Propagation of S-wave in a non-homogeneous anisotropic incompressible and initially stressed medium under influence of gravity field

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A R T I C L E  I N F O

Keywords:
Incompressible
Initial stress
Anisotropic
S-wave
Gravity field

A B S T R A C T

In this paper, propagation of shear waves in a non-homogeneous anisotropic incompressible, gravity field and initially stressed medium is studied. Analytical analysis reveals that the velocity of propagation of the shear waves depends upon the direction of propagation, the anisotropy, gravity field, non-homogeneity of the medium, and the initial stress. The frequency equation that determines the velocity of the shear wave has been obtained. The dispersion equations have been obtained and investigated for different cases. A comparison is made with the results predicted by Abd-Alla et al. [22] in the absence of initial stress and gravity field. The results obtained are discussed and presented graphically.

1. Introduction

In recent years, more attention has been given to using the anisotropic material in engineering applications in considerable research activity. The problem of shear waves in an orthotropic elastic medium is very important for the possibility of its extensive application in various branches of science and technology, particularly in optics, earthquake science, acoustics, geophysics and plasma physics.


Bouden and Datta [14] investigated Rayleigh waves in a granular medium over an initially stressed elastic half-space. Influence of gravity on propagation of waves in composite layer has been illustrated by Bhattacharya and Sengupta [15].

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0096-3003/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.amc.2010.10.029
Mahmoud [20] studied the effect of the non-homogeneity on wave propagation on orthotropic elastic media. Abd-Alla and Sengupta [17], investigated many problems of elastic waves and vibration under the influence of gravity field. Surface waves under the influence of gravity in a homogeneous medium were considered by De [18] and De and Sengupta [19]. Acharya and Sengupta [16] discussed the influence of gravity field on the propagation of waves in a thermoelastic layer. De et al. [18] and De and Sengupta [19] studied the effect of the non-homogeneity on wave propagation on orthotropic elastic media. Abd-Alla and Abo-Dahab [21] studied the time-harmonic sources in a generalized magneto-thermo-viscoelastic continuum with and without energy dissipation.

In this work, the effect of gravity field on the propagation of S-wave in a non-homogeneous anisotropic incompressible and initially stressed medium has been discussed using the wave equations which satisfied by the displacement potentials \( \phi \). The frequency equations that determines the velocity of shear wave have been obtained. The dispersion equations have been obtained, and investigated for different cases. Numerical results are presented for the variation of velocity of shear waves with respect to depth \( b \) and angle \( \theta \). Numerical results presented graphically. The effect of non-homogeneous, initial stress and gravity field are very pronounced.

2. Formulation of the problem

Most materials behave as incompressible media and the velocities of longitudinal waves in them are very high. The varieties of hard rocks present in the earth are also almost incompressible due to factors like external pressure, slow process of creep, difference in temperature, manufacturing processes, nitriding, pointing etc., the medium stays under high stresses. These stresses are regarded as initial stresses. Owing to the variation of elastic properties and the presence of these initial stresses, the medium becomes isotropic as well.

We consider an unbounded incompressible anisotropic medium under initial stresses \( s_{11} \) and \( s_{22} \) along the \( x \)-, \( y \)-directions, respectively. When the medium is slightly disturbed \((u, v)\), the incremental stresses \( s_{11}, s_{12} \) and \( s_{22} \) are developed, and the equations of motion in incremental state become [4]

\[
\begin{align*}
\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - p \frac{\partial u}{\partial y} - \rho g \frac{\partial v}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}, \\
\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + p \frac{\partial u}{\partial x} + \rho g \frac{\partial v}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2},
\end{align*}
\]

(1)

where, \( p = s_{22} - s_{11}, \ w = \frac{1}{2} \left( \frac{\partial s_{12}}{\partial x} - \frac{\partial s_{12}}{\partial y} \right) \), and \( \rho \) represents the density of the medium, also, \( s_{ij} \) are incremental stresses, \((u, v)\) are incremental deformation, \( w \) is the rotational component about the \( z \)-axis, and \( g \) is the acceleration due to gravity.

The incremental stress–strain relations for an incompressible medium may be taken as [4]

\[
\begin{align*}
s_{11} &= 2Ne_{xx} + s, \quad s_{22} = 2Ne_{yy} + s \quad \text{and} \quad s_{12} = 2Qe_{xy},
\end{align*}
\]

(3)

where, \( s = \frac{s_{11}+s_{22}}{2}, \ e_{ij} \) are incremental strain components, and \( N \) and \( Q \) are rigidities of medium.

The incompressibility condition \( e_{xx} + e_{yy} = 0 \) is satisfied by

\[
u = \frac{\partial \varphi}{\partial y} \quad \text{and} \quad \nu = \frac{\partial \varphi}{\partial x}.
\]

(4)

Substituting from Eqs. (3) and (4) into Eqs. (1) and (2), we get

\[
\begin{align*}
\frac{\partial s}{\partial x} - 2N \frac{\partial^2 \varphi}{\partial x^2} &= \frac{\partial}{\partial y} \left[ Q \left( \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial x \partial y} \right) \right] - \frac{p}{2} \left( \frac{\partial^3 \varphi}{\partial x^3} + \frac{\partial^3 \varphi}{\partial x^2 \partial y} \right) = \rho \left( \frac{\partial^2 \varphi}{\partial x^2} \frac{\partial^3 \varphi}{\partial t^2 \partial y} \right),
\end{align*}
\]

(5)

\[
\begin{align*}
\frac{\partial s}{\partial y} + Q \left( \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial x^2} \right) &= \frac{\partial}{\partial y} \left( 2N \frac{\partial^2 \varphi}{\partial x \partial y} \right) - \frac{p}{2} \left( \frac{\partial^3 \varphi}{\partial x \partial y^2} + \frac{\partial^3 \varphi}{\partial x^2 \partial y} \right) = \rho \left( \frac{\partial^2 \varphi}{\partial x \partial y} \frac{\partial^3 \varphi}{\partial t^2 \partial y} \right).
\end{align*}
\]

(6)

Assuming non-homogeneities as

\[
\begin{align*}
Q &= Q_0 (1 + ay), \\
N &= N_0 (1 + by), \\
\rho &= \rho_0 (1 + cy),
\end{align*}
\]

(7)

where, \( N_0 \) and \( Q_0 \) are rigidities, and \( \rho_0 \) is the density of the medium at the surface \( (y = 0) \). Substituting from (7) into Eqs. (5) and (6) we get

\[
\begin{align*}
&\left[ Q_0 (1 + ay) - \frac{p}{2} \frac{\partial^3 \varphi}{\partial x^3} + \left| 4N_0 (1 + by) - 2Q_0 (1 + ay) \right| \frac{\partial^3 \varphi}{\partial x^3 \partial y} + \left| 4N_0 b - 2aQ_0 \right| \frac{\partial^3 \varphi}{\partial x^2 \partial y^2} + \frac{N_0 b + 2aQ_0}{2} \frac{\partial^3 \varphi}{\partial x \partial y^3} + \frac{p}{2} \frac{\partial^3 \varphi}{\partial y^4} \\
&+ 2aQ_0 \frac{\partial^3 \varphi}{\partial y^3} = \rho_0 (1 + cy) \left[ \frac{\partial^4 \varphi}{\partial x \partial y^3} + \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} \right] - \rho_0 c \left[ \frac{\partial^4 \varphi}{\partial x \partial y^3} \frac{\partial^3 \varphi}{\partial t^2 \partial y} \right].
\end{align*}
\]

(8)
3. Solution of problem

For propagation of sinusoidal waves in any arbitrary direction, assuming harmonic time variation of Eq. (8) as
\[ \phi(x, y, t) = Ae^{ik(x\cos\alpha + y\sin\alpha - ct)} \]
where, \( p_1 \) and \( p_2 \) are cosines of angles made by direction of propagation with \( x \)- and \( y \)-axes, and \( c_1 \) and \( k \) are the velocity of propagation and the wave number, respectively.

Using Eq. (9) into Eq. (8) and equating real and imaginary parts separately, we get
\[
\left( \frac{c_1}{\beta} \right)^2 = \frac{1}{(1 + cy)} \left\{ \left[ (1 + ay) - \frac{p}{2Q_0} \right] p_1^4 + 2 \left[ \frac{N_0}{Q_0} (1 + by) - (1 + ay) \right] p_1^2 p_2^2 + \left[ 1 + ay + \frac{p}{2Q_0} \right] p_2^4 - \frac{gc}{k^2 \beta^2} p_1^2 \right\}
\]
and
\[
\left( \frac{c_2}{\beta} \right)^2 = 2 \left( \frac{N_0 b}{Q_0 c} - \frac{a}{c} \right) p_1^4 + 2 \left( \frac{a}{c} \right) p_2^2,
\]
where, \( \beta = \left( \frac{Q_0}{2Q_0} \right)^{\frac{1}{2}} \) is the velocity of shear waves in homogeneous isotropic medium. Eq. (10) gives the velocity of propagation of shear wave and Eq. (11) gives the damping. Eq. (10) shows that the velocity \( \left( \frac{c_1}{\beta} \right) \) depends much on the anisotropy factor, the initial stress factor and also on the direction of propagation denoted by \( (p_1, p_2) \).

4. Particular cases

4.1. Analysis of Eq. (10)

In order to gain more insight information the following cases have been discussed: analysis of Eq. (10) obtained by equating the real part of equation of motion.

**Case I.** In case \( Q \) is homogeneous \((a \to 0)\), i.e., rigidity along vertical direction is constant

\[
\left( \frac{c_1}{\beta} \right)^2 = \frac{1}{1 + cy} \left\{ \left[ 1 - \frac{p}{2Q_0} \right] p_1^4 + 2 \left[ \frac{N_0}{Q_0} (1 + by) - 1 \right] p_1^2 p_2^2 + \left[ 1 + \frac{p}{2Q_0} \right] p_2^4 - \frac{gc}{k^2 \beta^2} p_1^2 \right\}.
\]

The velocity in the \( x \)-direction is \((p_1 = 1, p_2 = 0, c = c_{11})\)

\[
\left( \frac{c_{11}}{\beta} \right)^2 = \frac{1}{1 + cy} \left\{ 1 - \frac{p}{2Q_0} - \frac{gc}{\beta^2 k^2} \right\},
\]

\[
c_{11}^2 = \frac{\beta^2}{1 + cy} \left\{ 1 - \frac{p}{2Q_0} - \frac{gc}{\beta^2 k^2} \right\}.
\]

In case the medium is free from initial stress \((p \to 0, c \to 0)\) and \( c_{11} = \beta \). Similarly the velocity of propagation along \( y \)-direction \((p_1 = 0, p_2 = 1, c_1 = c_{22})\) obtained as

\[
c_{22}^2 = \frac{\beta^2}{1 + cy} \left( 1 + \frac{p}{2Q_0} \right),
\]

Subtracting Eq. (15) from Eq. (14), we get

\[
c_{22}^2 - c_{11}^2 = \frac{1}{1 + cy} \left( \frac{p}{2Q_0} - \frac{gc}{\beta^2 k^2} \right),
\]

which is a function of initial stress, gravity field and density. It may also be observed that if \( p = S_{22} - S_{11} > 0 \), the effect of initial stresses on the body is compressive along \( x \)-direction and tensile along \( y \)-direction. The compressive initial stress reduces while tensile stress increases the velocity of wave shear along \( x \)-direction. A reverse effect is obtained along \( y \)-direction.

**Case II.** In case \( N \) is homogeneous \((b \to 0)\), i.e., rigidity along horizontal direction is constant

\[
\left( \frac{c_2}{\beta} \right)^2 = \frac{1}{1 + cy} \left\{ \left[ 1 + ay - \frac{p}{2Q_0} \right] p_1^4 + 2 \left[ \frac{N_0}{Q_0} (1 + by) - (1 + ay) \right] p_1^2 p_2^2 - \frac{gc}{k^2 \beta^2} p_1^2 + \left[ 1 + ay + \frac{p}{2Q_0} \right] p_2^4 \right\},
\]

the velocity along \( x \)-direction \((p_1 = 1, p_2 = 0, c_1 = c_{11})\) is given by

\[
c_{11}^2 = \frac{\beta^2}{1 + cy} \left\{ 1 + ay - \frac{p}{2Q_0} - \frac{gc}{k^2 \beta^2} \right\},
\]
which depends on the depth \( y \) and gravity and the wave is dispersive, the velocity along \( y \)-direction is \((p_1 = 0, p_2 = 1, c_1 = c_{22})\)

\[
c_{22}^2 = \frac{\beta}{1 + cy} \left[ 1 + ay + \frac{p}{2Q_0} \right],
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\(y\)
Fig. 4. Variation in velocities of shear wave in the direction of $\theta = 60^\circ$ with $x$-axis at different depth and different values of rigidities parameter $\alpha : c = 8; g = 0.3, N = 2.5; \alpha = 3.5, \ldots$.

Fig. 5. Variation in velocities of shear wave in the direction of $\theta = 30^\circ$ with $x$-axis at different depth and different values of $N$ (anisotropy): $\alpha = 8; P = 0.5, g = 0.3; \alpha = 4; N = 2.5, 3.5, \ldots$.

Fig. 6. Variation in velocities of shear wave in the direction of $\theta = 60^\circ$ with $x$-axis at different depth and different values of $N$ (anisotropy): $\alpha = 8; P = 0.5, g = 0.3; \alpha = 4; N = 2.5, 3.5, \ldots$. 
Fig. 7. Variation in velocities of shear wave in the direction of $\theta=30^\circ$ with $x$-axis at different depth and different values of initial stress $\bar{P}$: $\varepsilon = 8, \bar{P} = 0.5, g = 0.3, \alpha = 4, N = 2.5, \bar{P} = -0.8, -0.0, 0.8, -0.3$.

Fig. 8. Variation in velocities of shear wave in the direction of $\theta=60^\circ$ with $x$-axis at different depth and different values of initial stress $\bar{P}$: $\varepsilon = 8, \bar{P} = 0.5, g = 0.3, \alpha = 4, N = 2.5, \bar{P} = -0.8, -0.0, 0.8, -0.3$.

Fig. 9. Variation in velocities of shear wave in the direction of $\theta=30^\circ$ with $x$-axis at different depth and different values of gravity parameter $g$: $\varepsilon = 8, \bar{P} = 0.5, \alpha = 4, N = 2.5, g = 0.1, -0.3, 0.5, -0.3$. 
for \( p > 0 \), the velocity along \( y \)-direction may increase considerably at a distance \( y \) from free surface and the wave becomes dispersive.

**Case III.** In case \( N, Q \) and \( c \) are homogeneous

\[
\frac{c_1}{b} = \beta \left[ \left( 1 - \frac{p}{2Q_0} \right) p_1^2 + 2 \left[ \frac{2N_0}{Q_0} - 1 \right] p_1^2 p_2^2 + \left[ 1 + \frac{p}{2Q} \right] p_2^2 \right].
\]

(20)

In the absence of initial stress the velocity equation becomes

\[
\left( \frac{c_1}{b} \right)^{1/2} = \left\{ 1 - 4 \left( 1 - \frac{N_0}{Q_0} \right) p_1^2 p_2^2 \right\}
\]

(21)

in \( x \)-direction and \( y \)-direction \( c_1 = \beta \), then the velocity does not depend on an isotropy effects the velocity for isotropic medium \( N_0 = Q_0, c_1 = \beta \).

**Case IV.** In the absence of initial stress \( P \to 0 \), the velocity is obtained as

\[
\left( \frac{c_1}{b} \right)^{2} = \frac{1}{1 + cy} \left\{ (1 + ay)p_1^2 + 2 \left[ \frac{2N_0}{Q_0} (1 + by) - (1 + ay) \right] p_1^2 p_2^2 - \frac{gc}{k^2 b^2} p_1^2 + (1 + ay)p_2^2 \right\}
\]

(22)

in \( x \)-direction \( (p_1 = 1, p_2 = 0, c_1 = c_{11}) \), Eq. (22) reduces to

\[
\left( \frac{c_{11}}{b} \right)^{2} = \frac{1}{1 + cy} \left\{ 1 + ay \right\}
\]

(23)

and along \( y \)-direction \( (p_1 = 0, p_2 = 1, c = c_{22}) \), Eq. (22) tends to

\[
\left( \frac{c_{22}}{b} \right)^{2} = \frac{1 + ay}{1 + cy}.
\]

(24)

**Case V.** In the absence of gravity field the velocity is obtained as

\[
\left( \frac{c_1}{b} \right)^{2} = \frac{1}{1 + cy} \left\{ \left[ 1 + ay - \frac{P}{2Q_0} \right] p_1^2 + 2 \left[ \frac{2N_0}{Q_0} (1 + by) - (1 + ay)p_1^2 p_2^2 + \left( 1 + ay + \frac{P}{2Q_0} \right) p_2^2 \right] \right\}
\]

(25)

4.2. Analysis of Eq. (11)

Analysis of Eq. (11) obtained by equating imaginary parts of equation of motion.

In the absence of \( \rho \) and \( P \) in Eq. (11) three cases have been analyzed as follows:

**Case I.** In case \( Q \) is homogeneous \( (a \to 0) \) i.e., rigidity along vertical direction is constant, one may obtain

\[
\left( \frac{c_1}{b} \right)^{2} = 2 \left( \frac{2N_0}{Q_0} \frac{b}{c} \right) p_1^2.
\]

(26)

![Fig. 10. Variation in velocities of shear wave in the direction of \( \theta = 60^\circ \) with \( x \)-axis at different depth and different values of gravity parameter](image)
This shows that velocity of shear wave is always damped.
The velocity of wave along $x$-direction ($p_1 = 1, p_2 = 0, c = c_{11}$) is obtained as
\[ \left( \frac{c_{11}}{\beta} \right)^2 = 2 \left( \frac{2N_0 b}{Q_0 c} \right). \] \[(27)\]
This shows that actual velocity in $x$-direction is damped by $2 \left( \frac{2N_0 b}{Q_0 c} \right)$, and no damping takes place along $y$-direction.

**Case II.** In case $N$ is homogeneous ($b \to 0$) i.e., rigidity along horizontal direction is constant
\[ \left( \frac{c_1}{\beta} \right)^2 = 2 \left( -\frac{a}{c} \right) p_1^2 + 2 \left( \frac{a}{c} \right) p_2^2, \] \[(28)\]
the velocity of wave along $x$-direction ($p_1 = 1, p_2 = 0, c = c_{11}$) is given by

---

**Fig. 11.** Variation in velocities of shear wave with respect to the angle $\theta$ and different values of density parameter $c$ : $g = 0.3$; $a = 4$; $\bar{N} = 2.5$; (a) $\bar{P} = -0.8$; (b) $\bar{P} = 0$, (c) $\bar{P} = 0.8$; $\bar{c} = 0.7...$; $0.8---0.9...$
\[
\left( \frac{c_{11}}{\beta} \right)^2 = 2\left( -\frac{a}{c} \right).
\]  
(29)

Existence of negative sign shows that damping does not take place along the \(x\)-direction for \((b \rightarrow 0)\), the velocity along \(y\)-direction is given by
\[
\left( \frac{c_{22}}{\beta} \right)^2 = 2\left( \frac{a}{c} \right)
\]  
(30)
indicating a damping of magnitude \((\frac{2a}{c})\) takes place along \(y\)-direction.

**Case III.** In case \(N\) and \(Q\) are homogeneous but density is linearly varying with depth
\[
\left( \frac{c_1}{\beta} \right) = 0
\]  
(31)
i.e., no damping takes place.

**Fig. 12.** Variation in velocities of shear wave with respect to the angle \(\theta\) and different values of rigidities parameter \(a\) : \(\varepsilon = 0.8, g = 0.3, N = 2.5\); (a) \(P = -0.8\), (b) \(P = 0.0\), (c) \(P = 0.8\); \(a = 3\ldots; 3.5\ldots; 4\ldots\).
5. Numerical results and discussions

To get numerical information on the velocity of shear wave in the non-homogeneous initially stressed medium we introduce the following non-dimensional parameters:

\[ a = \frac{a}{b}, \quad b = by, \quad c = \frac{c}{b}, \quad c_1 = \frac{c_1}{b}, \quad N = \frac{N_0}{Q_0}, \quad p = \frac{P}{2Q_0}, \quad \bar{g} = \frac{gb}{k^2p}\]

Using these parameters in the Eq. (10) we obtain:

\[ c_1^2 = \frac{1}{(1 + cb)} \left[ (1 + ab - \bar{p})p_1^2 + 2N(1 + \bar{b}) - (1 + \bar{a}\bar{b})p_2^2 + (1 + \bar{a}\bar{b})p_1^2 - \bar{c}\bar{g}p_1^2 \right]. \]  

One may calculate \( c_1 \) for different values of \( a, c, N, \bar{p}, \bar{g}, \bar{b}, \) and \( \theta \), and the results are presented in Figs. 1–14.

Fig. 13. Variation in velocities of shear wave with respect to the angle \( \theta \) and different values of anisotropy parameter \( N \) : \( c = 8, \ a = 4, \ g = 0.3; (a) \bar{p} = -0.8, \) (b) \( \bar{p} = 0.0, \) (c) \( \bar{p} = 0.8; \ N = 2, 3, 5 \ldots \)
Figs. 1, 3, 5, 7 and 9 show the effects of density, rigidities, anisotropy, initial stress, respectively, if the angle \( \theta = 30^\circ \) on shear wave velocity \( c \) with respect to depth \( \beta \), Figs. 2, 4, 6, 8 and 10 show the effects of density, rigidities, anisotropy, initial stress, respectively, if the angle \( \theta = 60^\circ \) on shear wave velocity \( c \) with respect to depth \( \beta \). It is clear that shear wave velocity \( c \) increases with an increase of the depth \( \beta \).

From Figs. 1–9, it is obvious that shear wave velocity \( c \) decreases with an increase of the density parameter \( \bar{a} \) and the gravity \( g \) but increases with an increase of the rigidity parameter \( \bar{a} \) and anisotropy parameter \( N \) if \( \theta = \{30^\circ, 60^\circ\} \). Also, it appears that shear wave velocity \( c \) decreases with an increase of the initial stress if \( \theta = 30^\circ \) but increases with an increase of the initial stress if \( \theta = 60^\circ \). One can mention that shear wave velocity \( c \) has small values if \( \theta = 30^\circ \) compared to its values if \( \theta = 60^\circ \).

Figs. 11–14 display the variation in velocities of shear wave with respect to the angle \( \theta \) and different values of gravity parameter \( g \), rigidity parameter \( \bar{a} \), anisotropy parameter \( N \), and initial stress parameter \( P \).
It is seen that shear wave velocity $c_1$ decreases with an increase of angle $\theta$ and then increases with the high values of $\theta$. From Figs. 11 and 14, it is mentioned that shear wave velocity $c_1$ decreases with an increase of the density and gravity parameters. Also, it is obvious that shear wave velocity $c_1$ increases with an increase of rigidities and anisotropy parameters. Finally, from Figs. 11–14, it is concluded that the increasing values of initial stress (compression, without initial stress, and tensional initial stress), the shear wave velocity $c_1$ decreases.

6. Conclusions

The anisotropy, gravity field, non-homogeneity of the medium, the initial stress, the direction of propagation and the depth have considerable effect on the velocity of propagation of shear wave and attracts the attention of earth scientists in their work. Numerical computation shows that the presence of initial compressive stress in the medium gravity field, reduces the velocity of propagation while the tensile stress increases it. It is found that the variation in parameters associated with anisotropy and non-homogeneity of the medium directly affects the velocity of the wave. The velocity of propagation also depends on the inclination of the direction of propagation; an increase in the inclination angle decreases the velocity in the beginning, takes a minimum value before increasing. Finally, it is concluded that the increasing values of initial stress (compression, without initial stress, and tensional initial stress), the shear wave velocity $c_1$ decreases.

References