Local coordination and market equilibria

Shurojit Chatterji\textsuperscript{a} and Sayantan Ghosal\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} CIE, ITAM, Av. Camino Santa Teresa, 930 Mexico D.F. CP 10700
\textsuperscript{b} Department of Economics, University of Warwick, Coventry CV47AL, UK

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Abstract

We reformulate the local stability analysis of market equilibria in a competitive market as a local coordination problem in a market game, where the map associating market prices to best-responses of all traders is common knowledge and well-defined both in and out of equilibrium. Initial expectations over market variables differ from their equilibrium values and are not common knowledge. This results in a coordination problem as traders use the structure of the market game to converge back to equilibrium. We analyse a simultaneous move and a sequential move version of the market game and explore the link with local rationalizability.

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1. Introduction

While there is, by now, an extensive literature on the non-cooperative foundations of competitive equilibria,\textsuperscript{1} there has been very little work\textsuperscript{2} on the non-cooperative foundations for the stability analysis of competitive equilibria. This is surprising given that the traditional analysis of stability\textsuperscript{3} of competitive equilibria suffers from the problem that there is no explicit price formation rule that underpins the analysis of competitive equilibria.

\textsuperscript{*}Corresponding author.
\textit{E-mail addresses:} shurojit@itam.mx (S. Chatterji), s.ghosal@warwick.ac.uk (S. Ghosal).

\textsuperscript{1} See, for instance, Roberts [11], Dubey and Shapley [4] to name just two papers in this area.

\textsuperscript{2} A notable exception is [6] which is discussed later in the introduction.

\textsuperscript{3} See, for example, [1] and references therein.

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In this paper, we take a first step towards providing a non-cooperative foundation for the stability analysis of competitive markets using market games whose non-cooperative equilibria coincide with competitive equilibria. In a market game, the map that associates market prices to the actions of all individuals is well-defined both out of equilibrium and at equilibrium. We reformulate the stability problem of competitive equilibria as a coordination problem over the expectations that agents have over market variables, like prices. We assume that initial expectations over these market variables (a) differ from the equilibrium values of the same market variables and (b) are not common knowledge. This creates a coordination problem as traders try to use structure of the game to converge back to equilibrium. The coordination problem is local when the initial expectations over market variables are in the vicinity of their equilibrium values. The study of the resulting coordination dynamics, in the vicinity of the market equilibrium, is also an analysis of the local stability of competitive equilibria.

The model of non-cooperative exchange that we use belongs to a line of research initiated by Shapley and Shubik [14]. In this approach, each participant sends out quantity signals, bids and offers of the commodities they own, which indicate how much of each commodity they are willing to put up for trade. The market is modelled as a mechanism that consists of a price formation rule and an allocation rule: after receiving all the bids and offers, the mechanism determines prices and final holdings of each participant. In our set-up, there is a continuum of traders, two commodities and single market in which traders exchange the two commodities. Traders make bids in units of the first commodity and offers in units of the second commodity. The price formation rule sets the market price equal to the ratio of the aggregate bid of commodity one over the aggregate offer of commodity two. The allocation rule, then, assigns commodity bundles and two commodities. We study two versions of the market game: the simultaneous move market game, where both bids and offers are made simultaneously, and the sequential move market game, where offers are made after the market (aggregate) bid is observed.

We require all participants in the market game to know the rules of the market game, the strategy sets of players, their payoff functions. We also assume that it is common knowledge that all agents respond to a market bid or offer by determining their best response and that the best-responses of all other agents is known by every participant. This ensures that the map which associates market prices to the best-responses of all individuals is common knowledge in both variants of the market game. In the simultaneous move market game, starting from rationalizability, we describe the coordination problem over the heterogeneous expectations of agents over the market price. In the sequential move market game, the coordination problem is formulated over the expectations over the market bid for traders making

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4 Several variants of this market game exist in the literature (see for instance [10,12]). All these different variants coincide in the case of two commodities that we study in this paper.

5 Importantly, throughout this paper, we assume that agents have point expectations over market variables. This assumption is used in the remainder of the paper without being explicitly invoked. How our analysis extends to the general case, where agents are allowed to have stochastic expectations, is an important topic we intend to pursue in future research.
the first move. This formulation of the coordination problem directly on expectations over market variables allows us to focus on the competitive flavour of the market game. However, in the sequential move market game, we also show that our approach to the coordination problem can be derived directly starting from extensive-form rationalizability.

In this paper, for simplicity, we focus on economies where all traders have corner endowments. This allows us to model a competitive market where any trader can only act on one side of the market either as buyer or a seller.

We have two sets of results in this paper.

The first set of results relate to the analysis of local coordination in the simultaneous move market game. We derive the coordination dynamics with heterogeneous expectations over the market price as consequence of local rationalizability and derive conditions for its local stability. In the special case of homogenous expectations, the local stability of the coordination dynamics requires that the two sides of the market respond in a similar way to small changes in prices in the vicinity of the equilibrium price; in other words, the difference between the elasticity of market bids and the elasticity of market offers, evaluated at the equilibrium price, is bounded. With heterogeneous expectations, we show that the local coordination problem, in general, is harder to resolve than in the case homogenous expectations. Nevertheless, the analysis in the two cases coincide when all the traders belonging to one side of the market, respond in the same direction: all traders on the same side of the market either increase (or decrease) the quantity, put up for exchange, of the commodity they own. Moreover, by appropriately choosing the speed of adjustment, the local stability of the classical discrete-time tatonnement dynamics can be derived from (and in one case, made equivalent to) the coordination dynamics with homogenous beliefs over the market price.

The second set of results relate to the analysis of local coordination in the sequential move market game. In this part of the paper, we restrict attention to those subgame perfect equilibria that sustain market equilibria. As a preliminary step, we show that there exist subgame perfect equilibria, where market offers are described by smooth offer functions, that sustain market equilibria. We focus on the coordination problem faced by traders who make bids in the first stage when they anticipate that the offers in the second stage of the game are made according to these smooth offer functions at the subgame perfect equilibrium. We, then, derive the coordination dynamics when traders in the first stage have homogenous beliefs over the market bid. We show that when the two sides of the market move in opposite directions in response to small changes in prices in the vicinity of the subgame perfect (and competitive) equilibrium, local stability of the coordination dynamics in the simultaneous move game with homogenous expectations implies the local stability of the coordination dynamics in the sequential move game. Moreover, by example, we also that local stability in the sequential move game may obtain while local stability in the simultaneous move game does not. However, when the two sides of the market

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6When offer functions are not smooth, as in the case of “knife-edge” strategies, we show that the local coordination problem is never solved.
move in the same direction to small changes in prices, in the vicinity of the equilibrium, the opposite conclusion goes through.

Next, we explore the link between local extensive-form rationalizability and the coordination dynamics in the sequential move game. We show that whenever the subgame perfect equilibria of the sequential move game is locally extensive-form rationalizable, the coordination dynamics locally converges in the sequential move game. Moreover, we also show that the local stability of the cobweb dynamics in the competitive market coincides with the local stability of the coordination dynamics in the sequential moves market game.

The paper which is closest to our approach is by Guesnerie [6]. There, in a one-commodity, partial equilibrium model where all agents are price-takers, he introduced the notion of eductive stability. In his analysis, individuals who supply the commodity in question must choose the quantity they supply before prices adjust to ensure market clearing, i.e. the quantity supplied to the market satisfies an exogenously specified demand curve. This creates the possibility that the suppliers of the commodity do not necessarily coordinate on the equilibrium price: if the suppliers of the commodity have heterogeneous expectations of the market-clearing price, they will need to revise their expectations in order to converge to the equilibrium price. Although this analysis is a definite advance on the traditional analysis of the stability of competitive equilibria, it suffers from the problem that it rules out the possibility of coordination problems arising for a subset of participants, namely those whose behaviour is summarized in the exogenously specified demand curve. Moreover, the adjustment rule that associates the equilibrium price to the actions of the suppliers of the commodity and allows the suppliers to revise expectations, actually requires the existence of an exogenously specified demand curve. Therefore, it is unclear how Guesnerie’s [6] analysis would extend to a situation where all participants, on both sides of the market, may have to solve a coordination problem. Moreover, our result on the equivalence of the local stability of the cobweb dynamics with the local stability of the coordination dynamics in the sequential moves market game also provides a strategic foundation for [6].

In the next section, we specify the model of the competitive market and the simultaneous move and sequential move market game. Section 3 studies local coordination in the simultaneous move market game; Section 4 is devoted to the study of coordination in the sequential move market game. All proofs (unless otherwise stated) and some of the more technical material are gathered in the appendix.

2. The competitive market

We begin by specifying the fundamentals of the competitive market. The set of agents consists of the interval $I = [0, M]$. Formally, there is an atomless measure space of individuals, $(I, \mathcal{I}, \mu)$, with $\mathcal{I}$ the $\sigma$-algebra on $I$ and $\mu$ an atomless measure defined on $I$. Null sets of individuals are systematically ignored throughout the
paper. A commodity bundle is \( x \in \mathbb{R}^2_+ \). For some finite-dimensional Euclidean space \( \mathbb{R}^k \), an assignment is any function \( x : I \rightarrow \mathbb{R}^k \) each coordinate of which is integrable. There are two types of individuals, \( I_1, I_2 \), of measure \( m_1 \) and \( m_2 \) respectively, \( m_1 + m_2 = M \), which partitions \( I \). There is a fixed initial assignment of commodities, \( w : I \rightarrow \mathbb{R}^2_+ \), the endowments of individuals with \( w^i = (w^i_1, 0) \), for all \( i \in I_1 \) and \( w^j = (0, w^j_2) \), for all \( j \in I_2 \) and \( w_l = \int w^l_i \, dn > 0 \) for \( l = 1, 2 \), where the range of the integral, the set \( I \), has been suppressed for convenience. Preferences of trader \( n \) are described by a utility function \( u^n : \mathbb{R}^2_+ \rightarrow \mathbb{R} \). We make the following assumptions throughout the paper on utility functions:

A1(i). For each trader \( n \in I_1 \), \( u^n \) satisfies strict monotonicity and is smooth with \( (u^n_1, u^n_2) > 0 \) and \( (u^n_{11}, u^n_{12}) \) is negative definite on \( \mathbb{R}^2_{++} \);

A1(ii). For each trader \( n \in I_1 \), \( \lim_{x_1 \rightarrow 0} u^n_1(x) = \lim_{x_2 \rightarrow 0} u^n_2(x) = \infty \);

A2. The family of utility functions \( \{u^n\} \) is uniformly smooth.

The requirement of uniform smoothness in (A2) follows Aumann [2, Sections 4 and 10] except that we do not require the utility functions of any trader to be bounded (Aumann [2, p. 630, footnote 26]). A consequence of (A2) is that \( \tilde{u}^i : \mathbb{R} \rightarrow \mathbb{R} \) viewed as a map from \((n, x)\) to real numbers is measurable as a function of \((n, x)\).

An allocation is any assignment \( x \) such that \( x^n \in \mathbb{R}^2_+ \), for all \( n \in I \). It is feasible if in addition \( \tilde{x} = \int x^n \, dn = \int w^n \, dn \). Prices are \( p \in \mathbb{R}^2_+ \). We normalize prices so that \( p_1 = 1 \) and \( p_2 = p \). At prices \( p \), an individual solves the following maximization problem:

\[
\max_{x \in \mathbb{R}^2_+} u^n(x) \text{ s.t. } x_1 + px_2 \leq w^n_1 + pw^n.
\] (1)

Under assumptions (A1)–(A2), it follows that for each \( p \in \mathbb{R}^2_+ \), there exists a unique commodity bundle \( \tilde{x}^n(p) \) that solves the maximization problem (1). A competitive equilibrium is a tuple \((\tilde{p}, \tilde{x})\) of prices and a feasible allocation such that given \( \tilde{p}, \tilde{x}^n \in \mathbb{R}^2_+ \) solves (1) for each \( n \in I \). Let \( C \) denote the set of competitive equilibrium allocations.

There are two different models of non-cooperative exchange that we study in this paper, simultaneous move market games and sequential move market games.

### 2.1. The simultaneous move market game

The exchange mechanism we study is due to [14]. Here all traders simultaneously make bids and offers of commodities at the trading post where the two commodities are exchanged. A strategy for a trader \( i \in I_1 \) is a bid of commodity 1, denoted by \( b^i \) while a strategy for a trader \( j \in I_2 \) is an offer of commodity 2, \( q^j \). The corresponding set of strategies for each trader \( i \in I_1 \) is \( S^i = \{b^i : 0 \leq b^i \leq w^i_1\} \) while the strategy set of trader \( j \in I_2 \) is \( S^j = \{q^j : 0 \leq q^j \leq w^j_2\} \). For each assignment of strategies \( s = (b, q) \), at
the trading post the market bid is $B = \int_b^d b_i \, di$ while the market offer is $Q = \int_q^l q_j \, dj$. Define the price at the trading post to be $p = \frac{B}{Q}$ if $Q > 0$ with $p = 0$ otherwise. For each trader $i \in I_1$, the allocation rule determines commodity holdings as follows: if $\pi \neq 0$, $x^*_i = w^*_i - b^i$ and $x^*_i = \frac{b^i}{\pi}$; otherwise, $x^*_i = w^*_i$, for $i = 1, 2$. For each trader $j \in I_2$, the allocation rule determines commodity holdings as follows: if $\pi \neq 0$, $x^*_j = q^*_j$, and $x^*_j = w^*_j - q^*_j$; otherwise, $x^*_j = w^*_j$, for $l = 1, 2$. Using this allocation rule, by substitution in the utility function, for each trader $n$, we have payoffs defined as follows. For each trader $i \in I_1$, the payoff over any strategy profile $s$ with $B > 0$ and $Q > 0$ is given by $u^i(w^*_i - b^i, \frac{b^i}{\pi})$. Similarly, for each trader $j \in I_2$, the payoff over any strategy profile $s$ with $B > 0$ and $Q > 0$ is given by $u^j(q^*_j, w^*_j - q^*_j)$. A non-trivial Nash equilibrium assignment of strategies is $s^*$ if (i) the associated $B^*$ and $Q^*$ are both strictly positive, (ii) for all $i \in I_1$, $u^i(w^*_i - b^i, \frac{b^i}{\pi}) \geq u^i(w^*_i - b^i, \frac{b^i}{\pi})$, for all $b^i \in S^i$ and (iii) for all $j \in I_2$, $u^j(q^*_j, w^*_j - q^*_j) \geq u^j(q^*_j, w^*_j - q^*_j)$, for all $q^*_j \in S^j$. In the remainder of the paper we restrict attention to non-trivial Nash equilibria. Again, in what follows we will refer directly to Nash equilibria, dropping the qualification non-trivial. Let $N$ denote the set of Nash equilibrium allocations. The following lemma is also direct consequence of [4] and is stated without proof.

**Lemma 1.** $N = C$.

Lemma 1 allows us to refer to Nash equilibria or competitive equilibria interchangeably. In the remainder of the paper, without confusion, we will use the term market equilibria. Consider a market equilibrium assignment of strategies $s^*$ and prices $\pi^*$. Under assumptions (A1)–(A2), the following first-order conditions characterize the best-response of each individual:

For $i \in I_1$: $-u^i_1 + \frac{1}{\pi^*} u^i_2 = 0$, \hspace{1cm} (2a)

For $j \in I_2$: $-u^j_2 + \pi^* u^j_1 = 0$. \hspace{1cm} (2b)

Under assumptions (A1)–(A2) it follows that the best-response for each individual is unique and a continuously differentiable function of $\pi$, for all $\pi \in \mathbb{R}_{++}$. For each $i \in I_1$ (respectively, $j \in I_2$) and for all $\pi \in \mathbb{R}_{++}$, let $b^i(\pi)$ (respectively, $q^i(\pi)$) denote the best-response function implicitly defined by the first-order conditions. Note that for deriving the results in our paper, we do not really require the boundary condition stated in assumption (A1(ii)). We can replace the boundary condition in (A1(ii)) with the alternative assumption (A1(ii)'): there exists an open interval around some equilibrium price $\pi^*$ such that for all prices in this interval, the first-order conditions (2) characterize the optimal choices of individuals. Under (A1(ii)'), it would follow that the best-response for each individual is unique and a continuously differentiable function of $\pi$ in the vicinity of $\pi^*$. In the examples (see below, Sections 3 and 4) the preferences of traders will in fact violate the boundary condition assumption (A1(ii)) but satisfy (A1(ii)').
We conclude our description of the market game by noting that at each \( x^* \in \mathbb{N} \), the following equation must hold:
\[
\frac{\int_{I_1} b^j(\pi^*) \, di}{\int_{I_2} q^j(\pi^*) \, dj} = \pi^*.
\] (3)

At this stage, we point out that under our assumption of uniform smoothness, we can differentiate under the integration sign to justify the expression \( B'(\pi) = \int_{I_1} b^j(\pi) \, di \) for all \( \pi \in (\pi^* - \epsilon, \pi^* + \epsilon) \) (see [3, Theorem 13.8.6]). At several other points in the paper, similar expressions will be used without explicit justification.

2.2. The sequential move market game

In this version of the market game, there are two stages. In stage 1, each trader \( i \in I_1 \), makes a bid \( b^i \in S^i \). In stage 2, each \( j \in I_2 \) observes the resulting market bid \( B = \int_{I_1} b^i \, di \) and chooses an offer \( q^j \in S^j \), resulting in a market offer \( Q = \int_{I_2} q^j \, dj \). The allocation rules are as in the simultaneous move version of the market game. Let \( \hat{B} = \{ B \in \mathcal{R}_{++} : B = \int_{I_1} b^i \, di \text{, for some assignment of bids } b \} \). A strategy for \( j \in I_2 \), is an offer function \( q^j : \hat{B} \to S^j \). Let \( \hat{Q}^j \) be the set of all strategies for each \( j \in I_2 \) with \( \hat{Q} = \times_{j \in I_2} \hat{Q}^j \). An assignment of offer functions, \( \hat{q} \), is a map \( \hat{q} \in \hat{Q} \) such that for each \( B \in \hat{B} \), the integral \( \int_{j \in I_2} \hat{q}^j(B) \, dj \) exists. A non-trivial subgame perfect equilibrium is an assignment \( (b^*, \hat{q}^*) \) of bids and offer functions such that (i) \( B^* > 0 \) and \( Q^* = \int_{j \in I_2} \hat{q}^j(B^*) \, dj > 0 \), (ii) for each \( B \in \hat{B} \), for all \( j \in I_2 \), \( u^j(\hat{q}^j(B)) = \int_{j \in I_2} \hat{q}^j(B) \, dj \), (iii) for all \( i \in I_1 \), \( u^i(w_i - b^i) \geq u^i(w_i - b^i + \frac{\partial b^i}{\partial q^j}) \), for all \( b^i \in S^i \), where \( \pi^* = \frac{\int_{I_1} b^i(\pi^*) \, di}{\int_{I_2} q^j(\pi^*) \, dj} \). Throughout the paper, we restrict attention to only those non-trivial subgame perfect equilibria that yields a market equilibrium allocation. At a non-trivial subgame perfect equilibrium, for each \( B \in \hat{B} \), there is an associated market offer \( Q > 0 \). Moreover, by definition, for \( \pi = \frac{B}{Q} \), the first-order conditions (2b) have to be satisfied for each \( j \in I_2 \). In other words, for each market bid \( B \in \hat{B} \), at a subgame perfect equilibrium, the associated market offer \( Q \) must satisfy the equation \( Q = \int_{j \in I_2} q^j(\frac{B}{Q}) \, dj = \int_{j \in I_2} q^j(\pi) \, dj \), where \( q^j(\pi) \) is the implicit solution to the first-order conditions (2b) for each \( j \in I_2 \). Consider the quantity \( \epsilon^j(\pi) = \frac{\int_{j \in I_2} q^j(\pi) \, dj}{Q} \). Remark that at a subgame perfect equilibrium \( Q^* = \int_{j \in I_2} q^j(\frac{B^*}{Q^*}) \, dj \). Suppose the subgame perfect equilibrium price \( \pi^* \) satisfies the regularity condition \( \pi^* \epsilon^j(\pi^*) \neq -1 \). Direct computation shows that the total derivative of \( \int_{j \in I_2} q^j(\frac{B}{Q}) \, dj = Q \) with respect to \( B \) and \( Q \), evaluated at \( B^* \) and \( Q^* \), is given by \( (1 + \pi^* \epsilon^j(\pi^*)) \frac{dQ}{Q} = \epsilon^j(\pi^*) \frac{dB}{B} \). Provided that \( \pi^* \epsilon^j(\pi^*) \neq -1 \), we
obtain that \( \frac{dQ}{dB} = \frac{\epsilon(x^*)}{(1 + \epsilon(x^*))}. \) By applying the implicit function theorem (and the Lebesgue-dominated convergence theorem), it follows that for all \( B \in [B^* - \epsilon, B^* + \epsilon] \) in the vicinity of \( B^* \) there is a implicit solution to the equation \( \int_{I_2} q^j(B) \, dj = Q \) denoted by \( Q(B) \). We summarize the above discussion by the following result:

**Lemma 2.** Suppose the subgame perfect equilibrium price \( x^* \) satisfies the regularity condition \( x^* \epsilon(x^*) \neq -1 \). Then, for all \( B \in [B^* - \epsilon, B^* + \epsilon] \) in the vicinity of \( B^* \) there is a implicit solution to the equation \( \int_{I_2} q^j(B) \, dj = Q \) denoted by \( Q(B) \).

Observe that, by setting \( \tilde{q}^j(B) = q^j(B) \) for each \( B > 0 \) and \( j \in I_2 \), at a subgame perfect equilibrium, the first-order conditions (2) must be satisfied. The following lemma shows that (a) the use by each \( j \in I_2 \) of the differentiable offer function \( \tilde{q}^j(B) \) in the vicinity of a market equilibrium bid \( B^* \) is a subgame perfect equilibrium of the sequential market game and (b) that such a subgame perfect equilibrium supports some market equilibrium allocation \( x^* \).

**Lemma 3.** Consider a market equilibrium allocation \( x^* \in \mathbb{N} \) and let \( B^* \) and \( Q^* \) be the associated market equilibrium market bids and offers. Then, there exists for each \( j \in I_2 \), a continuously differentiable offer function, given by \( \tilde{q}^j(B) \) that supports \( x^* \in \mathbb{N} \) as a subgame perfect equilibrium allocation. Moreover, \( Q(B) = \int_{I_2} \tilde{q}^j(B) \, dj \) is well-defined and smooth in the vicinity of \( B^* \).

### 3. Rationalizability and coordination in the simultaneous move market game

We begin by defining local rationalizability in the simultaneous move market game. We, then, provide a specification of coordination dynamics with heterogeneous expectations and show that the local stability of the coordination dynamics with heterogeneous expectations is equivalent to local rationalizability. Next, we study conditions on preferences under which the coordination dynamics with heterogeneous expectations simplifies to coordination dynamics with homogenous expectations and we study the connections between the latter and discrete-time tatonnement dynamics.

Fix a market equilibrium price \( x^* \) with associated assignment of bids \( b^* \) and offers \( q^* \). For each \( i \in I_1 \), (respectively, \( j \in I_2 \)) consider \( S_0^i \subset S^i \) such that \( b^i \in S_0^i \) while \( 0 \notin S_0^i \) (respectively, \( S_0^j \subset S^j \) such that \( q^j \in S_0^j \) while \( 0 \notin S_0^j \)). Let

\[
\hat{H}_0 = \left\{ \pi \in \mathbb{R}_{++}^{I_1} : \forall i \in I_1, j \in I_2, \exists b^i \in S_0^i, q^j \in S_0^j \text{ s.t. } \pi = \frac{\int_{I_1} b^i \, di}{\int_{I_2} q^j \, dj} \right\}.
\]
For \( n \geq 1 \), define the sequence of sets
\[
\tilde{S}_n^i = \left\{ b \in S^i : \exists \pi \in \tilde{\Pi}_{n-1} \text{ s.t. } b \in \arg \max_{b \in S^i} u^i \left( w^i_1 - b, \frac{b}{\pi} \right) \right\},
\]
\[
\tilde{S}_n^j = \left\{ q \in S^j : \exists \pi \in \tilde{\Pi}_{n-1} \text{ s.t. } q \in \arg \max_{q \in S^j} u^j (\pi q, b^j_2 - q) \right\},
\]
\[
\tilde{\Pi}_n = \left\{ \pi \in \mathcal{R}_{++} : \forall i \in I_1, j \in I_2, \exists b^i_j \in \tilde{S}_n^i, \exists q^j \in \tilde{S}_n^j \text{ s.t. } \pi = \frac{\int_{I_1} b^i_j \, d\pi}{\int_{I_2} q^j \, d\pi} \right\}.
\]

Then, the market equilibrium assignment of bids \( b^* \) and offers \( q^* \) is locally rationalizable if and only if for some initial restrictions for each \( i \in I_1 \) (respectively, \( j \in I_2 \)), \( \tilde{S}_0^i \subset S^i \) (respectively, \( \tilde{S}_0^j \subset S^j \)), for all \( i \in I_1 \), \( \bigcap_{n \geq 0} \tilde{S}_n^i = \{ b^* \} \) while for all \( j \in I_2 \), \( \bigcap_{n \geq 0} \tilde{S}_n^j = \{ q^* \} \). Although this definition of local rationalizability mimics the definition of rationalizability in [9], it differs in two ways. One, it is a local definition as the initial restrictions on strategies are required for each trader to lie in a neighbourhood of their Nash equilibrium choice of bid or offer. Two, it restricts attention to pure strategies. Observe that the market equilibrium assignment of bids \( b^* \) and offers \( q^* \) is locally rationalizable if and only if \( \bigcap_{n \geq 0} \tilde{\Pi}_n = \{ \pi^* \} \).

Next, we consider coordination dynamics when agents have heterogeneous expectations about the market price. Consider a market equilibrium price \( \pi^* \) and let \( \Pi_0 = [\underline{\pi}, \overline{\pi}] \) be an interval that contains \( \pi^* \). While it is common knowledge that every agent has deterministic beliefs over prices in \( \Pi_0 \), i.e. each agent has a point expectation of the market price denoted by \( \pi^{e,n} \in [\underline{\pi}, \overline{\pi}] \), no specific configuration of these possibly heterogenous expectations is common knowledge. Given a configuration of point expectations, \( \{ \pi^{e,n} : n \in I \} \), each \( i \in I_1 \) submits a bid \( b^i(\pi^{e,j}) \) and each \( j \in I_2 \) submits an offer \( q^j(\pi^{e,j}) \). Formally, we only consider admissible configurations of point expectations \( \pi^e : I \to [\underline{\pi}, \overline{\pi}] \) which are ones such that the ratio \( \frac{\int_{I_1} b^i(\pi^{e,j}) \, d\pi}{\int_{I_2} q^j(\pi^{e,j}) \, d\pi} = \pi \).

Performing this exercise for each admissible configuration of point expectations generates, using (4), a new set \( \Pi_1 \). In the next step, agents repeat the above process for each admissible configuration of market prices \( \pi^e : I \to \Pi_1 \), generating in turn, a new set \( \Pi_2 \). Iterating this process generates a sequence of intervals of market prices denoted by \( \Pi_t \) and we study the conditions under which \( \pi^* \) is stable or unstable under this coordination dynamics, i.e. the conditions under which \( \Pi_t \subset \Pi_{t+1} \) for all \( t = 1, 2, \ldots \), and \( \bigcap_{t \geq 0} \Pi_t = \{ \pi^* \} \). Given our common knowledge assumptions, remark that at each step \( t, t \geq 1 \), of the coordination dynamics, the computation of the interval \( \Pi_t \) is not agent specific and can be performed by any one agent.

Note that if there exist non-trivial initial restrictions for each \( i \in I_1 \) (respectively, \( j \in I_2 \)), \( \tilde{S}_0^i \subset S^i \) (respectively, \( \tilde{S}_0^j \subset S^j \)), such that the market equilibrium assignment...
of bids $b^*$ and offers $q^*$ is locally rationalizable, there is some neighbourhood of prices $[\bar{\pi}, \bar{\pi}]$ around $\pi^*$ such that the sequence of intervals of prices $\Pi_t$, $t = 1, 2, \ldots$ with $\Pi_0 = [\bar{\pi}, \bar{\pi}]$, generated by the coordination dynamics with heterogeneous expectations shrinks to $\pi^*$. Now suppose that the market equilibrium assignment of bids $b^*$ and offers $q^*$ is locally stable under the coordination dynamics with heterogeneous expectations. Then, for each $t \geq 1$, $\pi \in \Pi_{t-1}$, with $\Pi_0 = \Pi_0$, let

$$
\bar{S}_t = \left\{ b \in S^i : \exists \pi \in \Pi_{t-1} \text{ s.t. } b \in \arg \max_{b \in S^i} u^i(\frac{\pi}{b} - b, \frac{\pi}{\pi}) \right\},
$$

$$
\bar{S}_t = \left\{ q \in S^j : \exists \pi \in \Pi_{t-1} \text{ s.t. } q \in \arg \max_{q \in S^j} u^j(q, w^j - q) \right\}.
$$

Let $\tilde{\Pi}_t = \{ \pi \in \mathbb{R}_{++} : \forall i \in I_1, j \in I_2, \exists b^i \in \tilde{S}_t, q^j \in \tilde{S}_t \text{ s.t. } \pi = \frac{\int_{t} b^i \, d\xi_j}{\int_{t} q^j \, d\eta_j} \}$. Remark that for each $t \geq 1$, $\Pi_t = \tilde{\Pi}_t$. As the sequence of sets of prices $\Pi_t$, $t \geq 1$, shrink to $\pi^*$, the sequence of sets of prices $\tilde{\Pi}_t$ also shrinks to $\pi^*$ and therefore, for each $i \in I_1$, the sequence of sets of bids $\tilde{S}_t$, $t \geq 1$, (respectively, each $j \in I_2$, the sequence of sets of offers $\tilde{S}_t$, $t \geq 1$.) shrinks to $b^i$ (respectively, $q^j$). It follows that the market equilibrium assignment of bids $b^*$ and offers $q^*$ are locally rationalizable. We summarize the above discussion with the following proposition:

**Proposition 4.** The coordination dynamics with heterogeneous expectations is locally stable if and only if there exist non-trivial initial restrictions for each $i \in I_1$, (respectively, $j \in I_2$) $\bar{S}_0 \subset S^i$ (respectively, $S_0^j \subset S^j$), such that the market equilibrium assignment of bids $b^*$ and offers $q^*$ is locally rationalizable.

In order to facilitate the analysis, we need to make some additional assumptions on utility functions of individuals. By direct computation, using the first-order conditions (2), we get that

$$
\text{for } i \in I_1 : \quad b^i(\pi) = \frac{u^i_1 + u^i_2 x_2 - b^i u^i_{12}}{\pi^2 u^i_1 - 2\pi u^i_{12} + u^i_2}, \quad (5a)
$$

$$
\text{for } j \in I_2 : \quad q^j(\pi) = -\frac{u^j_1 + u^j_2 x_1 - q^j u^j_{12}}{\pi^2 u^j_1 - 2\pi u^j_{12} + u^j_2} \quad (5b)
$$

As the bordered hessian of utility evaluated at the optimum for each individual is negative definite, the denominator in both (5a) and (5b) is negative. Further, we must also have that for each $i \in I_1$, $b^i \geq 0$ and for each $j \in I_2$, $q^j \geq 0$. It follows that for each $i \in I_1$, the sign of $b^i(\pi)$ is the sign of $-(u^i_1 + u^i_2 x_2 - b^i u^i_{12})$. Similarly, it follows that for each $j \in I_2$, the sign of $q^j(\pi)$ is the sign of $(u^j_1 + u^j_2 x_1 - q^j u^j_{12})$.

At prices $\pi > 0$ and commodity bundle $x \in \mathbb{R}^2_{++}$, let $I_1^+(\pi, x)$ denote the set of agents in $I_1$ such that $u^i_1 + u^i_2 x_2 \leq 0$ and $u^i_{12} \geq 0$. For agents in $I_1^+(\pi, x)$, $b^i(\pi) \geq 0$ at

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7See [5] for a similar result in the context of linear rational expectations models.
the commodity bundle \( x \). Similarly, let \( I_1^- (\pi, x) \) denote the set of agents in \( I_1 \) such that \( u'_2 + u'_{22, x_2} \geq 0 \) and \( u'_1 \leq 0 \) with at least one strict inequality. For agents in \( I_1^- (\pi, x) \), \( b^j (\pi) < 0 \) at the commodity bundle \( x \). Analogously, for individuals in \( I_2 \), at prices \( \pi > 0 \) and commodity bundle \( x \in \mathcal{R}_{++}^2 \), we can also define \( I_2^+ (\pi, x) \) to be the set of agents such that \( q^j (\pi) \geq 0 \) at the commodity bundle \( x \) and \( I_2^- (\pi, x) \) to be set of individuals with \( q^j (\pi) < 0 \) at the commodity bundle \( x \). We now introduce an assumption that will enable us to sign these derivatives. Formally, we have:

(A3). (i) If \( i \in I_1^+ (\pi, x) \) at some \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \), then \( i \in I_1^- (\pi, x) \) for all \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \). Let \( I_1^- \) denote the collection of such individuals.

(ii) If \( i \in I_1^- (\pi, x) \) at some \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \), then \( i \in I_1^+ (\pi, x) \) for all \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \). Let \( I_1^+ \) denote the collection of such individuals.

(iii) \( \{I_1^-, I_1^+\} \) is a partition of \( I_1 \).

(iv) If \( j \in I_2^- (\pi, x) \) at some \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \), then \( j \in I_2^+ (\pi, x) \) for all \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \). Let \( I_2^- \) denote the collection of such individuals.

(v) If \( j \in I_2^+ (\pi, x) \) at some \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \), then \( j \in I_2^- (\pi, x) \) for all \( \pi > 0 \) and \( x \in \mathcal{R}_{++}^2 \). Let \( I_2^+ \) denote the collection of such individuals.

(vi) \( \{I_2^-, I_2^+\} \) is a partition of \( I_2 \).

Under assumption (A3), (a) for each \( i \in I_1^+ \), \( b^j (\pi) \) is weakly monotone over \( \pi > 0 \), i.e. the sign of \( b^j (\pi) \) is non-negative for all \( \pi > 0 \) and (b) for each \( i \in I_1^- \), \( b^j (\pi) \) is monotone over \( \pi > 0 \), i.e. the sign of \( b^j (\pi) \) is negative for all \( \pi > 0 \). Similarly, under the assumption (A3), (a) for each \( j \in I_2^+ \), \( q^j (\pi) \) is weakly monotone over \( \pi > 0 \), i.e. the sign of \( q^j (\pi) \) is non-negative for all \( \pi > 0 \) and (b) for each \( j \in I_2^- \), \( q^j (\pi) \) is monotone over \( \pi > 0 \), i.e. the sign of \( q^j (\pi) \) is negative for all \( \pi > 0 \). What assumption (A3) rules out is the possibility that for some \( i \in I_1 \) (respectively, \( j \in I_2 \)), the sign of \( b^j (\pi) \) changes sign over \( \mathcal{R}_{++} \) (respectively, \( q^j (\pi) \) changes sign over \( \mathcal{R}_{++} \)).

Which kind of utility functions satisfy assumption (A3)? Consider individuals in \( I_1^+ \). When the utility function of each individual in \( I_1^+ \) is assumed to be additively separable over \( (x_1, x_2) \), i.e. there exists a representation of the form \( u^i (x_1, x_2) = u^i (x_1) + w^i (x_2) \), for each \( i \in I_1^+ \), assumption (A3) requires that \( w^i + w'' x_2 \leq 0 \) for all \( x_2 > 0 \), and therefore \(-w'' x_2 \leq 1 \) for all \( x_2 > 0 \). As \(-w'' x_2 \) is the relative risk aversion coefficient, this is equivalent to requiring that the relative risk aversion coefficient is no less than 1. Some examples from the family of constant relative risk aversion utility functions are now used to illustrate (A3). When \( w^i (x_2) = \ln x_2 \), \(-w'' x_2 = 1 \).

In this case, \( b^j (\pi) = 0 \). When \( w^i (x_2) = \frac{x_2^{1-\gamma} - 1}{1-\gamma} \), \( \gamma \neq 1 \), then \(-w'' x_2 = \gamma \). Therefore, if \( i \in I_1^+ \), assumption (A3) requires that \( \gamma > 1 \). For the same class of utility functions, if \( i \in I_1^- \), assumption (A3) requires that \( \gamma < 1 \). A similar interpretation can be given for individuals belonging to \( I_2^- \) and \( I_2^+ \).
Under assumption (A3), observe that the assignment of market prices \( \pi^e : I \to \Pi_0 \) that maximizes the value of the numerator in (4) is the one where \( \pi^{ci} = \bar{\pi} \) for all \( i \in I_1^+ \) and \( \pi^{cj} = \bar{\pi} \) for all \( j \in I_2^+ \). Similarly, the assignment of market prices \( \pi^e : I \to \Pi_0 \) that minimizes the value of the denominator in (4) is the one where \( \pi^{cj} = \underline{\pi} \) for all \( j \in I_2^+ \) and \( \pi^{ci} = \underline{\pi} \) for all \( i \in I_1^+ \). It follows that the ratio on the left-hand side of (4) attains its maximum value for this assignment of expectations. Analogously, the assignment of market prices \( \pi^e : I \to \Pi_0 \) that maximizes the value of the denominator in (4) is the one where \( \pi^{cj} = \bar{\pi} \) for all \( j \in I_2^+ \) and \( \pi^{ci} = \bar{\pi} \) for all \( i \in I_1^+ \). Similarly, the assignment of market prices \( \pi^e : I \to \Pi_0 \) that maximizes the value of the denominator in (4) is the one where \( \pi^{cj} = \underline{\pi} \) for all \( j \in I_2^+ \) and \( \pi^{ci} = \underline{\pi} \) for all \( i \in I_1^+ \). It follows that the ratio on the left-hand side of (4) attains its minimum value for this assignment of expectations. We can express the interval in terms of deviations from \( \pi^* \) as \( \bar{\pi} = \pi^* + \varepsilon_1 \) and \( \underline{\pi} = \pi^* - \varepsilon_2 \) for some \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \). Let \( \varepsilon_{1,t} = \varepsilon_1 \) and \( \varepsilon_{2,t} = \varepsilon_2 \). It follows that the bounds of the sequence of intervals \( \Pi_t, t \geq 1 \), is generated by the following iterative map, a two-dimensional difference equation system,

\[
\begin{align*}
\frac{f_1(\varepsilon_{1,t}, \varepsilon_{2,t})}{f_2(\varepsilon_{1,t}, \varepsilon_{2,t})} &= \frac{\int_{I_1^+} b^i(\pi^* + \varepsilon_{1,t}) \, di + \int_{I_2^+} b^j(\pi^* - \varepsilon_{2,t}) \, dj}{\int_{I_2^+} q^j(\pi^* - \varepsilon_{2,t}) \, dj + \int_{I_2^+} q^j(\pi^* + \varepsilon_{1,t}) \, dj} - \pi^* = \varepsilon_{1,t+1}, \quad (6a) \\
\frac{f_2(\varepsilon_{1,t}, \varepsilon_{2,t})}{f_2(\varepsilon_{1,t}, \varepsilon_{2,t})} &= \pi^* - \frac{\int_{I_2^+} q^j(\pi^* - \varepsilon_{2,t}) \, dj + \int_{I_2^+} q^j(\pi^* + \varepsilon_{1,t}) \, dj}{\int_{I_2^+} q^j(\pi^* + \varepsilon_{1,t}) \, dj + \int_{I_2^+} q^j(\pi^* - \varepsilon_{2,t}) \, dj} = \varepsilon_{2,t+1}. \quad (6b)
\end{align*}
\]

What role does assumption (A3) play in our reduction of the coordination dynamics with heterogeneous expectations to (6) in the vicinity of \( \pi^* \)? Remark that when the preferences of individuals fail to satisfy (A3), there will be non-null sets of agents whose best-responses are no longer monotone in prices. As a consequence, we can no longer ensure that the value of the ratio in (4) is maximized by an assignment of expectations constrained to lie on the bounds of the set \( \Pi_0 \). As stated, (A3) is a global restriction on the preferences of individuals. Strictly speaking, for our analysis, we can replace (A3) with the following (local) assumption (A3') there exists a locally unique market equilibrium price \( \pi^* \) with the property that around \( \pi \) there is an open set of prices such that the best-responses of all individuals are monotone and continuously differentiable in that open set. Remark that our reduction of the coordination dynamics with heterogeneous expectations to (6) will continue to apply in the vicinity of a market equilibrium price \( \pi^* \) which satisfies (A3'). We choose not to make assumption (A3') as (A3') restricts both exogenous factors i.e. the fundamentals of the economy, and endogenously determined variables i.e. market equilibria.

Next, we examine the conditions under which (6) is locally stable. At each \( t \geq 1 \), in the coordination dynamics defined by (6), if we start with \( \varepsilon_{1,t} > 0 \) and \( \varepsilon_{2,t} > 0 \), \( \varepsilon_{1,t} > 0 \) and \( \varepsilon_{2,t} > 0 \). This is because at each \( t \geq 1 \), the maximum (respectively, the minimum) value of the ratio on the left-hand side of (4) is by construction, bounded below (respectively, bounded above) by \( \pi^* \). We are interested in the local stability of \( (0,0) \) under (6) for initial conditions which start in the positive orthant and stay there. Remark that under our assumptions, all ratios in (6) are continuously differentiable.
at \( \pi^* \) and hence (6) is itself continuously differentiable at \((0,0)\). It follows that for all \( \varepsilon_1 \) and \( \varepsilon_2 \) (and in particular, \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \)) in the vicinity of \((0,0)\), the local stability of \((0,0)\) under (6) can be deduced from the eigenvalues of its Jacobian evaluated at \((0,0)\). Given a market equilibrium price \( \pi^* \) we define the quantities
\[
\epsilon^b_+ (\pi^*) = \frac{\int_{t_1} b'(\pi^*) \, dt}{B} \quad (\text{respectively, } \epsilon^b_-(\pi^*) = \frac{\int_{t_1} b'(\pi^*) \, dt}{B})
\]
\[
\epsilon^b_+(\pi^*) = \frac{\int_{t_2} q'(\pi^*) \, dt}{Q} \quad (\text{respectively, } \epsilon^b_-(\pi^*) = \frac{\int_{t_2} q'(\pi^*) \, dt}{Q}).
\]
Let \( A_1 = \epsilon^b_+(\pi^*) - \epsilon^b_-(\pi^*) \) (respectively, \( A_2 = \epsilon^q_+ (\pi^*) - \epsilon^q_-(\pi^*) \)). Then, direct computation shows that
\[
\lambda_1^2 - 2\lambda \pi^* A_1 + (\pi^*)^2 [(A_1)^2 - (A_2)^2] = 0.
\]
(6′)

By computation, it follows that (6′) has two distinct real solutions, \( \lambda_1 = \pi^*(A_1 + A_2) \) and \( \lambda_2 = \pi^*(A_2 - A_1) \). The following proposition characterizes the local stability of the coordination dynamics with heterogeneous expectations described by (6). Given the preceding computation, the proof is a standard application of the theory of discrete dynamical systems and is omitted.

**Proposition 5.** Assume preferences satisfy (A3). Consider a market equilibrium price \( \pi^* \). If both \( |\lambda_1| < 1 \) and \( |\lambda_2| < 1 \), there exists an interval \([\underline{\pi}, \overline{\pi}]\) such that both the sequences \( \pi_t = \pi^* + \varepsilon_{1,t} \) and \( \pi_t = \pi^* - \varepsilon_{2,t} \), where \((\varepsilon_{1,t}, \varepsilon_{2,t})\) is generated by the map (6), converges to \( \pi^* \). If either \( |\lambda_1| > 1 \) or \( |\lambda_2| > 1 \), there exists an interval \([\underline{\pi}, \overline{\pi}]\) such that the sequences \( \pi_t = \pi^* + \varepsilon_{1,t} \) and \( \pi_t = \pi^* - \varepsilon_{2,t} \), where \((\varepsilon_{1,t}, \varepsilon_{2,t})\) is generated by the map (6), eventually leaves \([\underline{\pi}, \overline{\pi}]\).

It follows from Proposition 5 that a sufficient condition for the local stability of (6) is that \( |A_2 - A_1| \) be close to zero and \( |A_1| \) be small enough so that \( |A_2| \) is close to zero and \( |\lambda_1| \) less than one. For example, it can be immediately verified that when preferences are such that all \( \pi \in \Pi_+ \) have utility functions \( u^b(x_1, x_2) = \log x_1 + \log x_2 \), \( b^b(\pi) = q^f(\pi) = \frac{1}{2} \) for all \( \pi \in \Pi_+ \) and therefore, \( b^l(\pi) = q^f(\pi) = 0 \) for all \( \pi > 0 \). By computation, it follows that \( \lambda_1 = \lambda_2 = 0 \) is the only solution to (6′) and therefore, the coordination dynamics described by (6) converges in one step.

More generally, in order to interpret the conditions under which (6) is locally stable, it is useful to consider a situation with homogenous expectations i.e. it is common knowledge that \( \pi^{e,i} = \pi^e \in [\underline{\pi}, \overline{\pi}] \) such that \( \pi^e \in [\underline{\pi}, \overline{\pi}] \). Then, (6) reduces to
\[
\frac{\int_{t_1} b'(\pi^e_t) \, dt}{\int_{t_2} q'(\pi^e_t) \, dt} = \pi^{e,i}_{t+1}
\]
in the vicinity of \( \pi^e \). As before, we are interested in finding out whether the sequence of intervals of prices \( \Pi_t \), \( t \geq 0 \), shrink to \( \pi^e \); this is equivalent to requiring that the discrete map defined by (7) is locally stable. Given a market equilibrium price \( \pi^e \) we define the quantities
\[
\epsilon^b(\pi^e) = \frac{\int_{t_1} b'(\pi^e) \, dt}{B} \quad \text{and} \quad \epsilon^q(\pi^e) = \frac{\int_{t_2} q'(\pi^e) \, dt}{Q}. \]
We are now in a
position to state the following proposition that characterises the conditions under which the Nash equilibrium market price $\pi^*$ is locally stable under the dynamics defined by (7). The proof is again an application of the local stability theory of one-dimensional difference equations and is omitted.

**Corollary 6.** Consider a market equilibrium price $\pi^*$ and consider the quantity $a = \pi^* [e^b(\pi^*) - e^q(\pi^*)]$. Then if $|a| < 1$, there exists an interval $[\pi, \tilde{\pi}]$ such that given any $\pi_0 \in [\pi, \tilde{\pi}]$, the sequence $\pi^n_0$ generated by the map (7) converges to $\pi^*$. If $|a| > 1$, there exists an interval $[\pi, \tilde{\pi}]$ such that given any $\pi_0 \in [\pi, \tilde{\pi}]$, the sequence $\pi^n_0$ generated by the map (7) eventually leaves $[\pi, \tilde{\pi}]$.

Note that $\pi^* e^b(\pi^*) = \pi^* \int_{I^+} \frac{b'(\pi^*)}{\pi^*} \, di = \int_{I^+} \frac{b'(\pi^*)}{\pi^*} \, di$ is the elasticity of the market bids with respect to the market price evaluated at the equilibrium price. It is a measure of how responsive market bids are to market prices close to the equilibrium price. Similarly, $\pi^* e^q(\pi^*) = \pi^* \int_{I^-} \frac{q'(\pi^*)}{\pi^*} \, di = \int_{I^-} \frac{q'(\pi^*)}{\pi^*} \, di$ the elasticity of the market offers with respect to the market price evaluated at the equilibrium price. It is a measure of how responsive market offers are to market prices close to the equilibrium price. The quantity $a$ is a measure of the difference in the response of the two sides of the market to price changes in the vicinity of an equilibrium price. The condition $|a| < 1$ puts a bound on this difference between the elasticity of market bids and the elasticity of market offers in the vicinity of an equilibrium.

Next, we compare the conditions under which the local stability obtains for homogeneous expectations with the corresponding conditions under which local stability obtains under heterogeneous expectations.

**Proposition 7.** Assume preferences satisfy (A3). Then (i) $|\lambda_1| > |a|$ while $|\lambda_2| = |a|$; (ii) if $\mu(I^-) = \mu(I^+) = 0$, the dynamics described by (6) degenerates into a one-dimensional map whose local stability coincides with (7); (iii) if $\mu(I^+) = \mu(I^-) = 0$, then $|\lambda_1| = |\lambda_2| = |a|$ and although the dynamics described by (6) is two dimensional, the local stability conditions coincide with that of (7); (iv) suppose either (a) $\mu(I^+) > 0$ and $\mu(I^-) > 0$ or (b) $\mu(I^-) > 0$ and $\mu(I^+) > 0$ holds, then $|\lambda_1| > |a|$.

The preceding proposition shows that the local stability of (6) is, in general, more demanding than the locally stability of (7).\(^8\) Note further that the conditions for local stability for (6) and (7) coincide when both $I^+_1$ and $I^-_2$ are sets of measure zero. The

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\(^8\)Our analysis of heterogenous expectations is very different from the analysis of Evans and Guesnerie [5] who also study the effect of introducing heterogenous expectations in a linear rational expectations model. They show that heterogenous expectations makes the stability properties of the coordination dynamics harder if agents are heterogenous. Here, in contrast, as agents are, by definition, heterogenous the link between the stability of the coordination dynamics with homogenous and heterogenous expectations is more subtle.
following example,\(^9\) where \(\mu(I_1^+) = \mu(I_2^-) = 0\), therefore demonstrates that the local stability of (6) and (7) are both robust possibilities.

**Example 8.** Set \(I_1 = I_2 = [0, 1]\) with \(w^i = (1, 0)\), \(u^i(x) = x_1 + x_1^2\), \(0 < \alpha < 1\), for all \(i \in I_1\) and \(w^j = (0, 1)\), \(u^j(x) = x_2 + x_1^n\), \(0 < \beta < 1\), for all \(j \in I_2\). From the first-order conditions we get that for all \(i \in I_1\), \(b^i = \alpha^{-1} - \frac{1}{\alpha - 1} - \frac{x_1^2}{1 - \gamma} = B\), \(\pi^i(\pi) = \frac{\pi}{\alpha(\alpha - 1)}\) while for all \(j \in I_2\), \(q^j = \beta^{1-\beta} - \frac{\beta}{1-\beta} - \frac{p}{1-\beta} = Q\), \(\pi^j(\pi) = \frac{\beta}{\pi(1-\beta)}\) with \(\pi^* = \frac{1}{\beta} - \frac{1}{\beta(1-\beta)}\), \(\mu(I_1^+) = \mu(I_2^-) = 0\) and \(|\lambda| = |\alpha| = \left|\frac{\beta}{\pi(1-\beta)}\right|\), if \(0 < \alpha < 1\), \(|\alpha| < 1\) while if \(\frac{1}{2} < \alpha < 1\), \(|\alpha| > 1\). More generally, as long as \(\frac{\alpha}{1-\gamma} + \frac{\beta}{1-\beta} > \frac{1}{2}\), \(0 < \alpha < 1\), \(0 < \beta < 1\), (6) and (7) are locally stable while if \(\frac{\alpha}{1-\gamma} + \frac{\beta}{1-\beta} > \frac{1}{2}\), (6) and (7) are locally unstable.

The following example however shows that the conditions under which (7) is locally stable while (6) is locally unstable is also a robust possibility.

**Example 9.** Set \(I_1 = I_2 = [0, 1]\). Partition \(I_1\) into two sets of equal measure \(I'_1, I''_1\). For \(i \in I'_1\), set \(w^i = (1, 0)\) and \(u^i(x) = x_1 + x_1^\gamma\), \(\alpha > 1\), while for \(i \in I''_1\), set \(w^i = (1, 0)\) and \(u^i(x) = x_1 + x_1^\gamma\), \(0 < \gamma < 1\). For all \(j \in I_2\), set \(w^j = (0, 1)\) with \(u^j(x) = x_2 + x_1^\gamma\), \(0 < \delta < 1\). From the first-order conditions we get that for all \(i \in I'_1\), \(b^i = \pi^{\frac{\gamma-1}{\beta}}\), \(\pi^i(\pi) = \frac{\pi}{\beta} - \frac{1}{\beta(\beta - 1)}\), while for all \(i \in I''_1\), \(b^i = \frac{\gamma}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} = \frac{1}{\beta}\), \(\pi^i(\pi) = \frac{\beta}{\pi(\beta - 1)}\), with \(B = \frac{1}{\beta} - \frac{1}{\beta(\beta - 1)}\) and \(\frac{1}{\beta}\). For \(j \in I_2\), from the first-order conditions we get that \(q^j = \pi^{\frac{1-\delta}{\delta}}\), \(\pi^j(\pi) = \frac{\beta^{1-\delta}}{\beta(\beta - 1)}\), \(\frac{1}{\beta}\). It follows that \(I'_1 = I_1^+\) while \(I''_1 = I_1^-\) and \(I_2 = I_2^+\). Market equilibrium prices are given by the solutions to the equation \(\pi^* = f(\pi) = \frac{B(\pi)}{Q(\pi)} = \left(\frac{\gamma-1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta} - \frac{1}{\beta}\right)\). Evidently, \(\pi^* = 1\) is a solution to this equation and as long as \(\frac{\gamma-1}{\beta} + \frac{1}{\beta} - \frac{1}{\beta} > 0\), it is also locally unique. At \(\pi^* = 1\), \(|\alpha| = \left|\frac{\gamma-1}{\beta} + \frac{1}{\beta} - \frac{1}{\beta}\right|\) while \(|\lambda| = \left|\frac{\gamma-1}{\beta} + \frac{1}{\beta} - \frac{1}{\beta}\right|\). For instance, when \(\alpha = 3, \gamma = \frac{1}{2}, \delta = \frac{3}{4}\), \(|\alpha| = \frac{3}{2} < 1\) while \(|\lambda| = 2 > 1\), implying that although (7) is locally stable, (6) fails to be locally stable. Further, both \(\alpha\) and \(\lambda\) (and therefore, \(|\alpha|\) and \(|\lambda|\)) are continuous in \(\alpha, \gamma, \) and \(\delta\). It follows that the above property is robust to small perturbations in the utility parameters \(\alpha, \gamma, \) and \(\delta\).

\(^9\)At this stage, we remind the reader that all examples in this section and the following section satisfy (A1(ii)) and not (A1(ii)).
Finally, the following remark makes a link between (7) and discrete-time tatonnement.

**Remark 10.** Fix a market equilibrium price $\pi^*$. The classical discrete-time tatonnement dynamics (see for instance, [7, pp. 233–235]), in the vicinity of $\pi^*$, is described by the map

$$
\pi_{t+1} = \pi_t + \gamma \left( \int_I x^n_I(1, \pi_t) \, dn - \hat{w}_2 \right). \quad \gamma > 0.
$$

(8)

Although the choice of $\gamma$ the speed of adjustment of the discrete-time tatonnement dynamics is arbitrary and cannot be pinned down by specifying the fundamentals of the economy, Hildenbrand and Kirman [7] point out that diminishing the adjustment speed $\gamma$ smooths out the price path generated by iterated applications of the map (8). From the allocation rules, we obtain that

$$
x^n_I(1, \pi_t) \, dn - \pi_t \int_I b^j(\pi_t) \, dj.
$$

Using this expression, the right-hand side of (8) can be rewritten as

$$
s(\pi_t, \gamma) = \pi_t + \frac{\gamma}{\pi_t} (\int_I b^j(\pi_t) \, dj - \pi_t \int_I q^j(\pi_t) \, dj).$$

By computation, observe that $s'(\pi^*, \gamma) = 1 + \frac{\gamma}{\pi^*}(a-1)$. It follows that if $a > 1$, for all $\gamma > 0$, $s'(\pi^*, \gamma) > 1$ while if $a < 1$, there exits $\gamma > 0$ such that if $\gamma < \gamma^*$, $|s'(\pi^*, \gamma)| < 1$. Moreover, if $\gamma = \frac{\gamma^*}{\pi^*}$, the local stability of (8) coincides with the local stability of (7). For an arbitrary choice of $\gamma$, local stability under (7) does not imply or rule out the local stability of (8).

4. Coordination in the sequential move market game

We restrict attention to only those subgame perfect equilibria that support market equilibrium allocations and study a coordination problem that is faced by the agents who move in the first stage of the sequential game. At a subgame perfect equilibrium, agents $i \in I_1$ anticipate correctly the market offer resulting from the strategies of $j \in I_2$. We ask what happens when the expected market bid for all $i \in I_1$ is not equal to the subgame perfect equilibrium market bid. Specifically, we assume that it is common knowledge that all individual agent’s expected market bid lies in some interval that contains exactly one subgame perfect equilibrium market bid. We make the assumption that each $i \in I_1$ anticipates correctly that all $j \in I_2$ make offers according to $\tilde{q}^j$. We, then, construct a specific dynamic process, based on the best-response functions of $i \in I_1$, that generates an iterated sequence of such intervals. We investigate conditions under which such a sequence converges to the subgame perfect equilibrium market bid. In Section 4.1 we study the case with homogeneous expectations while in Section 4.2 we study the case with heterogeneous expectations. Throughout this section, we restrict attention to subgame perfect equilibria which (a) satisfy the regularity condition $\pi^* \in d(\pi^*) \neq -1$ at the subgame perfect equilibrium price $\pi^*$ and (b) supports some market equilibrium allocation $x^*$. 
4.1. Stage 1 coordination dynamics

It worth noting that there are assignments of subgame perfect equilibrium strategies for which there will always be a coordination problem in stage 1. For instance, consider the following \textit{knife-edge} strategies, where at stage 2, the offer function of each \( j \in I_2 \), has the form: \( \tilde{q}^{ij} = Q^* \) if \( B = B^* \); otherwise \( \tilde{q}^{ij} = 0 \). Under the allocation rules of the market game with a continuum of players, as in our formulation, this assignment of strategies supports \( \pi^* = \frac{B^*}{\int_{j \in I_2} \tilde{q}^{ij}dj} = \frac{B^*}{Q^*} \), where \( \pi^* \) is a competitive equilibrium price, as a subgame perfect equilibrium price. However, observe that with these strategies, there will always be a coordination problem in stage 1. If traders in stage 1 expect that the market bid \( B^* \) is different from \( B^* \), they anticipate that the response of traders in the second stage of the game will be to set \( \tilde{q}^{ij} = 0 \) for all \( j \in I_2 \), resulting in final allocations equal to their initial endowments. It follows that the only best-response for each \( i \in I_1 \), is to put \( b^i = 0 \). But, then, traders will never converge back to \( B^* \). Therefore, in order to admit the possibility of coordination in stage 1, we restrict ourselves to assignments of subgame perfect equilibrium strategies where, in stage 2, traders submit offer functions that are locally continuously differentiable in the market bid.

Let \((B^*, \tilde{q}^*)\) denote a subgame perfect equilibrium and let \([B, \tilde{B}]\) be an interval that contains \( B^* \). We assume that all agents expect the same market price \( B^*_t \in [B, \tilde{B}] \). Given \( B^*_0 \), each \( i \in I_1 \) submits a bid \( b^i \left( \frac{B^*_i}{\int_{j \in I_2} \tilde{q}^{ij} (B^*_i) dj} \right) \), which in turn generates a new market bid for each \( t > 0 \) denoted by \( B^*_t \) according to

\[
\int_{i \in I_1} b^i \left( \frac{B^*_i}{\int_{j \in I_2} \tilde{q}^{ij} (B^*_i) dj} \right) di = B^*_t. \tag{9}
\]

We study the dynamics generated by the iterative map given by (9). As discussed immediately after Lemma 2, we set \( \tilde{q}^{ij} (B) = q^j (\frac{B}{Q(B)}) \). Therefore, for the purpose of studying local coordination dynamics around subgame perfect equilibria that support competitive equilibrium allocations, we can replace \( \tilde{q}^{ij} (B) \) by \( q^j (\frac{B}{Q(B)}) \) in (9). Further, as we must have \( \int_{j \in I_2} q^j (\frac{B}{Q(B)}) dj = Q(B) \) for all \( B \in [B^* - \epsilon, B^* + \epsilon] \) for some \( \epsilon > 0 \), we can replace (9) with (9'):

\[
\int_{i \in I_1} b^i \left( \frac{B^*_i}{Q(B^*_i)} \right) di = B^*_t. \tag{9'}
\]

Given a subgame perfect equilibrium price \( \pi^* \) we define the quantity \( \tilde{\alpha} = \frac{\pi^* e^h(\pi^*)}{1 + \pi^* e^h(\pi^*)} \). It can be verified by direct computation that the derivative of the map (9') evaluated at \( \pi^* \) is the quantity \( \tilde{\alpha} \). We are now in a position to state the following proposition that characterizes the conditions under which the subgame perfect equilibrium market bid \( B^* \) is locally stable under the dynamics defined by (9'). As in Section 3, with homogenous expectations, the shrinking of intervals depends on the stability of
(9). Again, the proof is a standard application of the local stability theory of one-dimensional difference equations and is omitted.

**Proposition 11.** Consider a subgame perfect equilibrium market bid \( B^* \) and price \( \pi^* \). If \( |\bar{a}| < 1 \), there exists an interval \( [B, \bar{B}] \) such that given any \( B_0 \in [B, \bar{B}] \), the sequence \( B'_i \) generated by the map (9) converges to \( B^* \). If \( |\bar{a}| > 1 \), there exists an interval \( [B, \bar{B}] \) such that given any \( B_0 \in [B, \bar{B}] \), the sequence \( B'_i \) generated by the map (9) eventually leaves \( [B, \bar{B}] \).

Next, we compare the conditions under which the coordination dynamics in the simultaneous move market game, described by (7), is locally stable, to the conditions under which the coordination dynamics in the sequential move market game, described by (9’), is locally stable.

**Proposition 12.** Assume preferences satisfy (A3). (i) If \( |a| < 1 \) and \( \mu(I^+_1) = \mu(I^-_2) = 0 \), then \( |\bar{a}| < 1 \); (ii) If \( |a| < 1 \) and \( \mu(I^-_1) = \mu(I^+_2) = 0 \), then \( |\bar{a}| < 1 \).

The following example shows, under the same assumptions of the preceding proposition, it is a robust possibility that (9’) is locally stable while (7) is locally unstable.

**Example 13.** Consider the same economy as in Example 8. Observe that \( \mu(I^+_1) = \mu(I^-_2) = 0 \), while \( |\bar{a}| = \frac{1}{\alpha(1-\beta)} \left[ \frac{1}{\alpha^2} + \frac{\beta}{1-\delta} \right] = |a| \) as \( 0 < \beta < 1 \). If \( \beta > \frac{1}{2} \), \( |a| > 1 \) while \( \alpha < \frac{1}{2-\beta} \). \( |\bar{a}| < 1 \).

What happens when both sides of the market behave in the same way, i.e. when both market bids and market offers are increasing in market prices or when both market bids and market offers are decreasing in market prices? The following example shows that it is a robust possibility that (9’) may be locally unstable, while (7) may be locally stable.

**Example 14.** Set \( I_1 = I_2 = [0, 1] \). For \( i \in I_1 \), let \( w'_1 = (1, 0) \) and \( u'_i(x) = x_1 + \frac{x^{1-a}}{1-x} \), \( 0 < \alpha < 1 \), while for \( j \in I_2 \), set \( w'_2 = (0, 1) \) with \( u'_j(x) = x_2 + \frac{x^{1-\delta}}{1-\delta} \), \( \delta > 1 \). From the first-order conditions we get that for all \( i \in I_1 \), \( \alpha^j = \frac{\alpha^1 - \beta}{\alpha} = B \), \( \alpha^j = \frac{\alpha^1 - \beta}{\alpha} = \frac{\alpha^{1-\delta}}{1-\delta} \). For \( j \in I_2 \) we get that \( q^j = \frac{1}{\alpha} = Q \) and \( q^j = \frac{1-\delta}{\alpha} = \frac{1-2\delta}{\alpha} \). It follows that \( I_1 = I^-_1 \) and \( I_2 = I^-_2 \). Therefore, both market bids and market offers are decreasing in the market price \( \pi \). There is a unique market equilibrium price \( \pi^* = 1 \). A computation shows that at \( a = \frac{\alpha^{1-\delta}}{\alpha} - \frac{1}{\alpha} \) while \( \bar{a} = \frac{\alpha^{1-\delta}}{1+\frac{1}{\alpha}} \). Observe that when \( \alpha = \frac{3}{\delta} \), \( \bar{a} = -2 \), \( |\bar{a}| > 1 \), implying that (9’) is locally unstable. Further, as both \( a \) and \( \bar{a} \) and therefore, \( |a| \) and \( |\bar{a}| \) are continuous in \( a \) and \( \delta \), this property is robust to small perturbations in the utility parameters \( \alpha \) and \( \delta \).
4.2. Coordination and local extensive-form rationalizability

Here, we study the link between the coordination dynamics in the market game with sequential moves and extensive-form rationalizability. In addition to assumptions (A1) and (A2), we will find it convenient to assume (A3) as well. A preliminary complication that we need to deal with arises from the fact that, as stated, the coordination dynamics described by (9') applies only to the case with homogeneous expectations. Let $(\mathbf{B}^e, \tilde{q}^e)$ denote a subgame perfect equilibrium and let $[\tilde{B}, \bar{B}]$ be an interval that contains $\mathbf{B}^e$. As before, we assume that it is common knowledge that every trader in $I_1$ has some point expectation of the market bid denoted by $\mathbf{B}^{e,i} \in [\tilde{B}, \bar{B}]$. Remark that in the sequential market game, all traders $j \in I_2$ observe the same market bid resulting from stage 1. As this fact is common knowledge for all traders in stage 1, it follows that given a configuration of market bids $\{\mathbf{B}^{e,i} : i \in I_1\}$, each $i \in I_1$ submits a bid $b^i(\int_{j \in I_2} \tilde{q}^j(\mathbf{B}^{e,i}) dj)$ according to admissible configurations $\{\mathbf{B}^{e,i} : i \in I_1\}$ such that $\int_{i \in I_1} b^i(\int_{j \in I_2} \tilde{q}^j(\mathbf{B}^{e,i}) dj) di$ is well-defined i.e. each admissible $\{\mathbf{B}^{e,i} : i \in I_1\}$ generates, in analogy with (4), a new market bid for each $t > 0$ denoted by $\mathbf{B}$ according to

$$\int_{i \in I_1} b^i\left(\frac{\mathbf{B}^{e,i}}{\tilde{q}^i(\mathbf{B}^{e,i})}ight) di = \mathbf{B}. \quad (10)$$

As before, using Lemma 2, we must have $\int_{j \in I_2} q^j(\frac{B_j}{Q(B_j)}) dj = Q(B)$ for all $B \in [\tilde{B}, \bar{B}]$. Therefore, for all $B \in [\tilde{B}, \bar{B}]$, we can replace (10) with (10'):

$$\int_{i \in I_1} b^i\left(\frac{\mathbf{B}^{e,i}}{Q(\mathbf{B}^{e,i})}\right) di = \mathbf{B}. \quad (10')$$

In what follows, we assume that $\mu(I^+_1) = \mu(I^-_2) = 0$ which implies that $\epsilon^i(\pi) > 0$ and $b^i(\pi) < 0$ for all $i \in I_1$, for all $\pi$ in the vicinity of $\pi^e$. Therefore, $\frac{db^i}{d\pi} = \frac{\pi^{e^i}(\pi)}{(1 + \pi^{e^i}(\pi))}$ for all $\pi$ in the vicinity of $\pi^e$. It immediately follows that $\int_{i \in I_1} b^i\left(\frac{\mathbf{B}^{e,i}}{Q(\mathbf{B}^{e,i})}\right) di$ is maximized by setting $\mathbf{B}^{e,i} = \tilde{B}$ and minimized by setting $\mathbf{B}^{e,i} = \bar{B}$. As before, we express the interval in terms of deviations from $\mathbf{B}^e$ as $\tilde{B} = \mathbf{B}^e + \varepsilon_1$ and $\bar{B} = \mathbf{B}^e - \varepsilon_2$ for some $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Let $\varepsilon_{1,0} = \varepsilon_1$ and $\varepsilon_{2,0} = \varepsilon_2$. It follows that the bounds of the sequence of intervals $[\tilde{B}_t, \bar{B}_t]$, $t \geq 1$, is generated by the following iterative map, a two-dimensional difference equation system,

$$\tilde{f}_1(\varepsilon_{1,t}, \varepsilon_{2,t}) = \int_{i \in I_1} b^i\left(\frac{\mathbf{B}^e - \varepsilon_{2,t}}{Q(\mathbf{B}^e - \varepsilon_{2,t})}\right) di - \mathbf{B}^e = \varepsilon_{1,t+1}, \quad (11a)$$

$$\bar{f}_2(\varepsilon_{1,t}, \varepsilon_{2,t}) = \mathbf{B}^e - \int_{i \in I_1} b^i\left(\frac{\mathbf{B}^e + \varepsilon_{1,t}}{Q(\mathbf{B}^e + \varepsilon_{1,t})}\right) di = \varepsilon_{2,t+1}. \quad (11b)$$
By computation, the characteristic polynomial of the dynamical system (11), evaluated at the Nash equilibrium price $\pi^*$, is given by

$$v^2 = \left( \frac{\pi^* c_b(\pi^*)}{1 + \pi^* \epsilon^b(\pi^*)} \right)^2.$$

(11')

But, then, $|v| = |\tilde{a}|$. We summarize this discussion as the following lemma.

**Lemma 15.** Assume preferences satisfy (A3). Then, if $\mu(I_1^+) = \mu(I_2^-) = 0$, then $|v| = |\tilde{a}|$ and although the coordination dynamics under heterogeneous expectations given by (11a) and (11b) is two dimensional, the local stability conditions are the same as in the case with homogeneous expectations given by (9').

Next, we compare the coordination dynamics defined by (9') to local extensive-form rationalizability. Fix a subgame perfect equilibrium market bid and assignment of offer functions. Heuristically, local extensive-form rationalizability requires that (a) for every market bid in the vicinity of the subgame perfect equilibrium market bid, type 2 traders are able to coordinate on the assignment of subgame perfect equilibrium offer functions and (b) given that type 2 traders are choosing offer functions that “close” (in an appropriately chosen topology, see appendix for details) to their subgame perfect equilibrium offer functions, type 1 traders are able to coordinate on the subgame perfect equilibrium market bid. A formal definition of local extensive-form rationalizability is postponed to the appendix. For the moment, we note that our definition of local extensive-form rationalizability differs from the definition of extensive-form rationalizability in [9] in two ways. One, it is a local definition as the initial restrictions on strategies are required for each trader to lie in a neighbourhood of their subgame perfect equilibrium strategy assignments. Two, it restricts attention to pure strategies. The following corollary compares the coordination dynamics defined by (9') to local extensive-form rationalizability under the assumption that $\mu(I_1^+) = \mu(I_2^-) = 0$. It shows that whenever the subgame perfect equilibrium $(b^*, q^*)$ is locally extensive-form rationalizable, the coordination dynamics described by (11) (and hence, (9)) converges.

**Corollary 16.** Suppose (A1)–(A3) is satisfied and $\mu(I_1^+) = \mu(I_2^-) = 0$. Fix a subgame perfect equilibrium $(b^*, q^*)$ with $\pi^*$ the associated market price. Then, if there exist non-trivial initial restrictions for each $i \in I_1$, (respectively, $j \in I_2$) $\tilde{S}_0$ (respectively, $\tilde{Q}_{S,0}$), such that the subgame perfect equilibrium $(b^*, q^*)$ is locally extensive-form rationalizable, then there is some neighbourhood $[\bar{B}, \tilde{B}]$ around $B^*$ such that the sequence generated by the coordination dynamics (9') converges to $B^*$.

The intuition for this result is as follows. Local extensive-form rationalizability implies that in stage 1, agents have to converge to $b^*$ even when they anticipate that agents in stage 2 are using any assignment of offer functions that are in the vicinity (in an appropriate chosen topology, see appendix for details) of the subgame perfect
equilibrium assignment of offer functions \( \tilde{q}^* \). In particular, this implies that in stage 1, agents are able to converge to \( b^* \) when they anticipate that in stage 2 agents use the subgame perfect equilibrium assignment of offer functions \( \tilde{q}^* \). The result immediately follows.

Establishing the converse of the preceding corollary is more problematic. In fact, it can be shown that the common knowledge restrictions implied by the coordination dynamics (11) (and hence, (9')) are stronger than the corresponding common-knowledge restrictions implied by local extensive-form rationalizability.

4.3. Coordination and cobweb dynamics

In our set-up, for traders in \( I_1 \) at all positive prices \( p \), the optimal choice of commodity bundles imply an excess supply of commodity 1 and an excess demand for commodity 2; reciprocally, for traders in \( I_2 \) at all positive prices \( p \), the optimal choice of commodity bundles imply an excess supply of commodity 2 and an excess demand for commodity 1. Therefore, traders in \( I_1 \) supply commodity 1 while traders in \( I_2 \) demand commodity 1 and traders in \( I_1 \) demand commodity 1 while traders in \( I_2 \) supply commodity 1. It follows that there are two separate possibilities for the study of cobweb dynamics. The classical model of cobweb dynamics is one where supply reacts to changes in prices only after a given time lag while prices themselves adjust instantaneously to equate the quantity supplied to demand. For \( l = 1, 2 \), let \( \tilde{x}_{t,t} = S_l(p_{t-1}) = D_l(p_t) \) denote the market quantities traded at the \( t-th \) iteration of the cobweb dynamics in the market for commodity \( l \), where \( S_l(p_t) \) (respectively, \( D_l(p_t) \)) denotes the market excess supply for commodity \( l \) (respectively, market excess demand for commodity \( l \)). Let \( \tilde{x}_l(p) \) denote the excess demand (equivalently, excess supply) of commodity \( l \) at a competitive equilibrium price \( \hat{p} \). Then, the map describing the cobweb dynamics\(^{12}\) in the market for commodity \( l \), \( l = 1, 2 \), in the vicinity of the competitive equilibrium excess demand, is given by

\[
(\tilde{x}_{t,t} - \tilde{x}_l(p)) = \frac{S_l(p)}{D_l(p)} (\tilde{x}_{t,t-1} - \tilde{x}_l(p)).
\]

(8')

Evidently, the local stability of the cobweb dynamics in the market for commodity \( l \) is determined by the absolute value of the ratio \( \frac{S_l(p)}{D_l(p)} \). Using the allocation rules, a direct computation shows that \( \left| \frac{S_l(p)}{D_l(p)} \right| = \frac{1}{1 + \pi^e(\pi^c)} = |\tilde{a}| \). The following corollary summarizes the preceding discussion.

**Corollary 17.** Fix a market equilibrium assignment of allocation \( x^* \) and price \( \pi^* \). Then, the cobweb dynamics (8') is locally stable if and only if the coordination dynamics described by (9') is locally stable.

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\(^{10}\) An explicit example demonstrating this point is available from the authors.

\(^{11}\) We remind the reader that \( p = \frac{p_2}{p_1} \).

\(^{12}\) The specification of the cobweb dynamics follows [13, pp. 265].
In the sequential move market game the coordination dynamics described by \((9')\) requires traders in \(I_1\) to solve a simultaneous move coordination problem, anticipating that traders in \(I_2\) submit offers according to the function \(Q(B)\). This is consistent with the sequence of moves implicit in the description of the cobweb dynamics in the market for commodity 1.

**Remark 18.** The above corollary provides a strategic foundation for the stability analysis in [6]. In the cobweb dynamics for the market for commodity \(l, l = 1, 2\), supply adjusts only after a one-period time lag while prices adjust instantaneously to clear the market. This adjustment process requires that the supply side of the market moves before the demand side of the market. Guesnerie [6] studied the coordination problem for competitive suppliers, who face an exogenous demand curve, and who have to decide on the quantity they choose to supply before prices adjust to clear the market. He showed that the stability of the coordination dynamics studied by him (defined as eductive stability) coincided with that of the cobweb dynamics. In our setup, the dynamics described by \((9')\) corresponds to the sequential market game where type 1 traders choose their bids (equivalently, the quantity they choose to supply of commodity 1) and the behaviour of type 2 traders is summarized by the market offer curve. Applying the definition of eductive stability to the market for commodity 1, would, then, yield the same (local) stability analysis as the coordination dynamics described by \((9')\). Evidently, the analysis of the dynamics studied by eductive stability is very different from the local stability of the coordination dynamics in the simultaneous-move market game studied by (4).

### 5. Conclusion and a discussion of possible extensions

We reformulate the stability analysis of competitive equilibria as a coordination problem in a market game whose non-cooperative equilibria coincide with competitive equilibria. In the market game, the map which associates market prices to the best-responses of all traders is common knowledge and is well-defined both in and out of equilibrium. Initial beliefs over market variables, like prices, differ from the equilibrium values of the same market variables. This creates a coordination problem as traders use the structure of the game to converge back to equilibrium. We derive and study the resulting coordination dynamics in both simultaneous and sequential move versions of the market game and are able to link, in a common framework, the local stability analysis of discrete-time tatonnement (respectively, cobweb dynamics) with local rationalizability (respectively, local extensive-form rationalizability). Several open questions remain which we intend to pursue in future research: (1) extending the analysis of coordination dynamics to stochastic beliefs; (2) the generalization of our analysis of coordination to more than two commodities and continuous time: with three or more commodities and continuous time, we can study better the link with tatonnement stability; (3) the study of coordination dynamics in large finite economies and relating it to the coordination dynamics in the limit case studied by us here.
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Appendix A

A.1. Proofs of propositions

Proof of Proposition 7. (i) By comparing the expressions for $a$, $A_1$ and $A_2$, it follows that $|\pi'(A_2 - A_1)| = |a|$. Therefore, $|\lambda_2| = |a|$. As $|\lambda_1| = |\pi'(A_2 + A_1)|$, as both $A_1 \geq 0$ and $A_2 \geq 0$,

$|\pi'(A_2 + A_1)| \geq |\pi'(A_2 - A_1)| = |a|.

(ii) As $\mu(I_1^+) = \mu(I_2^-) = 0$, observe that the ratio in the middle equality of (6a) is equal to the ratio on the left-hand side of (7) when $\pi_0^e = \pi$. Similarly, the ratio in the middle equality of (6b) is equal to the ratio on the left-hand side of (7) when $\pi_0^e = \pi$. This implies that the coordination dynamics described by (6a) and (6b) degenerates to the corresponding coordination dynamics given by (7) when $|a| \neq 1$.

(iii) When $\mu(I_1^+) = \mu(I_2^-) = 0$, $A_1 = 0$. Therefore, $|\lambda_1| = |\lambda_2| = |A_2 \pi^e| = |a|$.

(iv) When either condition (a) or condition (b) holds, we must have either $A_1 > 0$ or $A_2 > 0$ or both, which, in turn, implies that $|(A_2 + A_1) \pi^e| > |a|$ and therefore, $|\lambda_1| > |a|$.

Proof of Proposition 12. As $\tilde{a} = \frac{\pi' \bar{e}(\pi^e)}{1 + \pi' \bar{e}(\pi^e)}$, it follows that $\tilde{A} = \tilde{a} - a = \frac{\pi' \bar{e}(\pi^e)}{1 + \pi' \bar{e}(\pi^e)} [1 - a]$.

We want to check that $|a| < 1$ implies that $|\tilde{a}| < 1$ as well. Suppose (i) holds where, by assumption, $\mu(I_1^+) = \mu(I_2^-) = 0$. Then, $e^b(\pi^e) < 0$ while $e^g(\pi^e) > 0$. It follows that $a < 0$, $\tilde{a} > 0$ which in turn implies that $\tilde{A} > 0$ and therefore, $\tilde{a} > a$ or $|\tilde{a}| < |a|$ and therefore, if $|a| < 1$, then $|\tilde{a}| < 1$. Now, suppose (ii) holds where, by assumption, $\mu(I_1^-) = \mu(I_2^+) = 0$. It follows that $e^b(\pi^e) > 0$ while $e^g(\pi^e) < 0$. This implies that $a > 0$.

There are two sub-cases to consider: (a) where $1 + \pi' e^g(\pi^e) < 0$ and (b) where $1 + \pi' e^g(\pi^e) > 0$. When (a) obtains, $-\pi' e^g(\pi^e) > 1$, and as $e^b(\pi^e) > 0$, $a > 1$, which contradicts the assumption that $|a| < 1$. When (b) obtains, we have $-\pi' e^g(\pi^e) < 1$, and as $a < 1$, we have $\tilde{a} > 0$, $\tilde{A} < 0$, $\tilde{a} < a$ and therefore, $|\tilde{a}| < 1$. □

A.2. The definition of (local) extensive-form rationalizability

As in Section 4, we restrict attention to subgame perfect equilibria which (a) satisfy the regularity condition $\pi' e^g(\pi^e) \neq -1$ at the subgame perfect equilibrium price $\pi'$.
and (b) supports some market equilibrium allocation $x^*$. For each $i \in I_1$, consider $S_0^i \subset S_i$ such that $b^* \in S_0^i$ with $S_0^i$ as in the definition of local rationalizability. It follows that there exists an interval $[\bar{B}, \tilde{B}]$ around $B^*$ such that $B \in \mathcal{B}_0 = [\bar{B}, \tilde{B}]$ if and only if $B = \int_{I_1} b^i \, di$ for some assignment $b : I_1 \rightarrow \times_{i \in I_1} S_0^i$. For the purposes of defining (local) extensive-form rationalizability, without loss of generality, we may restrict the strategy space of each $j \in I_2$, to be continuously differentiable offer functions $\tilde{q}_j : \mathcal{B}_0 \rightarrow S_j$ with $\mathcal{B}_0$. At a subgame perfect equilibrium $(\mathbf{B}^*, \mathbf{\tilde{q}}^*)$, each trader $j \in I_2$ is choosing a continuously differentiable offer function $\tilde{q}_j^* (B) = q_j^* \left( \frac{B}{Q_j(B)} \right)$ where $\int_{I_2} q_j^* \left( \frac{B}{Q_j(B)} \right) \, dj = Q(B)$ for all $B$ in the vicinity of $B^*$. Let $\tilde{Q}_S^j$ be the set of all continuously differentiable offer functions that are feasible for each $j \in I_2$. We endow the set $\tilde{Q}_S^j$ with the topology of $C^1$ uniform convergence on compacta for each $j \in I_2$. Remark that as $\mathcal{B}_0$ is compact, this is equivalent to the $C^1$-topology.\(^{13}\) Let $\tilde{Q}_S = \times_{j \in I_2} \tilde{Q}_S^j$. An assignment of continuously differentiable offer functions is $\tilde{\mathbf{q}}$, $\tilde{\mathbf{q}} \in \tilde{Q}_S$, such that for each $B$ in the vicinity of $B^*$, the integral $\int_{I_2} \tilde{q}_j^* (B) \, dj = \tilde{Q}(B)$ exists and moreover, $\tilde{Q}^j (B) = \int_{I_2} \tilde{q}_j^* (B) \, dj$. In the norm associated with the $C^1$-topology, let $\tilde{Q}_S^{j*} (\mathbf{\tilde{q}}^{j*}, \epsilon)$ denote the set of all continuously differentiable offer functions that lie within an $\epsilon$-neighbourhood of $\mathbf{\tilde{q}}^{j*}$ for each $j \in I_2$. Let $\tilde{Q}_{S0}^j \subset \tilde{Q}_S^{j*} (\mathbf{\tilde{q}}^{j*}, \epsilon)$ for some $\epsilon > 0$ denote an initial restriction on the set of offer functions for each $j \in I_2$. For each $B \in \mathcal{B}_0$ and each assignment of continuously differentiable offer functions is $\tilde{\mathbf{q}}$, $\tilde{\mathbf{q}} \in \tilde{Q}_{S0}$, there corresponds a market offer $\tilde{Q}(B) = \int_{I_2} \tilde{q}_j^* (B) \, dj$. Moreover, as we are using the norm associated with the $C^1$-topology to define neighbourhoods, for every $\delta_1 > 0$, $\delta_2 > 0$, for each $B \in \mathcal{B}_0$, there exists $\epsilon > 0$ such that if for all $j \in I_2$, $\tilde{q}_j \in \tilde{Q}_S^{j*} (\mathbf{\tilde{q}}^{j*}, \epsilon)$, then $| \tilde{Q}(B) - \tilde{Q}^j (B) | < \delta_1$ and $| \tilde{Q}(B) - \tilde{Q}^j (B) | < \delta_2$. For each $B \in \mathcal{B}_0$, let $Q_0(B) = \{ Q \in \mathfrak{R}_{++} : \exists \text{ an assignment of continuously differentiable functions } \tilde{q}, \tilde{q} \in \tilde{Q}_{S0}, \text{ s.t. } Q = \int_{I_2} \tilde{q}_j^* (B) \, dj \}$. For $n \geq 1$, given $B \in \mathcal{B}_0$, let $Q_n (B) = \{ Q \in \mathfrak{R}_{++} : \exists \text{ an assignment of continuously differentiable functions } \tilde{q}, \tilde{q} \in \tilde{Q}_{S0}, \text{ s.t. } Q = \int_{I_2} \tilde{q}_j^* (B) \, dj \}$. For each assignment of continuously differentiable offer functions $\tilde{\mathbf{q}}$, $\tilde{\mathbf{q}} \in \tilde{Q}_{S0}$, and an assignment of $B \in \mathcal{B}_0$, for $n \geq 1$, let $S_n^j \{ b \in S^j : \exists \text{ an assignment of continuously differentiable functions } \tilde{q}, \tilde{q} \in \tilde{Q}_{S0}, \text{ s.t. } \tilde{Q}(B) = \int_{I_2} \tilde{q}_j^* (B) \, dj \text{ and } b = b^j (\frac{\tilde{B}}{Q(B)}) \}$, $B_n = \{ B \in \mathcal{B} : \exists B = \int_{I_1} b^i \, di \text{ for some assignment } b : I_1 \rightarrow \times_{i \in I_1} S_i \}$.

**Definition 19.** The subgame perfect equilibrium $(\mathbf{b}^*, \mathbf{\tilde{q}}^*)$ is locally extensive-form rationalizable if for initial non-trivial restrictions on strategy spaces $S_i^j$ for $i \in I_1$ and

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\(^{13}\) See, for instance, [8, pp. 50] for a definition.
$\tilde{Q}_{S,0}^j$ for $j \in I_2$ (with the associated sets $\tilde{B}_0, \tilde{Q}_{S,0}(B)$ for each $B \in \tilde{B}_0$), $\bigcap_{n \geq 0} \tilde{S}_n^i = \{b^i\}, \bigcap_{n \geq 0} \tilde{S}_n^j = \{q^j\}$ for each $i \in I_1$ and $j \in I_2$.

**Proof of Corollary 16.** As the subgame perfect equilibrium $(b^*, q^*)$ is locally extensive-form rationalizable, consider the initial restriction $\tilde{Q}_{S,0}^i = \{q^i\}$ and $B_0^i = \tilde{B}_0$. Remark that $\tilde{Q}_{S,0}^i \subset \tilde{Q}_S$ and therefore, at each $n \geq 1$, and for each $i \in I_1$, $\tilde{S}_n^i = \{b \in S^i : b = b^i(\frac{B}{Q(B)})\}$ and $\tilde{B}_n^i = \{B \in B : \exists B = \int_{I_1} b^i \, \text{d}i\}$ for some assignment $b : I_1 \to \times_{i \in I_1} \tilde{S}_n^i$ are such that $\tilde{S}_n^i \subset \tilde{S}_n^i$ and $\tilde{B}_n^i \subset \tilde{B}_n^i$. As the sequence of sets $\tilde{S}_n^i$ and $\tilde{B}_n^i$ shrink to $b^i$ for all $i \in I_1$ and $B^*$ respectively, it follows that the sequence of sets $\tilde{S}_n^i$ and $\tilde{B}_n^i$ shrink to $b^i$ for all $i \in I_1$ and $B^*$ respectively as well. Therefore, both the dynamics defined by (11) and (9) on $[\tilde{B}, \tilde{\tilde{B}}]$ converges to $B^*$. \qed

**References**


