n \cdot X := \langle \{0^n X, 1^n X\} \cup n \cdot X', 0^n X, 1^n X, \leq \rangle,
\]

where \( z_1 \leq z_2 \) whenever \( z_1 = 0^n X \) or \( z_2 = 1^n X \) or there exists \( i \in \{1, \ldots, n\} \) such that \( z_1 = \langle i, x_1 \rangle \), \( z_2 = \langle i, x_2 \rangle \) and \( x_1 \leq x_2 \). The associated injection morphisms are given by

\[
c^n_i(x) := \begin{cases} 
0^n X & \text{if } x = 0^X, \\
1^n X & \text{if } x = 1^X, \\
\langle i, x \rangle & \text{otherwise.}
\end{cases}
\]
To dualize clones of operations over \( A \) to clones of dual operations over \( X \), we need to find the (unique) family of isomorphisms \( \eta_n: D(A^n) \to n \cdot X \) for all \( n \in \mathbb{N}_+ \). Therefore, we first need to understand how the powers of \( A \) (together with their associated projection morphisms) are dualized under \( D \). We have

\[
D(A^n) = \langle \text{Spec}(A^n), \emptyset, A^n, \subseteq \rangle
\]

and it is not hard to show (and well-known) that we have

\[
\text{Spec}(A^n) = \{ A^{i-1} \times A^{n-i} \mid i \in \{1, \ldots, n\}, x \in \text{Spec}(A) \}
\]

(a proof for this can be found in [12], for instance). Concerning the projection morphisms, it is obvious that \( D(\pi^n_i) \) is given as follows:

\[
D(\pi^n_i): D(A) \to D(A^n): x \mapsto A^{i-1} \times x \times A^{n-i}.
\]

It is now fairly obvious how we have to define \( \eta_n \) (recall that it is the inverse of \( \eta^{-1}_n \)).

\[
\eta_n(x) := \begin{cases} 
0^n \times & \text{if } x = \emptyset, \\
1^n \times & \text{if } x = A^n, \\
(i, y) & \text{if } x = A^{i-1} \times y \times A^{n-i} \text{ for some } y \in \text{Spec}(A) \setminus \{\emptyset, A\}.
\end{cases}
\]

We now obtain our clone duality \((-)^\circ : O_A \to O_X\) by setting \( f^\circ := \eta_n \circ D(f) \) for \( f \in O_A^n \). Stating this explicitly, \((-)^\circ\) maps each \( f \in O_A^n \) to

\[
f^\circ(x) = \begin{cases} 
0^n \times & \text{if } f^{-1}(x) = \emptyset, \\
1^n \times & \text{if } f^{-1}(x) = A^n, \\
(i, y) & \text{if } f^{-1}(x) = A^{i-1} \times y \times A^{n-i} \text{ for some } y \in \text{Spec}(A) \setminus \{\emptyset, A\}.
\end{cases}
\]

Let us now introduce the two typeclasses \( T \subseteq A \) and \( T' \subseteq X \) that we have already used in Example 38:

\[
T := \{\{0, 1\}, \lor, \land\}^n \mid n \in \mathbb{N}_+\}, \quad T' := \{\{\emptyset, \{1\}, \ldots, \{n\}, \emptyset, \emptyset, \subseteq\} \mid n \in \mathbb{N}_+\}.
\]

These two typeclasses are equivalent under \( D \), and it follows immediately from Lemma 17 that all subsets of \( O_A, O_X, R_A^+ \) and \( R_X^+ \) are locally closed. Constructing \((-)^\circ\) is very easy. For each \( B \in T \) we denote by \( Z_B \) the element of the typeclass \( T' \) that is isomorphic to \( D(B) \) and we define

\[
\psi_B: D(B) \to Z_B: x \mapsto \begin{cases} 
\emptyset & \text{if } x = \emptyset, \\
\{i\} & \text{if } \land x = (0, \ldots, 0, 1, 0, \ldots, 0), \\
\mathbb{N} & \text{if } x = B.
\end{cases}
\]
We leave it to the reader to verify that \((\psi_B)_B \in T\) is in fact a well defined family of isomorphisms. We obtain \((-)^0\) by setting 
\[
(-)^0 : R^\sigma_A \to R^\tau_X,
\]
\[
\sigma \in R^B_A \mapsto \{\psi_B \circ D(r) \mid r \in \sigma\},
\]
and we end up with the following commuting diagram:

\[
\begin{array}{ccc}
L^*_A & \overset{(-)^0}{\longrightarrow} & L^*_X \\
\text{Pol}_A & \downarrow_{\text{Inv}_A} & \downarrow_{\text{Inv}_X} \\
L_A & \underset{(-)^0}{\longrightarrow} & L_X \\
\end{array}
\]

Note that these results also enable us to dualize sets of operations without using \((-)^0\). As the following example shows, this comes in handy whenever applying \((-)^0\) would be rather difficult.

**Example 46.** Assume the scenario outlined in the last example. Define \(F \subseteq O_A\) to be the set of strictly monotone operations, that is, \(f(a_1, \ldots, a_n) > f(b_1, \ldots, b_n)\) whenever \(a_i > b_i\) for all \(i \in \{1, \ldots, n\}\). It seems as if dualizing \(F\) via \((-)^0\) requires some nontrivial calculation. So, let us use the above results to dualize \(F\) in another way. We have seen in Example 12 that we have \(F = \text{Pol}_A\{\sigma\}\) for 
\[
\sigma := \{r \in A(B, A) \mid r(0) \neq r(1)\},
\]
where \(B := \langle\{0, 1\}, \lor, \land\rangle \in T\). Dualizing \(\sigma\) via \((-)^0 : R^A \to R^X\) is easy. We have \(D(B) \cong Z_B := \langle\{0, 1\}, \lor, \land\rangle\) and 
\[
\sigma^0 = \{r \in X(Z_B) \mid \exists x \in X\setminus\{0, A\} : r(x) = \{1\}\}.
\]
But now, Example 27(iv) yields 
\[
\text{Pol}_X(\sigma^0) = \{g \in \overline{O}_X \mid \exists i \in \{1, \ldots, n\} : g[X] = \iota^i[X]\}.
\]
By Theorem 43, we have \(F^0 = \text{Pol}_X(\sigma^0)\), so we have successfully dualized \(F\).

In the same way, one can dualize relations without using \((-)^0\).

At this point of the paper, the reader is hopefully convinced that the name “general duality theory” is justified. However, generality does of course not ensure applicability. In fact, we have not yet discussed how to apply the theory in practice. This will be done in Secs. 5 and 6. First, we outline how to dualize clones over sets (after all, these are the clones that one is usually interested in), and afterwards we put the duality to work, illustrating how it can be used to produce concrete results.
5. Dualizing Clones Over Sets

In the last section, we have presented how to dualize clones of operations, clones of relations and the corresponding Galois theory in a category \( \mathcal{A} \) to their dual counterparts in any category \( X \) that is dually equivalent to \( \mathcal{A} \).

So far, our approach has been mainly theoretical, with the exception of one concrete example. At the end of the last section, we have used the Priestley duality to dualize clones over distributive lattices (together with their relational counterparts and the relationship between them) to clones of dual operations over bounded posets. Thus, if we have a clone over a finite set that happens to be the centralizer clone of a distributive lattice, then we can use this particular technique, and we will learn in Sec. 6.2 that this indeed a powerful tool to investigate these clones. But what about all the other clones? Is there a similarly efficient technique to dualize arbitrary clones over (finite) sets? These are the questions that we are about to discuss in this section.

First, let \( A \) be a finite set and let \( C \subseteq O_A \) be a clone. Clearly, we can dualize \( C \) to a clone of dual operations \( C^\partial \) in any category dually equivalent to \( \text{Set} \). However, the categories that are dually equivalent to \( \text{Set} \) (such as the category of complete atomic Boolean algebras) have horrible copowers and, consequently, very complicated dual operations. Hence, dualizing \( C \) in this way will probably not be of great help. It seems more promising to interpret \( C \) as a clone in another category and then dualize this category instead. Preferably we want a category that is easily accessible and in which we can regard \( C \) as the full clone \( O_A \) for some object \( A \) (so we do not have to build the clone duality in a way such that it can also dualize clones that are greater than \( C \)). A possible way to do so is to interpret \( C \) as the full clone over some structure \( A \) in a quasivariety. This is precisely what we did in Example 45, and it will be the approach we will pursue in this section.

As a first step, let us note that we can always think of \( C \) as the set of homomorphisms \( \bigcup_{n \in \mathbb{N}} \text{Hom}(A^n, A) \) for some finite structure \( A = \langle A, F, H, R \rangle \). We can always make this work by choosing \( F = H = \emptyset \) and \( R = \text{Inv } C \), although we will shortly understand that this might not be the best choice. Moreover, let \( \mathcal{A} \) be the finite part of any quasivariety that contains \( A \). Recall that we can understand \( \mathcal{A} \) as a category by defining the objects to be the structures and the morphisms to be the homomorphisms between the structures (see Example 4). In this setting, we have \( O_A = \bigcup_{n \in \mathbb{N}} \text{Hom}(A^n, A) \), so \( O_A \) is essentially our clone \( C \).

Additionally, we can choose \( \mathcal{A} \) to be a quasivariety that is generated by a single finite structure. In other words, there always exists a finite structure \( M \) such that

\[
\mathcal{A} := \text{ISP}(M)_{\text{fin}} = \{ B \in \text{ISP}(M) \mid B \text{ finite} \}
\]

contains our structure \( A \). The most obvious choice would be \( M := A \), but again, other choices of \( M \) might be more promising, as we will see in a moment.

In any way, we have succeeded in finding a category \( \mathcal{A} \) of comparably easy structure such that \( C \) can be written as \( O_A \) for some \( A \in \mathcal{A} \). All that remains to be
done is to find a dual equivalence between \( \mathcal{A} \) and some (preferably easy) category \( X \). For this, we will use the theory of natural dualities for structures from [6].

**Theorem 47.** Let \( C \) be a clone over a finite set \( A \). There exists a category of finite structures \( X \) and some \( X \in X \) such that the order-ideal \( \langle C \rangle \) is isomorphic to \( \overline{\mathcal{L}}_X \).

**Proof.** As explained above, there always exists a structure \( A \) with carrier set \( A \) and a finite structure \( M \) such that \( A \in \mathbb{ISP}(M)_{\text{fin}} \). Understanding \( \mathcal{A} := \mathbb{ISP}(M)_{\text{fin}} \) as a category gives us \( C = O_A \). By the Brute Force construction for natural dualities (see [6]), there exists an alter ego \( \mathcal{M} \) of \( M \) equipped with the discrete topology such that \( \mathcal{A} \) and some full subcategory \( X \) of \( \mathbb{ISP}_{\text{fin}}(M) \) are dually equivalent via a natural dual equivalence whose two functors \( D, E \) are given by \( \mathcal{A}(-, M)_X \) and \( X(-, M) \), respectively, where the indices of the two hom-functors mean that the images of \( \mathcal{A}(-, M) : \mathcal{A} \to \mathcal{Set} \) and \( X(-, M) : X \to \mathcal{Set} \) must be equipped with the structure that is inherited from suitable powers of \( M \) and \( \mathcal{M} \), respectively.\( ^{d} \) Since all structures in \( X \) are finite, we can safely omit the discrete topology. Let \( X := D(M) \).

By Theorem 34, we have a clone-isomorphism \( (-)^{0} : \mathcal{L}_A \to \overline{\mathcal{L}}_X \). Thus, we obtain \( \langle C \rangle \cong \mathcal{L}_A \cong \overline{\mathcal{L}}_X \).

By the way the natural dualities work, it is now easy to see why choosing \( M := A \) might not be the best choice: the functor \( D \) dualizes \( \mathcal{A}(A, M)_X \), and the latter is obviously of the easiest form if \( M \) is chosen to be as small as possible.

Moreover, as outlined above, we can always choose the structures \( A \) and \( M \) to be total (that is, no partial operations) or purely relational, in which case \( X \) becomes a category of finite total structures. However, the theorem is not stated in this way, and this has a very good reason: it is often advantageous to minimize the number of relations in \( M \) by including operations or even partial operations. This is due to the fact that operations, unlike relations, reduce the number of substructures of finite powers of \( M \), which simplifies the alter ego \( \mathcal{M} \) and consequently the category \( X \).

This theorem is a generalization of a result appearing in [17], where it was shown that the statement above holds for all centralizer clones \( C \). In fact, it was already proposed in [17] to use natural dualities to dualize centralizer clones and a clone isomorphism \( (-)^{0} : O_A \to \overline{\mathcal{L}}_X \) similar to the one we constructed in Sec. 4 was built for the specific scenario of \( A \) being a finite algebra. As mentioned in Sec. 1, [17] was the starting point for our general theory.

If \( C \) is a clone over an infinite set \( A \), things are more complicated and arguments similar to those from the proof of Theorem 47 are not necessarily applicable. In fact, the theory of natural dualities might fail to construct a desirable duality. One might not be able to find a structure \( A \) in quasivariety \( \mathcal{A} \) such that we have \( C = O_A \) and \( \mathcal{A} \) is generated by a finite structure \( M \), and if one does, then it is not guaranteed

\( ^{d} \)In fact, almost all well-known dual equivalences arise via such structured hom-functors and pairs of objects \( (M, M) \). Dualities of this fashion are also known as Isbell dualities or concrete dualities, see [1, 19].
that one can construct an alter ego $M$ such that $\mathcal{A}$ and some full subcategory of $\mathcal{I}S^o_{CP} (M)$ are dually equivalent. However, there are still many cases in which this is possible and $C$ can be dualized to the full clone of dual operations over some topological structure $X$ in a category of profinite topological structures (the profiniteness comes from the fact that $M$ carries the discrete topology). A discussion of these cases would go deep into the theory of natural dualities and may not be elaborated here. Let us simply note that there are many clones that can be dualized in this way. For instance, it is possible if $M$ has a near-unanimity term.

However, one has to keep in mind that the construction of Theorem 47 and the use of natural dualities is only one method to dualize a clone. Applying other dual equivalences can also be helpful. In particular, it is worthwhile to think about those dualities that connect two different fields of mathematics, as they might allow us to transfer a clone from the framework of universal algebra into something different, for instance into a topological framework. Example 52 will illustrate how this can be beneficial.

6. An Illustration of the Approach

In this section, we will finally put all the results of the previous sections to work and present several applications of the theory in order to (hopefully) convince the reader that the approach is not only general, but also useful. We choose a collection of results (some are taken from [12], some only appear here), aiming for an illustration of how clone dualities can produce some very general results as well as specific and technical ones. It should be noted that many more examples of results that are achieved by using clone dualities can be found in [12].

6.1. General results

In this subsection, we will aim for some general results that arise by studying the clones of dual operations and transferring the obtained information back to clones of operations.

Let $\mathcal{A}$ be a category with an object $A$ such that all non-empty finite powers of $A$ also exist in $\mathcal{A}$. Moreover, let $X$ be a category that is dually equivalent to $\mathcal{A}$, let $X := D(\mathcal{A})$ be the dual of $\mathcal{A}$ under some dual equivalence $\langle D, E, e, \epsilon \rangle$, and let $(-)^\partial : O_{\mathcal{A}} \rightarrow \mathcal{O}_X$ be the clone duality with respect to $D$.

Our goal is to illustrate how the change of perspective that comes with looking at $\mathcal{O}_X$ instead of $O_{\mathcal{A}}$ can make some problems easier to solve. Of course, in an entirely abstract framework, studying $\mathcal{O}_X$ instead of $O_{\mathcal{A}}$ would be nothing more than a change of notation. Since this is not what we want, let us leave the purely abstract setting, and let us assume that $\mathcal{A}$ and $X$ are concrete categories. Recall from the preliminaries that we use (without loss of generality) the convention of dismissing the forgetful functor and directly assuming that all objects from the two categories have underlying sets and that all morphisms are mappings between these sets.
At first, let us take a look at essential variables. It is an obvious consequence of Proposition 35 that the $i$th variable of an operation $f \in O_A$ is essential if and only if the $i$th variable of $f^0 \in \mathcal{O}_X$ is essential. Thus, instead of investigating the essentiality of the variables of $f$, we can investigate those of $f^0$. Essential variables of dual operations were studied in [13] without connection to clone dualities. In particular, the following result is presented.

**Lemma 48 ([13]).** Let $m, k \in \mathbb{N}_+ \cup \{0\}$ and let $g \in O^{(k)}_X$. For $k \geq 2$ and $\varphi : m \to k$, the following two statements are equivalent:

(a) $g[X] \subseteq \varphi \cdot X[m \cdot X]$.
(b) For each $t \in \{1, \ldots, k\} \setminus \{m\}$, the $t$th variable of $g$ is nonessential.

If $g$ is a unary dual operation, then its only variable is nonessential if and only if $\iota_1^1(x) = \iota_2^1(x)$ for all $x \in g[X]$.

By looking at this lemma, it becomes evident that dual operations offer a different, potentially easier, view of essential variables. In fact, it is illustrated in [13] that this lemma is a handy tool to determine which variables of a dual operation are essential and which are not. In view of our clone dualities, it seems therefore promising to investigate the essentiality of the variables of some $f \in O_A$ by applying the lemma for $f^0 \in \mathcal{O}_X$. However, this technique is not the end of the road. For some (very common kind of) dual operations, condition (a) from the lemma can be made much simpler. To understand the idea, we will now define a property for sets of dual operations $G \subseteq \mathcal{O}_X$ that will be the key definition for the remaining work in this subsection.

**Definition 49.** A $k$-ary dual operation $g \in \mathcal{O}_X$ is said to respect the images of the injection morphisms to the degree $n$ provided that each $y \in g[X]$ can be written as $\varphi \cdot X(x)$ for some $\varphi : n \to k$ and $x \in n \cdot X$. A set of dual operations $G$ is said to respect the images of the injection morphisms to the degree $n$ whenever each $g \in G$ respects the images of the injection morphisms to the degree $n$.

Equivalently formulated, a set of dual operations $G \subseteq \mathcal{O}_X$ respects the images of the injection morphisms to the degree $n$ if we have

$$\bigcup_{g \in G^{(k)}} g[X] \subseteq \bigcup_{\varphi : n \to k} \varphi \cdot X[n \cdot X]$$

for all $k \geq n$.

As one can see, the question of whether a dual operation respects the images of the injection morphisms to the degree 1 depends heavily on the form of the copowers of $X$. This connection is extensively studied in [12, 13], but may not be discussed here. Let us simply note that a quick look at the copowers of $X$ will often
very quickly reveal whether all dual operations over \( X \) respect the images of the injection morphisms to a certain degree.

**Example 50.** In many well-known categories, all dual operations respect the images of the injection morphisms to the degree 1. Among them are the category of sets, the category of (bounded) posets, the category of pointed spaces, the category of graphs, the category of topological spaces (with important subcategories such as the category of all compact Hausdorff spaces), any quasivariety of unary algebras, and any quasivariety of relational structures.

As we will now see, dual operations that respect the images of the injection morphisms to a degree \( n \) have special properties. In particular, we can formulate a stronger version of Lemma 48.

**Lemma 51 ([12, 13]).** Let \( k, n \in \mathbb{N}^+ \) with \( k \geq 2 \) and let \( g \) be a \( k \)-ary dual operation that respects the images of the injection morphisms to the degree \( n \). For \( t \in \{1, \ldots, k\} \), the following two statements are equivalent:

(i) \( g[X] \subseteq \bigcup_{\varphi \in \mathcal{P}_n} \varphi \cdot X[n \cdot X] \).

(ii) The \( t \)th variable of \( g \) is nonessential.

Note that the direction \((a) \Rightarrow (b)\) is already given by Lemma 48, but the other direction does in fact require that \( g \) respects the images of the injection morphisms to the degree \( n \). In particular, for \( n = 1 \) (recall that this applies to many well-known categories), this lemma boils down to the following easy characterization of nonessential variables: the \( t \)th variable is nonessential if and only if

\[
g[X] \cap \nu_i^{-1}[X] \neq \emptyset.
\]

As the following example shows, taking this characterization and the clone duality allows us to investigate the essential arity of operations over \( A \) with rather unusual methods.

**Example 52.** If \( C \) is a clone over a commutative unital \( C^* \)-Algebra \( A \), then it dualizes to a clone of dual operations over a compact Hausdorff space \( X \) by building the clone duality on the dual equivalence of Gelfand and Naimark (since this is only supposed to be a small example, we will not introduce the duality and all the corresponding notions; they can be found in virtually any textbook on functional analysis). Since each dual operation in the category of compact Hausdorff spaces respects the images of the injection morphisms to the degree 1, we can apply the last lemma, and it follows that the \( i \)th variable of a dual operation \( g \in C^0 \) is essential if and only if \( g[X] \cap \nu_i^{-1}[X] \neq \emptyset \). However, \( \nu_i^{-1}[X] \) is clopen with respect to the topology of \( n \cdot X \), so \( g^{-1}[\nu_i^{-1}[X]] \) must be clopen as well. Hence, it follows that \( \{g^{-1}[\nu_1^{-1}[X]], \ldots, g^{-1}[\nu_n^{-1}[X]]\} \setminus \emptyset \) is a partition of \( X \) into clopen sets whose cardinality gives the essential arity of \( g \). Consequently, the tight bound on the
essential arity of operations over $A$ is the number of connected components of $X$. This, in turn, is the integer $n$ such that there are precisely $2^n$ idempotent elements in $A$.

In Sec. 6.2, we will also use Lemma 51 to obtain some concrete results for clones over distributive lattices.

Now, as the high point of our discussion of essential variables of dual operations, let us note the following theorem.

**Theorem 53 ([12, 13]).** Let $C \leq O_A$ and assume that $X$ is finite. The following two statements are equivalent:

(i) there exists $k \in \mathbb{N}$ such that every $f \in C$ is essentially at most $k$-ary.

(ii) There exists $n \in \mathbb{N}^+$ such that $C^{\partial}$ respects the images of the injection morphisms to the degree $n$.

It follows that all clones over the object $A$ have bounded essential arity if $X$ is a finite object in one of the categories from Example 50. For instance, it follows immediately from the Priestley duality that each clone over a finite distributive lattice has bounded essential arity (a sharp bound of this arity will be determined in Sec. 6.2.1).

Stepping away from essential variables, one can also show that a dual operation that respects the images of the injection morphisms to the degree 1 is restricted with respect to the identities that it satisfies.

**Theorem 54.** Let $f \in O^{(n)}_A$ and let $h \in O^{(r)}_A$. Assume that $f^{\partial}$ respects the images of the injection morphisms to the degree 1 and that the following identity holds for $\varphi : n \rightarrow n$ and $\varphi' : r \rightarrow n$:

$$f \circ A^\varphi = h \circ A^\varphi'.$$

Then, the $t$th variable of $f$ is nonessential whenever $\varphi(t) \notin \varphi'[r]$.

**Proof.** Let $t \in \{1, \ldots, n\}$ such that $\varphi(t) \notin \varphi'[r]$. Without loss of generality, we can assume $t = 1$ and $\varphi(1) = 1$. We will prove the claim by showing that the identity

$$\varphi \cdot X \circ f^{\partial} = \varphi' \cdot X \circ h^{\partial}$$

implies that the first variable of $f^{\partial}$ is nonessential. For contradiction, assume that the first variable is essential. Lemma 51 yields $f^{\partial}[X] \notin \bigcup_{i=2}^n t_i^n[X]$, so there exists $x \in X$ such that $f^{\partial}(x) \notin t_i^n[X]$ for all $i \in \{2, \ldots, n\}$. Since $f^{\partial}[X] \subseteq \bigcup_{i=1}^n t_i^n[X]$, this implies that there exists $y \in X$ such that $f^{\partial}(x) = t_i^n(y)$. Hence,

$$(\varphi \cdot X \circ f^{\partial})(x) = (\varphi \cdot X \circ t_1^n)(y) = t_{\varphi(1)}^n(y) = t_1^n(y) = f^{\partial}(x) \notin t_i^n[X]$$
for all $i \in \{2, \ldots, n\}$. On the other hand, we can write $h^\delta(x)$ as $i_j^\delta(z)$ for some integer $j \in \{1, \ldots, r\}$ and some $z \in X$. Thus,

$$(\varphi' \cdot X \circ h^\delta)(x) = (\varphi' \cdot X \circ i_j^\delta)(z) = i_{\varphi'(j)}^\delta(z) \in i_{\varphi'(j)}^\delta[X].$$

Since $\varphi'(j) \in \varphi'[r] \subseteq \{2, \ldots, n\}$, we have successfully contradicted

$$\varphi \cdot X \circ g^\delta = \varphi' \cdot X \circ h^\delta$$

as the two sides of the equation do not coincide for $x$. \hfill \square

Note that this is a very strong condition on the identities that an operation can satisfy. For instance, the theorem implies that a nontrivial operation $f \in O_A$ cannot be a majority operation, a minority operation or a proper semiprojection whenever $f^\delta$ respects the images of the injection morphisms to the degree 1 (a remark to the reader familiar with minimal clones: Note that this narrows the possible classes of minimal clones down to the unary and binary case). However, such an operation can still be an idempotent operation, that is, it may hold $f \circ (\text{id}_A, \ldots, \text{id}_A) = \text{id}_A$. But now, an idempotent operation $f$ also has some interesting properties if $f^\delta$ respects the images of the injection morphisms to the degree 1.

To study this phenomenon, let us first recall that we can define the notion of a dual idempotent operation by saying that $g \in \overline{O}_X^{(n)}$ is idempotent whenever $[\text{id}_X, \ldots, \text{id}_X] \circ g = \text{id}_X$. For dual operations that respect the injection morphisms to the degree 1, we can characterize idempotency as follows.

**Theorem 55.** Assume that $g \in \overline{O}_X^{(n)}$ respects the images of the injection morphisms to the degree 1. The following two statements are equivalent:

(a) $g$ is idempotent.
(b) For each $x \in X$, $g(x)$ can be written as $i_i^\delta(x)$ for some $i \in \{1, \ldots, n\}$.

**Proof.** (a) $\Rightarrow$ (b). Let $x \in X$. Since $g$ respects the images of the injection morphisms to the degree 1, we can write $g(x)$ as $i_i^\delta(y)$ for some $i \in \{1, \ldots, n\}$ and $y \in X$. Moreover, we have $i_i^\delta = [i_i^\delta, \ldots, i_i^\delta] \circ g$ as $g$ is idempotent. Hence,

$$i_i^\delta(x) = ([i_i^\delta, \ldots, i_i^\delta] \circ g)(x) = ([i_i^\delta, \ldots, i_i^\delta] \circ i_i^\delta)(y) = i_i^\delta(y) = g(x).$$

(b) $\Rightarrow$ (a). For each $x \in X$, there exists $j_x \in \{1, \ldots, n\}$ such that $g(x) = i_{j_x}^\delta(x)$. Hence,

$$x = \text{id}_X(x) = ([\text{id}_X, \ldots, \text{id}_X] \circ i_{j_x}^\delta)(x) = ([\text{id}_X, \ldots, \text{id}_X] \circ g)(x). \hfill \square$$

With this theorem, we can establish a close connection between dual idempotent operations and partitions.

**Definition 56.** Let $X^\sharp$ be the set of all $x \in X$ such that $i_i^\delta(x) \neq i_j^\delta(x)$ for all $n \in \mathbb{N}_+$ and $i \neq j$.

It is an easy exercise to show that $x \notin X^\sharp$ is equivalent to $i_1^\delta(x) = i_2^\delta(x) = \cdots = i_n^\delta(x)$ for all $n \in \mathbb{N}_+$. 
Example 57.

(i) If $X$ is a set in the category of sets, then $X^2 = X$.

(ii) If $X = \langle X, 0, 1, \leq \rangle$ is a bounded poset in the category of bounded posets, then $X^2 = X \setminus \{0, 1\}$.

Definition 58. For a dual idempotent operation $g \in \mathcal{O}_X^{(n)}$ that respects the images of the injection morphisms to the degree 1, we denote by $\Pi(g)$ the partition of $X^2$ defined as follows:

$$\Pi(g) := \{X_1, \ldots, X_n\} \setminus \{\emptyset\}, \text{ where } x \in X_i : \iff g(x) = i^g(x).$$

Note that $\Pi(g)$ is well-defined since, for each $x \in X^2$, $g(x)$ is contained in exactly one of the sets $i^g_1[X], \ldots, i^g_n[X]$. To obtain more results for dual idempotent operations, let us denote by $\leq$ the finer-than order relation on partitions.

Lemma 59. Let $f, h \in O_X$ be idempotent operations such that $f^0$ and $h^0$ respect the images of the injection morphisms to the degree 1. We have $f \in \text{Clo}(h)$ if and only if $\Pi(f^0) \leq \Pi(h^0)$. Hence, $\text{Clo}(f) = \text{Clo}(h)$ if and only if $\Pi(f^0) = \Pi(h^0)$.

Proof. For notational simplicity, set $g := f^0$ and $g' := h^0$.

"$\Rightarrow$". Let $n$ be the arity of $g'$ and denote by $r(g)$ the minimal number of appearances of the operational symbol $g'$ needed to write $g$ as a superposition of $g'$ and the injection morphisms. We will show the claim by induction over $r(g)$.

For $r(g) = 0$, $g$ is an injection morphism, and we obtain $\Pi(g) = \{X^1\}$, which clearly implies $\Pi(g) \leq \Pi(g')$. Now suppose $r(g) > 0$. Then, $g$ can be written as $[g_1, \ldots, g_n] \circ g'$ where $r(g_1), \ldots, r(g_n) < r(g)$. By the induction hypothesis, we have $\Pi(g_1), \ldots, \Pi(g_n) \leq \Pi(g')$. But now, applying Theorem 55 evidently establishes $\Pi(g) \leq \Pi(g')$.

"$\Leftarrow$". Let $\Pi(g) = \{Y_1, \ldots, Y_p\}$ and $\Pi(g') = \{Z_1, \ldots, Z_q\}$ be partitions of $X^2$ into $p$ and $q$ parts, respectively, and let $\Pi(g) \leq \Pi(g')$. Without loss of generality, we can assume that all variables of $g$ and $g'$ are essential (that is, $g$ is $p$-ary and $g'$ is $q$-ary) and that there exist $i_1, \ldots, i_p \in \mathbb{N}_+$ such that $i_1 + \cdots + i_p = q$ and

$$Z_1 \cup \cdots \cup Z_{i_1} = Y_1,$$
$$Z_{i_1+1} \cup \cdots \cup Z_{i_1+i_2} = Y_2,$$
$$\vdots$$
$$Z_{(\sum_{j=1}^{p-1} i_j)+1} \cup \cdots \cup Z_{\sum_{j=1}^p i_j} = Y_p.$$
But now, in view of Theorem 55, we obtain
\[ g = [t^p_1, \ldots, t^p_1, t^p_2, \ldots, t^p_2, \ldots, t^p_n, \ldots, t^p_n] \circ g'. \]
Hence, \( g \in \text{Clo}(g') \) and thus \( f \in \text{Clo}(h) \). \( \square \)

With this lemma, it also follows that each clone of idempotent operations \( C \) is generated by a single operation whenever \( X \) is finite and \( C^0 \) respects the images of the injection morphisms to the degree 1.

**Lemma 60.** Assume that \( X \) is finite and let \( C \) be a clone of idempotent operations such that \( C^0 \) respects the images of the injection morphisms to the degree 1. Then, \( C \) is generated by a single operation.

**Proof.** Since each \( g \in C^0 \) respects the images of the injection morphisms to the degree 1 and \( X \) is finite, we can immediately infer from Theorem 53 that the essential arity of the dual operations among \( C \) is bounded by some integer \( k \in \mathbb{N} \).

Let \( (C^0)^{(k)} = \{g_1, \ldots, g_s\} \). For the next step of the proof, let us introduce the following notation for a \( k \)-ary \( g \in \mathcal{O}_X \):
\[ [g_1, \ldots, g_s] := [t^m_1 \circ g, t^m_{k+1} \circ g, \ldots, t^m_{l(m-1)k+1} \circ g, \ldots, t^m_{nk+1} \circ g]. \]

Now, we set
\[ g := [g_1, \ldots, g_s] \circ \cdots \circ [g_1, \ldots, g_s] \circ [g_1, \ldots, g_s] \circ g_1. \]
Note that \( g \) also respects the images of the injection morphisms to the degree 1 since \( g \in C^0 \). In view of Theorem 55, it is easy to see that \( \Pi(g_1), \ldots, \Pi(g_s) \leq \Pi(g) \).

But now, there exist \( f, f_1, \ldots, f_s \in O_\mathcal{A} \) such that \( f^0 = g \) and \( f^0_i = g_i \) for all \( i \in \{1, \ldots, s\} \), and Lemma 59 establishes \( f_1, \ldots, f_s \in \text{Clo}(f) \). Thus, \( C = \text{Clo}(f, \ldots, f_s) = \text{Clo}(f) \).

With the last two results, it is now rather easy to show that the lattice of subclones of the clone of all dual idempotent operations that respect the images of the injection morphisms to the degree 1 can be order-embedded into the partition-lattice \( \text{Part}(X^2), \leq \). Again, if \( X \) is some object in one of the categories mentioned in Example 50, then this implies that the lattice of subclones of the clone of all idempotent operations over \( \mathcal{A} \) can be order-embedded into the aforementioned partition lattice. For instance, this is true for clones over finite distributive lattices, where we will further describe this phenomenon in Sec. 6.2.2. Moreover, for the reader familiar with minimal clones, let us extend our remark below Theorem 54: If \( \mathcal{O}_X \) respects the images of the injection morphisms to the degree 1, then our observations immediately imply that the minimal clones among \( \mathcal{L}_\mathcal{A} \) are precisely all unary minimal clones (which are completely characterized by Rosenberg’s classification theorem [27]) and all clones generated by nontrivial binary idempotent operations.
6.2. Clones over distributive lattices

In the last section, we have obtained some rather general results. Let us now try to obtain some more specific and detailed results by focusing on one particular example: clones over distributive lattices. Until the end of this section, let \( \mathcal{A} \) always be the category of finite distributive lattices, let \( \mathcal{X} \) be the category of finite bounded posets, let \( A \) be a finite distributive lattice with at least two elements, and let \( X \) be the dual of \( A \) under the Priestley duality \( \langle D, E, e, \epsilon \rangle \) between \( \mathcal{A} \) and \( \mathcal{X} \). In Example 45, we have already outlined how to determine the clone duality \( (-)^{\partial} : \mathcal{O}_A \rightarrow \mathcal{O}_X \) that we will use in this section. Recall that we are also equipped with the generalized Galois connection \( \text{Pol}_X \rightarrow \text{Inv}_X \) whenever we are dealing with the dual operations over \( X \). In fact, when it comes to the last theorem of this section we will take advantage of this tool and apply it to prove the desired claim.

During our investigation, the distributive lattice and its dual from Fig. 2 will serve as a running example.

Before we start attacking specific problems, let us introduce the following helpful definition.

**Definition 61.** Let \( Y = \langle Y, 0^Y, 1^Y, \leq^Y \rangle \in \mathcal{X} \). Define \( G_Y \) to be the undirected graph whose set of vertices is \( Y \setminus \{0^Y, 1^Y\} \) and in which two vertices \( y_1 \) and \( y_2 \) are connected by an edge if and only if \( y_1 \leq^Y y_2 \) or \( y_2 \leq^Y y_1 \). A subset \( Z \subseteq Y \setminus \{0^Y, 1^Y\} \) is said to be **connected** if there exists a path in \( Z \) between each two elements of the set. For \( y \in Y \setminus \{0^Y, 1^Y\} \), denote by \( \langle y \rangle_{G_Y} \) the largest connected subset of \( Y \setminus \{0^Y, 1^Y\} \) that contains \( y \). Then,

\[
\text{Con}(Y) := \{ \langle y \rangle_{G_Y} \mid y \in G_Y \}
\]

is called the set of **connected components** of \( Y \).

![Fig. 2. A distributive lattice and its dual.](image-url)
Example 62. Consider the poset $X$ given in Fig. 2. The set $\{x_1, x_2, x_3\}$ is connected, whereas $\{x_1, x_2\}$ is not. Moreover, the connected components of $X$ are the three sets $\{x_1, x_2, x_3\}$, $\{y\}$ and $\{z_1, z_2\}$.

Note that, for $n \in \mathbb{N}_+$, the connected components of $n \cdot X$ are precisely the images of the connected components of $X$ under the injection morphisms. That is,

$$\text{Con}(n \cdot X) = \{\iota^n_i[Y] \mid Y \in \text{Con}(X), i \in \{1, \ldots, n\}\}.$$ 

In particular, two elements $x_1, x_2 \in n \cdot X \setminus \{0^n \cdot X, 1^n \cdot X\}$ necessarily belong to two different connected components of $n \cdot X$ whenever they belong to different sets among $\iota^n_1[X], \ldots, \iota^n_n[X]$.

6.2.1. Essential variables

We have already observed that the essential arity of operations among $O_A$ is bounded (see the remark after Theorem 53. Let us now use the clone duality $(-)^O$ to obtain the sharp bound.

Proposition 63. The cardinality of the longest antichain of join-irreducible elements of $A$ is a sharp bound on the essential arity of operations among $O_A$.

Proof. Let $k$ be the cardinality of the longest antichain of join-irreducible elements of $A$. It suffices to show that each dual operation in $\overline{O}_X$ has at most $k$ essential variables and that there exists some $g^* \in \overline{O}_X$ with exactly $k$ essential variables. To do so, let us first note that $a_1, \ldots, a_k$ is an antichain of join-irreducible elements in $A$ if and only if $\uparrow a_1, \ldots, \uparrow a_k$ is an antichain in $X$ (where $\uparrow a$ denotes the order-filter generated by $a$, that is, $\uparrow a = \{b \in A \mid b \geq a\}$). Therefore, $k$ is also the length of the longest antichain in $X$. Let us assume that there exists some $g \in \overline{O}_X$ with at least $k + 1$ essential variables. The dual operations over $X$ respect the images of the injection morphisms to the degree 1. Hence, we can apply Lemma 51, and it follows that the $i$th variable of $g$ is nonessential if and only if $g[X] \cap \iota^n_i[X] = \{0^n \cdot X, 1^n \cdot X\}$. Thus, there exist $x_1, \ldots, x_{k+1} \in g[X] \setminus \{0^n \cdot X, 1^n \cdot X\}$ such that $g(x_1), \ldots, g(x_{k+1})$ pairwise belong to different connected components of $n \cdot X$. Since this implies that $g(x_1), \ldots, g(x_{k+1})$ is an antichain in $n \cdot X$, it follows that $x_1, \ldots, x_{k+1}$ is an antichain in $X$. This contradicts our assumption that an antichain in $X$ can contain at most $k$ elements.

For the second part, let $x_1, \ldots, x_k$ be an antichain in $X$ and define $g^* \in \overline{O}_X^{(k)}$ as follows:

$$g^*(x) := \begin{cases} 
0^n \cdot X & \text{if } x < x_i \text{ for some } i \in \{1, \ldots, k\}, \\
1^n \cdot X & \text{if } x > x_i \text{ for some } i \in \{1, \ldots, k\}, \\
\langle i, x \rangle & \text{if } x = x_i.
\end{cases}$$
We have to show that $g^*$ is well-defined. For $x \in X$, we must have $x \leq x_i$ or $x \geq x_i$ for at least one $i \in \{1, \ldots, k\}$ since otherwise we would get an antichain of length $k + 1$. Hence, at least one of the three cases in the definition of $g^*$ is satisfied. Moreover, it is straightforward to show that two of the three cases cannot hold simultaneously (for instance, if we have $x < x_i$ and $x > x_j$ for some $i, j \in \{1, \ldots, k\}$, then we obtain $x_j < x < x_i$, which contradicts that $x_1, \ldots, x_k$ is an antichain). Thus, $g^*$ is a well-defined mapping from $X$ to $k \cdot X$. Moreover, we evidently have $g^*(0_X) = 0_{k \cdot X}$, $g^*(1_X) = 1_{k \cdot X}$ and $x \leq y$ implies $g^*(x) \leq g^*(y)$. Thus, $g^* \in O_{k_X}$.

This finishes the proof since it is obvious that $g^*$ has no nonessential variables. 

With very similar methods (see [12] for details), one can also show the following result.

**Proposition 64 ([12]).** Let $O_{A_0}$, $O_{A_1}$ and $O_{A_{01}}$ be the clone of all 0-homomorphisms, 1-homomorphisms and 01-homomorphisms among $O_A$, respectively. Then, a sharp bound on the essential arity of operations

- among $O_{A_0}$ is given by the number of atoms of $A$,
- among $O_{A_1}$ is given by the number of coatoms of $A$,
- among $O_{A_{01}}$ is given by $|\text{Con}(X)|$, that is, the greatest integer $n$ such that there exist pairwise distinct elements $a_1, \ldots, a_n \in A \setminus \{0\}$ with $\bigvee a_i = 1$ and $a_i \land a_j = 0$ for $i \neq j$.

**Example 65.** For the distributive lattice from Fig. 2, the longest antichain of join-irreducible elements contains four elements, $A$ has three atoms, four coatoms, and we have $|\text{Con}(X)| = 3$. Thus, we obtain that 4, 3, 4 and 3 are sharp bounds on the essential arity of the operations among $O_A$, $O_{A_0}$, $O_{A_1}$ and $O_{A_{01}}$, respectively.

### 6.2.2. Idempotent operations

Let us denote by $I_A$ the clone of all idempotent operations over $A$ and by $I_X$ the clone of all dual idempotent operations over $X$. Recall that $(I_A)^0 = I_X$.

In Sec. 6.1, we have already noted some general results about dual idempotent operations that respect the images of the injection morphisms to the degree 1. Since this applies to all dual operations over $X$, we can build on these results to collect some facts about the idempotent operations over $A$ and to explore $(I_A)$.

First, let us use Theorem 55 to obtain the following characterization of the dual idempotent operations over $X$.

**Lemma 66.** Let $X_1, \ldots, X_k$ be the connected components of $X$. An $n$-ary dual operation $g \in O_X$ is idempotent if and only if there exists a (unique) function $j_g : \{1, \ldots, k\} \rightarrow \{1, \ldots, n\}$
such that

\[ g(x) = \begin{cases} 
0^n X & \text{if } x = 0^X, \\
1^n X & \text{if } x = 1^X, \\
\langle j(g(i), x) \rangle & \text{if } x \in X_i.
\end{cases} \]

**Proof.** "\( \Rightarrow \)." The idempotency of \( g \) follows immediately from Theorem 55.

"\( \Rightarrow \)." By Theorem 55, there must exist a function \( j : X \to \{1, \ldots, n\} \) such that \( g(x) = \iota_{ji(x)}^n(x) \) for all \( x \in X \). This implies

\[ g(x) = \begin{cases} 
0^n X & \text{if } x = 0^X, \\
1^n X & \text{if } x = 1^X, \\
\langle j(x), x \rangle & \text{otherwise}.
\end{cases} \]

It remains to show that we have \( j(x) = j(y) \) whenever \( x \) and \( y \) belong to the same connected component of \( X \).

Let \( x, y \in X_i \) for some \( i \in \{1, \ldots, k\} \). By the definition of connected components, there must exist \( z \in X_i \) such that we either have \( x, y \geq z \) or \( z \geq x, y \). Without loss of generality, we assume the latter case. It follows \( g(z) \geq g(x) \) and \( g(z) \geq g(y) \). As \( g(z) \neq 1^n X \), this implies that \( g(x) \) and \( g(y) \) belong to the same connected component. Thus, \( j(x) = j(y) \).

By the results from Sec. 6.1, we know that \( \langle I_A \rangle \) is isomorphic to some sublattice of \( \langle \text{Part}(X \setminus \{0^X, 1^X\}), \preceq \rangle \) (see the discussion at the end of Sec. 6.1. Let us now try to characterize this sublattice more precisely.

**Definition 67.** Let \( f \in O_A^{(n)} \) be idempotent. For \( i \in \{1, \ldots, n\} \), set

\[ P_i := \{ Y \in \text{Con}(X) | f^0[Y] \subseteq \iota_i^n[X] \} \]

and \( \Pi(f) := \{ P_1, \ldots, P_n \} \setminus \{\emptyset\} \).

Note that \( \Pi(f) \) is well-defined since, by Lemma 66, the image of each \( Y \in \text{Con}(X) \) under \( f^0 \) can only be contained in one of the sets \( \iota_1^n[X], \ldots, \iota_n^n[X] \). Moreover, it is easy to show that, for each partition \( \{P_1, \ldots, P_n\} \) of \( \text{Con}(X) \), there exists an idempotent operation \( f \in O_A^{(n)} \) such that \( \Pi(f) = \{P_1, \ldots, P_n\} \).

**Theorem 68.** The ideal \( \langle I_A \rangle \) is isomorphic to the lattice \( \langle \text{Part}(\text{Con}(X)), \preceq \rangle \) of partitions of \( \text{Con}(X) \).

**Proof.** Let \( \Phi \) be a partition of \( \text{Con}(X) \). As noted above, there exists an operation \( f_\Phi \) such that \( \Pi(f_\Phi) = \Phi \). This operation is not unique, but in view of Lemma 66 it is evident that all operations with this property arise from each other by a suitable permutation of variables. Hence, they generate the same clone and the mapping

\[ \varphi : \text{Part}(\text{Con}(X)) \to \mathcal{L}_A : \Phi \mapsto \text{Clo}(f_\Phi) \]
is well-defined. Furthermore, by arguments analogous to those in the proof of Lemma 59, one can show that we have \( f \in \text{Clo}(h) \) if and only if \( \Pi(f) \preceq \Pi(h) \). Thus, \( \varphi \) is an order embedding from \( \langle \text{Part} (\text{Con}(X)), \preceq \rangle \) to \( \langle I_A \rangle \). It remains to show that \( \varphi \) is bijective. Define the mapping \( \varphi' : L_A \rightarrow \text{Part}(\text{Con}(X)) \) by setting \( \varphi'(C) = \Pi(f) \) where \( f \) is one of the idempotent operations that generate \( C \) (the existence of such an operation is given by Lemma 60). Evidently, \( \varphi' \) is the inverse of \( \varphi \), and the claim follows.

Note that this theorem also implies that the number of essentially different idempotent operations over \( A \) and the number of idempotent clones over \( A \) are both the Bell number \( B_{|\text{Con}(X)|} \).

**Example 69.** Once again, let us turn to our running example from p. 492. We have \(|\text{Con}(X)| = 3\), so \( \langle I_A \rangle \) is isomorphic to \( \langle \text{Part} (\{1, 2, 3\}), \preceq \rangle \) and there are exactly five clones of idempotent operations in \( L_A \).

For a given \( n \in \mathbb{N}_+ \), we can also easily infer the number of essentially different, essentially \( n \)-ary idempotent operations over \( A \). For this, denote by \( S^n_l \) the number of partitions of an \( l \)-element set into \( n \) non-empty subsets (the so-called Stirling number of the second kind).

**Corollary 70.** Let \( l := |\text{Con}(X)| \). The number of essentially different, essentially \( n \)-ary idempotent operations over \( A \) is \( S^n_l \).

**Proof.** As presented above, \( f \) and \( g \) generate a distinct clone if and only if \( \Pi(f) \neq \Pi(g) \). Thus, the number of essentially different, essentially \( n \)-ary idempotent operations over \( A \) is the number of possible partitions of \( \text{Con}(X) \) into \( n \) parts, i.e. the number is \( S^n_l \).

As a curiosity, we will now look at the clone generated by the union of \( I_A \) and \( \text{End} A \). That is, we look at the least clone that contains all unary and all idempotent operations over \( A \). In the lattice of clones over sets, this clone is the full clone. One way to see this is to apply the Shupecki criterion [28]. The same is true for clones over Boolean algebras [17] and even Boolean lattices [12]. However, in the lattice of clones over distributive lattices, this is only true in certain cases. To characterize these cases, we have to introduce some more notation.

**Definition 71.** For \( Y \in \text{Con}(X) \), denote by \( \text{Spl}(Y) \) the set of pairs

\[ (Y_1, Y_2) \in (\mathfrak{P}(Y)\backslash\{\emptyset\}) \times (\mathfrak{P}(Y)\backslash\{\emptyset\}), \]

such that each of the two sets \( Y_1 \) and \( Y_2 \) is connected (see Definition 61) and we have \( y_1 \not\leq y_2 \) and \( y_2 \not\leq y_1 \) for all \( y_1 \in Y_1, y_2 \in Y_2 \).
Example 72. Let $X$ be the bounded poset illustrated by the following diagram:

$$\begin{align*}
\text{Spl}(\{x_0, \ldots, x_4\}) &= \{\{x_1\}, \{x_3\}\}, \\
& \quad \{\{x_3\}, \{x_1\}\}, \\
& \quad \{\{x_2\}, \{x_3\}\}, \\
& \quad \{\{x_3\}, \{x_2\}\}, \\
& \quad \{\{x_1, x_2\}, \{x_3\}\}, \\
& \quad \{\{x_3\}, \{x_1, x_2\}\}\}.
\end{align*}$$

Clearly, Spl($Y$) is a symmetric relation. The notation Spl($Y$) is due to the fact that $(Y_1,Y_2) \in \text{Spl}(Y)$ indicates that $Y_1$ and $Y_2$ can be split into different connected components via some dual operation $g \in \overline{O}_X$. This is shown in the next lemma.

Lemma 73. Let $Y \in \text{Con}(X)$. For two connected subsets $Y_1,Y_2 \subseteq Y$, we have $(Y_1,Y_2) \in \text{Spl}(Y)$ if and only if there exists a dual operation $g \in \overline{O}_X^{(n)}$ such that $g[Y_1] \subseteq Z_1$ and $g[Y_2] \subseteq Z_2$ for some $Z_1,Z_2 \subseteq \text{Con}(n \cdot X)$, $Z_1 \neq Z_2$.

Proof. Let $(Y_1,Y_2) \in \text{Spl}(Y)$. We define the following mapping:

$$g(x) := \begin{cases}
0^2X & \text{if } x < y \text{ for some } y \in Y_1 \cup Y_2, \\
1^2X & \text{if } x > y \text{ for some } y \in Y_1 \cup Y_2, \\
\langle 1, x \rangle & \text{if } x \in Y_1, \\
\langle 2, x \rangle & \text{if } x \in Y_2, \\
\langle 1, x \rangle & \text{otherwise.}
\end{cases}$$

Since $y_1 \not< y_2$ and $y_2 \not< y_1$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$, $g$ is a well-defined binary dual operation over $X$. Conversely, let $g \in \overline{O}_X^{(n)}$ be an operation that maps $Y_1$ and $Y_2$ into two different connected components of $n \cdot X$. Then, clearly, for $y_1 \in Y_1$ and $y_2 \in Y_2$, we have $g(y_1) \not< g(y_2)$ and $g(y_2) \not< g(y_1)$. Since $g \in \overline{O}_X$, this implies $y_1 \not< y_2$ and $y_2 \not< y_1$. Thus, $(Y_1,Y_2) \in \text{Spl}(Y)$.

With this notation, we can now formulate the desired characterization. Since the proof of the theorem is a little bit too long to present it entirely, we only prove the first direction as it gives a nice application of how to use dual relations in a
proof. For the other direction (which is rather technical and far from exciting), we refer to [12].

**Theorem 74.** The following two statements are equivalent:

(a) \( \text{Clo}(\mathcal{I}_A \cup \text{End} \mathbf{A}) = O_A \).

(b) For each \( Y \in \text{Con}(\mathbf{X}) \) and \( (Y_1, Y_2) \in \text{Spl}(Y) \) there exists \( Y' \in \text{Con}(\mathbf{X}) \setminus \{Y\} \) such that \( Y_1 \) or \( Y_2 \) can be order-embedded into \( Y' \).

**Proof.** (a) \( \Rightarrow \) (b). Assume that there exist \( Y \in \text{Con}(\mathbf{X}) \) and \( (Y_1, Y_2) \in \text{Spl}(Y) \) such that neither \( Y_1 \) nor \( Y_2 \) can be order-embedded into any \( Y' \in \text{Con}(\mathbf{X}) \setminus \{Y\} \). We will show \( \overline{\text{Clo}(\mathcal{I}_X \cup \text{End} \mathbf{X})} \neq \overline{\mathcal{O}_X} \). For \( (Z_1, Z_2) \in \text{Spl}(Y) \), we write \( (Z_1, Z_2) \sim (Y_1, Y_2) \) to indicate that \( Z_i \) and \( Y_i \) are order-isomorphic for \( i \in \{1, 2\} \). Define \( g \in \overline{\mathcal{O}_X(2)} \) as follows:

\[
g(x) := \begin{cases} 
0^2 \mathbf{X} & \text{if } x < y \text{ for some } y \in Y_1 \cup Y_2, \\
1^2 \mathbf{X} & \text{if } x > y \text{ for some } y \in Y_1 \cup Y_2, \\
(1, x) & \text{if } x \in Y_1, \\
(2, x) & \text{if } x \in Y_2, \\
(1, x) & \text{otherwise}.
\end{cases}
\]

Since \( (Y_1, Y_2) \in \text{Spl}(Y) \), \( g \) is well-defined (see Lemma 73). We will show

\[ g \notin \overline{\text{Clo}(\mathcal{I}_X \cup \text{End} \mathbf{X})} \]

by defining a dual relation \( \sigma \) of type \( 2 \cdot \mathbf{X} \) such that \( \mathcal{I}_X \cup \text{End} \mathbf{X} \not\supseteq \sigma \), whereas \( g \not\supseteq \sigma \). Let \( \sigma \) contain all morphisms \( r : \mathbf{X} \to 2 \cdot \mathbf{X} \) such that, for all \( (Z_1, Z_2) \in \text{Spl}(Y) \) with \( (Z_1, Z_2) \sim (Y_1, Y_2) \), \( r \) satisfies one of the following two cases:

(i) it does not map \( Z_1 \) and \( Z_2 \) into different connected components,

(ii) it is not order-reflecting on \( Z_1 \) or \( Z_2 \).

Let \( r_1, \ldots, r_n \in \sigma \) and let \( h \in \mathcal{I}_X \cup \text{End} \mathbf{X} \). If \( h \) happens to be idempotent, then \( [r_1, \ldots, r_n] \circ h \in \sigma \) is immediate since Lemma 66 implies that there exists \( i \in \{1, \ldots, n\} \) such that

\[
([r_1, \ldots, r_n] \circ h)[Y] = ([r_1, \ldots, r_n] \circ \iota_i^n)[Y] = r_i[Y].
\]

Assume that \( h \) is unary. Consequently, we have \( n = 1 \) and it remains to show that \( r_1 \circ h \in \sigma \). Let \( (Z_1, Z_2) \in \text{Spl}(Y) \) with \( (Z_1, Z_2) \sim (Y_1, Y_2) \). Since, by assumption, \( Z_1 \) and \( Z_2 \) both cannot be order-embedded into different connected components of \( \mathbf{X} \), it follows that \( h \) satisfies (i) or (ii). If \( h \) satisfies (ii), then so does \( r_1 \circ h \) and we obtain \( r_1 \circ h \in \sigma \). Hence, it only remains the case in which \( h \) satisfies (i) and is order-reflecting on \( Z_1 \) and \( Z_2 \). Since neither \( Z_1 \) nor \( Z_2 \) can be order-embedded into a connected component different from \( Y \), this implies that we have \( h[Z_1], h[Z_2] \subseteq Y \) or there exists \( z \in Z_1 \cup Z_2 \) such that \( h(z) \in \{0^X, 1^X\} \). In the latter case, we have \( (r_1 \circ h)(z) \in \{0^2 \mathbf{X}, 1^2 \mathbf{X}\} \), and it follows that \( r_1 \circ h \) satisfies (i). Hence, let us assume
If \( (h[Z_1], h[Z_2]) \notin \text{Spl}(Y) \), then it follows by Lemma 73 that \( h[Z_1] \) and \( h[Z_2] \) cannot be mapped into different connected components, whence we have \( r_1 \circ h \in \sigma \). If \( (h[Z_1], h[Z_2]) \in \text{Spl}(Y) \), then the assumption that \( h \) is order-reflecting implies \( (h[Z_1], h[Z_2]) \sim (Z_1, Z_2) \sim (Y_1, Y_2) \), and the claim follows since \( r_1 \) must satisfy (i) or (ii) for \( (h[Z_1], h[Z_2]) \). Thus it turns out that, \( \mathcal{I}_X \cup \text{End}X \nsubseteq \sigma \).

However, since \( g \) does evidently not satisfy (i) or (ii) for \( (Y_1, Y_2) \), we obtain \[ [i_1^2, i_2^2] \circ g = g \notin \sigma, \]
and this means that \( g \) does not preserve \( \sigma \).

\( (b) \Rightarrow (a) \). See [12]. \( \square \)

Example 75.

(i) Let \( A \) be our running example from Fig. 2. It is easy to see that condition (2) from the theorem is satisfied. Thus, \( \text{Clo}(\mathcal{I}_A \cup \text{End}A) = O_A \).

(ii) Let \( A \) be a distributive lattices that dualizes to the poset \( X \) from Example 72. For \( (\{x_1, x_2\}, \{x_3\}) \in \text{Spl}(\{x_0, \ldots, x_4\}) \), there exists no \( Y' \in \text{Con}(X) \setminus \{Y\} \) such that \( \{x_1, x_2\} \) or \( \{x_3\} \) can be order-embedded into \( Y' \). Thus it turns out that, \( \text{Clo}(\mathcal{I}_A \cup \text{End}A) \) is not the full clone.

(iii) Let \( A \) be a distributive lattice with exactly one atom or one coatom. Since this implies that \( X \) has only one connected component, condition (2) of the theorem is satisfied if and only if \( X \) is totally ordered, which, in turn, is equivalent to \( A \) being totally ordered. Thus, \( \text{Clo}(\mathcal{I}_A \cup \text{End}A) = O_A \) if and only if \( A \) is a chain.

6.2.3. Minimal clones

The determination of the minimal clones over a given distributive lattice is a rather simple task. It is well-known that these clones are precisely all unary minimal clones and all nontrivial binary idempotent clones, and it also follows from our results in Sec. 6.1 (see the concluding remark on p. 492). A much more subtle task is to solve the problem of whether the join (with respect to the clone lattice) of all minimal clones over \( A \) is the full clone \( O_A \). While the question has a positive answer for clones over sets (in fact, it was shown in [5] that the join of two minimal clones is enough), the answer for distributive lattices is: it depends on the form of the lattice. A characterization can be given by using the clone duality.

**Proposition 76.** The join of all minimal clones over \( A \) is the full clone if and only if the following two conditions are met:

(i) For each \( Y \in \text{Con}(X) \) and \( (Y_1, Y_2) \in \text{Spl}(Y) \) there exists \( Y' \in \text{Con}(X) \setminus \{Y\} \) such that \( Y_1 \) or \( Y_2 \) can be order-embedded into \( Y' \).

(ii) The join of all minimal clones contains \( \text{End}A \).
Proof. “⇒”. It is obvious that (ii) must hold. Since all minimal clones are generated by unary operations or binary idempotent operations, we must have $\text{Clo}(\mathcal{I}_A \cup \text{End} \ A) = O_A$. By Theorem 74, this is equivalent to (i).

“⇐”. By (ii), the join of minimal clones contains End $A$. Moreover, it follows directly from Theorem 68 that $\mathcal{I}_A^{(2)}$ generates $\mathcal{I}_A$. Since we have already noted that each operation in $\mathcal{I}_A^{(2)}$ is either minimal or trivial, we can infer that the join of all minimal clones contains $\mathcal{I}_A$. Thus, the join of all minimal clones contains $\text{Clo}(\mathcal{I}_A \cup \text{End} \ A)$. But now, by (i) and Theorem 74, the latter is the full clone. □

7. Remarks and Conclusion

In this paper, we introduced a general duality theory for clones (both operational and relational) and illustrated some applications of the theory. We gave examples of how clone dualities can be used to obtain some rather general results, and we have applied the theory on clones over distributive lattices to exemplify how it can help us to solve specific problems. We have also hinted at the provided possibility to transfer clone-theoretic problems into a framework very different from that of universal algebra. For instance, we have seen in Example 52 that topological arguments can be consulted, and it is shown in [14] that certain clones can be investigated in the framework of formal concept analysis [7].

In our approach, we did not consider nullary operations. We made this decision because clone theory is usually, with only very few exceptions, pursued without constants. However, it should be noted that the theory as presented here can be modified accordingly by including $\mathcal{J}(A^0, A)$ into the definition of $O_A$, requiring, of course, that the terminal object $A^0$ exists in $\mathcal{A}$. Note that this would make our notion of a clone more similar to the standard definition of a Lawvere theory. Similarly, to keep the duality theory, we had to include $X(X, 0 \cdot X)$ into $\mathcal{O}_X$, which evidently requires $X$ to contain an initial object. We will not elaborate the consequences of this change, but it should be noted that the theory outlined in this paper would stay essentially the same. However, some minor adjustments would be necessary. For instance, the empty relation would not necessarily be preserved by a given set of operations. Hence, condition (i) had to be removed from the definition of a clone of relations (see Definition 9), and the smallest relation (of any given type) preserved by a set of operations would not necessarily be the empty relation.

To conclude the paper, let us put some emphasis on the obvious fact that every clone duality is only as good as the underlying dual equivalence. For our purposes, dual equivalences for categories of relational structures are particularly desirable as every clone on a finite set can be written as the full clone over some relational structure (see Theorem 47). Unluckily for us, however, nice dual equivalences for categories of relational structures are very rare. Partly because several obstacles make dualities for relational structures more difficult to find, and partly because the interest in finding dualities for other classes (such as algebras or topological structures) seemed to be higher. In this regard, one might also see this paper as a
motivation for research in this direction. Indeed, every step towards a richer duality theory for relational structures provides us with a new opportunity to obtain more results for clones over sets by using the machinery introduced in this paper.

References


