A stable algorithm for Hankel transforms using hybrid of Block-pulse and Legendre polynomials

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1. Introduction

There are several integral transforms which are frequently used as a tool for solving numerous scientific problems. It is well known that the Fourier transform (FT) is used to obtain spatial spectrum of optical light\textsuperscript{1}. Fourier optics is widely used in optical instrument design, optical propagation through lenses and in quadratics graded index mediums. Most classical optical systems like mirrors or lenses are axially symmetrical devices. In many practical problems, data are often acquired in such a form that is desirable to perform a two-dimensional polar Fourier transform that is a Hankel transform (HT) rather than the Cartesian forms. So, we transform the Cartesian coordinates into the polar coordinates.

Let \( f(x, y) \) be an input field such that it can be separated as \( f(x, y) = f_1(x)f_2(y) \), where \( f_1 \) and \( f_2 \) are independent functions. Then its two-dimensional Fourier transform \( \tilde{f} \) is also separable as the same symmetry property is transposed through a linear FT. Hence, \( \tilde{f}(u, v) = \tilde{f}_1(u)\cdot \tilde{f}_2(v) \).

Changing to the polar coordinates and if \( f(r, \theta) = f(r) \) is axially symmetrical, then in [2], it was shown that

\[
\tilde{f}(k, \varphi) = \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} f(r) J_0(kr) r \, dr \, d\theta = F_0(k)
\]

which is also axially symmetrical in the Fourier frequency domain, where \( F_0 \) is the Hankel transform of order zero. The general Hankel transform pair with the kernel being \( J_v \) is defined as [3]

\[
F_v(p) = \int_{0}^{\infty} f(r) J_v(pr) dr,
\]

and HT being self reciprocal, its inverse is given by

\[
f(r) = \int_{0}^{\infty} F_v(p) J_v(pr) dp,
\]

where \( J_v \) is the \( v \)-th order Bessel function of first kind.

The Hankel transform arises naturally in the discussion of problems posed in cylindrical coordinates and hence, as a result of separation of variables, involving Bessel functions. The Hankel transform is frequently used as a tool for solving numerous scientific problems. It is widely used in several fields like, elasticity [4], optics [5,6], fluid mechanics [7], seismology [8], astronomy and image processing [9–16]. The Hankel transform becomes very useful in analysis of wave fields where it is used in mathematical handling of radiation, diffraction, and field projection. Recently, it has been
utilized to study pseudo-differential operators. Singh and Pandey [17] used HT of order \(v, v \in \mathbb{R}\) to study a special class of pseudo-differential operator (PDO) \((x^{-1}D)^v\), \(D = \frac{d}{dx}\) and proved that the (PDO) is almost an inverse of HT operator \(h_v\) in the sense that \(h_{-v}(x^{-1}D)^v \psi = h_v \psi\) over certain freshet space \(F\) thus representing the PDO as a Fourier-Bessel series. Further, in 1995, Singh [18], using the HT representation of the PDO, proved that \(e^{-\alpha x^2}\) and \(e^{-\alpha |x|^2/2}\) are the eigenfunctions and \(e^{-\alpha |x|^2/2}\) is a fixed point of \((x^{-1}D)^v\), \(v \in \mathbb{C}\).

Several papers have been written to the evaluation of the Hankel transform in general and the zero order in particular. Analytical evaluations of (2) and (3) are rare and their numerical computations are difficult because of the oscillatory behavior of the Bessel function and the infinite length of the interval. Since seminal work by Siegman [19] in 1977, a number of algorithms for the numerical evaluation of the Hankel transform have been published for both zero-order [5, 6, 9–12, 20–24] and high-order [25–32] Hankel transform. Unfortunately, the efficiency of a method for computing Hankel transform is highly dependent on the function to be transformed, and thus it is difficult to choose the optimal algorithm for given function. In [21], the authors used Filon quadrature to evaluate zero-order Hankel transform. They separated the integrand into the product of (assumed) slowly varying component and a rapidly oscillating one (in this case, former is the zero-order HT algorithm of Yu [23] to higher orders. Their algorithm is based on the orthogonality properties of Bessel functions. Postnikov [32], proposed, for the first time, a novel and powerful method for computing zero and first order HT by using Haar wavelets. Refining the idea of Postnikov [32], Singh et al. [34, 35] obtained three numerical algorithms for numerical evaluation of HT of order \(v \geq 1\) using linear Legendre multi-wavelets, Legendre wavelets and rationalized Haar wavelets which were shown to be superior to the other mentioned algorithms.

The data function \(f(r)\) when measured experimentally may contain some noise terms affecting the accuracy of the algorithms for computing the HT. Thus, it is desirable to have algorithms stable under small random perturbation in the data function.

The purpose of this communication is to present a new stable algorithm that is quite accurate and fast for numerical evaluation of the HT using hybrid of Block-pulse and Legendre polynomials which is either superior or comparable to the previous algorithms. Test functions with known analytic HT are used with random noise which is either superior or comparable to the previous algorithms.

### 2. Hybrid functions of Block-pulse and Legendre polynomials

Hybrid functions \(b_{nm}(t), n = 1, 2, \ldots, N, m = 0, 1, \ldots, M - 1\), have three arguments; \(n\) and \(m\) are the order of block-pulse functions and Legendre polynomials, respectively, and \(t\) is the normalized time. They are defined on the interval \([0, 1]\) as [36].

\[
b_{nm}(t) = \left\{ \begin{array}{ll} P_m(2Nt - 2n + 1), & \text{for } \frac{n-1}{N} \leq t < \frac{n}{N}, \\ 0, & \text{otherwise}. \end{array} \right. \tag{4}\]

Here \(P_m(t)\) are the well-known Legendre polynomials of order \(m\) which satisfy the following recursive formula:

\[
P_0(t) = 1, \quad P_1(t) = t, \quad P_{m+1}(t) = \frac{2m+1}{m+1} P_m(t) - \frac{m}{m+1} P_{m-1}(t), \quad m = 1, 2, \ldots. \tag{5}\]

These hybrid functions \(b_{nm}(t)\) form an orthogonal family on \([0, 1]\).

### 3. Out line of algorithm

The function \(f(r)\) representing physical fields are either zero or have an infinitely long decaying tail outside a disk of finite radius \(R\). Hence, in most practical applications either the signal \(f(r)\) has a compact support or for a given \(\varepsilon > 0\) there exists a \(R > 0\) such that \(\int_{R} f(r) J_v(pr) dr < \varepsilon\).

Therefore, in either case,

\[
\hat{F}_v(p) = \int_0^R f(r) J_v(pr) dr = \left( \int_0^1 g(r) J_v(pr) dr \right) \left( \int_0^1 \right) \tag{6}\]

known as the finite Hankel transform (FHT) is a good approximation of the HT as given by (2). Writing \(f(r) = g(r)\) in Eq. (6), we get

\[
\hat{F}_v(p) = \int_0^1 g(r) J_v(pr) dr. \tag{7}\]

We may expand \(g(r)\) as follows

\[
g(r) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm}(r), \tag{8}\]

where \(c_{nm} = \|g(r), b_{nm}(r)\|^2\).

By truncating the infinite series \(8\) at levels \(n = N\) and \(m = M - 1\), we obtain an approximate representation for \(g(r)\) as

\[
g(r) \approx \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm}(r) = C^T B(r), \tag{9}\]

where the matrices \(C\) and \(B\) are given by

\[
C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{N0}, \ldots, c_{NM-1}]^T \tag{10}\]

and

\[
B(r) = [b_{10}(r), b_{11}(r), \ldots, b_{20}(r), \ldots, b_{2M-1}(r), \ldots, b_{N0}(r), \ldots, b_{NM-1}(r)]^T. \tag{11}\]

Substituting (9) in (7), we get

\[
\hat{F}_v(p) \approx C^T \int_0^1 B(r) J_v(pr) dr. \tag{12}\]

Taking \(N = 4\) and \(M = 3\), Eq. (12) reduces to

\[
\hat{F}_v(p) \approx C^T \int_0^1 B(r) J_v(pr) dr. \tag{12}\]
where $b_{10}, b_{11}, ..., b_{42}$ are defined through Eqs. (4)–(5) are given by:

$$
\begin{align*}
  b_{10} &= 1, \\
  b_{11} &= (8r - 1), \\
  b_{12} &= \frac{3}{2}(8r - 1)^2 + \frac{1}{2}, \\
  b_{20} &= 1, \\
  b_{21} &= (8r - 3), \\
  b_{22} &= \frac{3}{2}(8r - 3)^2 - \frac{1}{2}, \\
  b_{30} &= 1, \\
  b_{31} &= (8r - 5), \\
  b_{32} &= \frac{3}{2}(8r - 5)^2 - \frac{1}{2}, \\
  b_{40} &= 1, \\
  b_{41} &= (8r - 7), \\
  b_{42} &= \frac{3}{2}(8r - 7)^2 - \frac{1}{2}.
\end{align*}
$$

(13)

We re-label and write Eq. (13) as,

$$
\tilde{F}_v (p) \approx c^T \left[ \int_0^{1} b_{10}(r) J_v (pr) dr, \int_0^{1} b_{11}(r) J_v (pr) dr, ..., \int_0^{1} b_{42}(r) J_v (pr) dr \right] T,
$$

(15)

where $I^l_j$'s are the $l$th place integral in Eq. (13).

The integrals arising in Eq. (13) are evaluated by using the following formulae:

$$\int f(r) dr = \lim_{L \to \infty} \sum_{z=0}^{L} f_{v+2z+1}(a), \quad \text{Re} v > -1 \quad [37, \text{p. 333}],$$

$$\int r^{1-v} J_v (t) dt = \frac{1}{2} \sum_{z=0}^{L} (v+2z+1) \Gamma \left( \frac{v+2z+1}{2} + z \right) \times J_{v+2z+1}(a), \quad \text{Re}(v+\mu+1) > 0 \quad [38, \text{p. 480}],$$

(16)

4. Numerical results

In this section we discuss, the implementation of our numerical method and investigate its accuracy and stability by applying it on numerical examples with known analytical HT.

In all the examples, the exact data function is denoted by $g(r)$ and the noisy data function $g^\varepsilon (r)$ is obtained by adding noise $\varepsilon$ to $g(r)$ such that $g^\varepsilon (r_i) = g(r_i) + \varepsilon_i$, where $r_i = ih$, $i = 1, 2, ... , N$, $Nh = 40$ and $\varepsilon_i$ is a uniform random variable with values in $[0, 1]$, such that $\max_{0 \leq i \leq N} |g^\varepsilon (r_i) - g(r_i)| \leq \delta$.

The following examples are solved with and without noise to illustrate the efficiency and stability of our method by choosing three different values of the noise terms $\varepsilon$ as $\varepsilon_0 = 0.000$, $\varepsilon_1 = 0.002$ and $\varepsilon_2 = 0.005$ and computing the error $E_j(p)$ = Approximate HT obtained from (15) with noise $\varepsilon_j$ − the exact HT, $j = 0, 1, 2$. The various $E_j(p)$'s are shown in Figs. 2, 4, 6, 8, 10, and 12. Note that in all the examples, the truncation is done at level $N = 4$. We observed that the accuracy of the method is very high even at such a low level of truncation. The various graphs in the following examples are plotted by choosing the sample points.
as \( p = 0.01(0.01)P \), where \( P = 40 \) in Figs. 1–12. We take \( L = 40 \) in (16) and latter in (24) to get approximate solutions of the numerical examples given in this section.

We also use the discrete \( l^2 \) norm and the continuous \( L^2 \) norm in \( l = [0, P] \) to measure errors as well. They are defined as:

\[
\| f \|_2 = \left( \frac{1}{N} \sum_{i=1}^{N} |f(r_i)|^2 \right)^{1/2}
\]

(17) and

\[
\| f \|_{L^2} = \left( \int_{0}^{P} |f(r)|^2 \, dr \right)^{1/2},
\]

(18)

respectively.

**Example 1 (Sombrero function).** Though this function was studied in [34,35], but as the stability of the algorithms in those references were not tested, we take it up again to check the stability and accuracy of the proposed algorithm. The Circ function is defined as

\[
\text{Circ}(r/a) = \begin{cases} 
1, & r \leq a, \\
0, & r > a.
\end{cases}
\]

(19)

The zeroth-order Hankel transform of Circ\((r/a)\) is the Sombrero function [34,35,39], given by

\[
S_0(p) = a^2 \frac{J_1(ap)}{ap}.
\]

Note that \( S_0(p) \) and \( F_0(p) \) are indicated by \( S_0(p) \) (solid line) and \( H_0(p) \) (dotted line) in the Fig. 1 and the various errors \( Ej(p) \) are shown in Fig. 2.

**Example 2.** Let

\[
f(r) = \frac{2}{\pi} \arccos(r) - r(1 - r^2)^{1/2}, \quad 0 \leq r \leq 1,
\]

then,

\[
F_0(p) = 2 \frac{J_1^2(p/2)}{p^2}, \quad 0 \leq p \leq \infty.
\]

(20)

A well-known result. The pair \((f(r), F_0(p))\) arises in optical diffraction theory [40]. The function \( f(r) \) is the optical transfer function of an aberration-free optical system with a circular aperture, and \( F_0(p) \) is the corresponding spread function.

Barakat et al. [21], evaluated \( F_0(p) \) numerically using Filon quadrature philosophy but the associated error is appreciable for \( p < 1 \); whereas our method gives almost zero error in that range.

Note that \( F_0(p) \) and \( F_1(p) \) are indicated by \( S_0(p) \) (solid line) and \( H_0(p) \) (dotted line) in Fig. 3. The errors \( Ej(p) \) are shown in Fig. 4.

**Example 3.** Let \( f(r) = (1 - r^2)^{1/2} \), \( 0 \leq r \leq 1 \), then,

\[
F_1(p) = \begin{cases} 
\pi \frac{2p}{2p}, & 0 < p < \infty, \\
0, & p = 0.
\end{cases}
\]

(21)

Barakat et al. [22], evaluated \( F_1(p) \) numerically using Filon quadrature philosophy but again the associated error is appreciable for \( p < 1 \); whereas our method gives almost zero error in that range. The comparison of the approximation \( H_1(p) \) (dotted line) with the exact Hankel transform \( F_1(p) \) (solid line) and the errors \( Ej(p) \) are shown in Figs. 5 and 6 respectively.

**Example 4.** In this example, we choose as a test function the generalized version of the top-hat function, given as

\[
f(r) = r^a [H(r) - H(r - a)],
\]

(22)

\( a > 0 \) and \( H(r) \) is the step function,

\[
H(r) = \begin{cases} 
1, & r \geq 0, \\
0, & r < 0.
\end{cases}
\]

Then,
example 5. the following example was solved numerically by

we take \( a = 1 \) and \( v = 4 \) for numerical calculations. we take \( a = 1 \), \( v = 2 \), and observe that the errors are quite small as shown in fig. 8 compare to [33].

example 5. the following example was solved numerically by

\[
F_v(p) = \frac{I_{v+1}(p)}{p}.
\]

(22)

\( F_v(p) \) is radius and \( \nu \) is time (\( u \) does not depend on \( \theta \) and \( t \) satisfying the differential equation

\[
D_r^2 u + \frac{1}{r} D_t u = D_t u \quad (0 < r < 1, 0 < t < \infty)
\]

(25)

and the following initial and boundary conditions:

(i) as \( t \to 0^+ \), \( u(r, t) \to f(r) = \frac{2}{\pi} [\arccos(r) - r(1 - r^2)^{1/2}] \), \( 0 \leq r \leq 1 \).

(ii) as \( r \to 1^- \), \( D_r u + H u \to 0 \) for each fixed \( t > 0 \), where \( H > 0 \).

5. application

as an application, we solve the heat equation in cylindrical coordinates inside an infinitely long cylinder of radius unity, by using the theory of hankel transform developed in the preceding pages. we seek a function \( u(r, t) \); where \( r \) is radius and \( t \) is time satisfying the differential equation

\[
D_r^2 u + \frac{1}{r} D_t u = D_t u \quad (0 < r < 1, 0 < t < \infty)
\]

(25)

and the following initial and boundary conditions:

(i) as \( t \to 0^+ \), \( u(r, t) \to f(r) = \frac{2}{\pi} [\arccos(r) - r(1 - r^2)^{1/2}] \), \( 0 \leq r \leq 1 \).

(ii) as \( r \to 1^- \), \( D_r u + H u \to 0 \) for each fixed \( t > 0 \), where \( H > 0 \).

When \( u \) denotes the temperature within the cylinder, \( H > 0 \) means that heat is being radiated away from the surface of the cylinder.

let \( \Omega_{r,t} \) denotes the differential operator \( D_r^2 + \frac{1}{r} D_t - \frac{\nu^2}{r^2} \), then the differential equation (25) can be written as

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}
\]

(26)

The Dini expansion associated with \( f(r) \) is

\[
B_0(r) + \sum_{m=1}^{\infty} b_m J_\nu(\lambda_m r) \quad [41, p. 596].
\]

(27)

where \( \lambda_m, m = 1, 2, 3, \ldots, \) are the positive roots (arranged in ascending order of magnitude) of the transcendental equation

\[
z J_\nu'(z) + H J_\nu(z) = 0, \quad \nu \geq -\frac{1}{2}.
\]

(28)

\( b_m, m = 1, 2, 3, \ldots, \) are given by
The exact transform, \( F_\nu(p) \) (solid line) and the approximate transform, \( H_\nu(p) \) (dotted line).

Fig. 8. Comparison of the errors.

The exact transform, \( S_0(p) \) (solid line) and the approximate transform, \( H_0(p) \) (dotted line).

Fig. 9. Comparison of the errors.

The condition of validity of (27) are given in the following theorem [41, p. 601].

**Theorem 1.** Let \( f(r) \) be a function defined over the interval \((0, 1)\), and let \( \int_0^1 r^{1/2} f(r) \, dr \) exist and (if it is improper integral) let it be absolutely

\[
b_m = \frac{2\lambda_m^2}{\lambda_m^2 - \nu^2} \int_0^1 r f(r) J_\nu(\nu r) \, dr
\]

\[
+ \lambda_m^2 J_\nu(\nu \lambda_m)
\]

and \( B_0(r) = 0 \) if \( H + \nu > 0 \), which is the case for the present problem as \( \nu = 0 \) and \( H > 0 \).
If \( f(r) \) has limited total fluctuation in \((a, b)\) where \( 0 < a < b \leq 1 \) then the series (27) converges to the sum \( \frac{1}{2} [f(r + 0) + f(r - 0)] \) at all points \( r \) such that \( a + \Delta \leq r < b - \Delta \) where \( \Delta \) is arbitrarily small; and the convergence is uniform if \( f(r) \) is continuous in \((a, b)\).

Substituting Eqs. (6) and (29) in (27) and using the above theorem, we obtain the following inversion theorem for the finite Hankel transform

**Theorem 2 (Inversion).** Let \( f \) be Hankel transformable. We obtain the following inversion theorem for the finite Hankel transform, we obtain the following inversion theorem for the finite Hankel transform

\[
f(r) = \lim_{N \to \infty} \sum_{m=1}^{N} \frac{2 \lambda_m^2 \hat{f}_2(\lambda_m)}{\lambda_m^2 - \nu^2} \int_0^\infty J_0(\lambda_m r) dr. \tag{30}
\]

From the well-known formula

\[
\Omega_0, f \int_0^\infty J_0(\lambda_m r) dr = -\lambda_m^2 f(\lambda_m), \tag{31}
\]

it follows from integration by parts that

\[
\int_0^1 [\Omega_0, f(r)] f \int_0^\infty J_0(\lambda_m r) dr = \int_0^1 f(r) \Omega_0, f \int_0^\infty J_0(\lambda_m r) dr \tag{32}
\]

\[
= -\lambda_m^2 \int_0^1 f(r) r J_0(\lambda_m r) dr,
\]

if we put some suitable condition on \( f(r) \) such that the limit terms in integration by parts in (32) vanish.

Applying the finite Hankel transform operator to Eq. (26) and using (32), we obtain

\[
-\lambda_m^2 U(\lambda_m, t) = \frac{\partial U(\lambda_m, t)}{\partial t}.
\]

where

\[
U(\lambda_m, t) = \int_0^1 U(r, t) r J_0(\lambda_m r) dr,
\]

so that

\[
U(\lambda_m, t) = \hat{F}_0(\lambda_m) e^{-\lambda_m^2 t}.
\]

The initial condition determines the constant \( A \). Thus

\[
A(\lambda_m) = \hat{F}_0(\lambda_m) \int_0^1 f(r) r J_0(\lambda_m r) dr.
\]

Hence

\[
U(\lambda_m, t) = \hat{F}_0(\lambda_m) e^{-\lambda_m^2 t}.
\]

Therefore, by Inversion Theorem 2, we have

\[
U(r, t) = \lim_{N \to \infty} \sum_{m=1}^{N} \frac{2 \hat{F}_0(\lambda_m) e^{-\lambda_m^2 t} f(\lambda_m)}{\lambda_m^2 + \lambda_m^2 f(\lambda_m)}, \tag{33}
\]

since \( f_0(r) = -f_1(r) \).

We want to prove that \( u(r, t) \), given by (33) is truly a solution of (25) that satisfies the given initial and boundary conditions. To achieve this, we need the following well-known estimates:

\[
F_0(\lambda_m) = O(\lambda_m^{-3/2}) \quad \text{as} \quad m \to \infty \quad \text{[41, p. 595]},
\]

\[
\lambda_m \sim \pi \left( m + \frac{1}{4} \right) \quad \text{as} \quad m \to \infty,
\]

\[
f_0^2(\lambda_m) + f_1^2(\lambda_m) \sim \frac{\pi}{2 \lambda_m} \quad \text{as} \quad m \to \infty.
\]

Hence

\[
F_0(\lambda_m) \left[ f_0^2(\lambda_m) + f_1^2(\lambda_m) \right]^{-1} = O(m^{-1/2}) \quad \text{as} \quad m \to \infty.
\]
Hence by applying Eq. (25).

Using the above estimates, we see that the series (33) and the series obtained by applying $\Omega_{D_2}$ and $D_1$ separately under the summation sign of (33) converges uniformly on $0 < r < 1$ and $t > 0$. Hence by applying $\Omega_{D_2} - D_1$ and using the fact $\Omega_{D_2} f_k(\lambda_{m\epsilon} t) = -\lambda_{m\epsilon}^2 f_k(\lambda_{m\epsilon} t)$, we see that (33) satisfies the differential equation (25).

Let us verify the boundary condition (ii), we have

$$\lim_{r \to 1^-} [D_r u + Hu] = \lim_{r \to 1^-} \left[ \sum_{m=1}^{\infty} D_r \left\{ 2F_0(\lambda_{m\epsilon}) e^{-\lambda_{m\epsilon}^2 t} J_0(\lambda_{m\epsilon} t) \right\} \right]$$

and since the convergence is uniform, we can take the $\lim_{r \to 1^-}$ inside the summation sign and arrive at the conclusion, since $\lambda_{m\epsilon}$‘s are the roots of the equation $\lambda J_0(\lambda) + H J_1(\lambda) = 0$.

The initial condition (i) is already taken care of as we evaluated the constant $A$ by using it. Through Figs. 13–17, we establish the accuracy of the propose method. All figures are drawn by truncating the series (33) at $N = 10$. The presence of $e^{-\lambda_{m\epsilon}^2 t}$ ensures that even ten terms give satisfactory solution for $t > 0$. While evaluating the solution $u(r, t)$ from (33), we have evaluated $\Phi_0(\lambda_{m\epsilon})$ first from its analytical expression given by (20) and denote the solution thus obtained by $u(r, t)$ in Figs. 13–17 and then evaluating $\Phi_0(\lambda_{m\epsilon})$ by using the proposed algorithm for evaluation of the finite Hankel transform as given by Eq. (15). This solution is denoted by $u_a(r, t)$ in the above mentioned figures.

Fig. 13 compares the given initial condition $f(r)$ with $u(r, t)$ as $t \to 0^+$ and Fig. 14 shows the error corresponding error $E(r) = u(r, 0) - f(r)$. Fig. 15 depicts the various profiles of $u_1(r, t)$ at times $t = 0, \frac{1}{10}, \frac{1}{5}$ and $\frac{1}{2}$, the various profiles are denoted by $u_0, u_1, u_2$ and $u_3$. As the maximum possible error occurs in the neighborhood of 0 and 0.001, we have restricted $t$ in $(0, 1)$ in Figs. 16 and 17 representing $u(r, t)$ and $u_a(r, t)$ respectively and note that they are in good agreement in the range.

6. Error analysis

The numerical stability property of the algorithm is illustrated in Tables 6.1–6.6 where the discrete $l^2$ norm as well as $L^2$ norm of the error is shown as functions of the amount of noise $\epsilon$ in...
Table 6.1 Error norm as function of \( \varepsilon \) in Example 1.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 10000 )</th>
<th>( N = 5000 )</th>
<th>( N = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.00000003a</td>
<td>0.00000003b</td>
<td>0.00000003a</td>
</tr>
<tr>
<td>0.002</td>
<td>0.00001682</td>
<td>0.00001683</td>
<td>0.00001689</td>
</tr>
<tr>
<td>0.005</td>
<td>0.00003031</td>
<td>0.00003032</td>
<td>0.00003041</td>
</tr>
</tbody>
</table>

Table 6.2 Error norm as function of \( \varepsilon \) in Example 2.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 10000 )</th>
<th>( N = 5000 )</th>
<th>( N = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.00001123</td>
<td>0.00001123</td>
<td>0.00001123</td>
</tr>
<tr>
<td>0.002</td>
<td>0.00001657</td>
<td>0.00001657</td>
<td>0.00001689</td>
</tr>
<tr>
<td>0.005</td>
<td>0.00003116</td>
<td>0.00003117</td>
<td>0.00003125</td>
</tr>
</tbody>
</table>

Table 6.3 Error norm as function of \( \nu \) in Example 3.

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( N = 10000 )</th>
<th>( N = 5000 )</th>
<th>( N = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.00071197</td>
<td>0.00071197</td>
<td>0.00071197</td>
</tr>
<tr>
<td>0.002</td>
<td>0.00070338</td>
<td>0.00070338</td>
<td>0.00070338</td>
</tr>
<tr>
<td>0.005</td>
<td>0.00071104</td>
<td>0.00071104</td>
<td>0.00071104</td>
</tr>
</tbody>
</table>

Table 6.4 Error norm as function of \( \varepsilon \) in Example 4.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 10000 )</th>
<th>( N = 5000 )</th>
<th>( N = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.00000043</td>
<td>0.00000043</td>
<td>0.00000043</td>
</tr>
<tr>
<td>0.002</td>
<td>0.00001314</td>
<td>0.00001314</td>
<td>0.00001314</td>
</tr>
<tr>
<td>0.005</td>
<td>0.00002103</td>
<td>0.00002103</td>
<td>0.00002103</td>
</tr>
</tbody>
</table>

Table 6.5 Error norm as function of \( \varepsilon \) in Example 5.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 10000 )</th>
<th>( N = 5000 )</th>
<th>( N = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.01736064</td>
<td>0.01736101</td>
<td>0.01736398</td>
</tr>
<tr>
<td>0.002</td>
<td>0.0173507</td>
<td>0.01735106</td>
<td>0.01735399</td>
</tr>
<tr>
<td>0.005</td>
<td>0.01736082</td>
<td>0.01736964</td>
<td>0.01737253</td>
</tr>
</tbody>
</table>

Table 6.6 Error norm as function of \( \varepsilon \) in Example 6.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 10000 )</th>
<th>( N = 5000 )</th>
<th>( N = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.0000146</td>
<td>0.0000146</td>
<td>0.0000146</td>
</tr>
<tr>
<td>0.002</td>
<td>0.00001798</td>
<td>0.00001798</td>
<td>0.00001798</td>
</tr>
<tr>
<td>0.005</td>
<td>0.00001819</td>
<td>0.00001819</td>
<td>0.00001819</td>
</tr>
</tbody>
</table>

the data function, for Examples 1–6 respectively. We notice that in all the cases, the numerical stability of the proposed algorithm is confirmed. Moreover, in the \( \varepsilon \) range 0.000 to 0.005 the discrete error norms are barely sensitive to changes in \( h = 1/N \) in all the examples.

7. Summary and conclusions

A new stable method based on the hybrid of Block-pulse and Legendre polynomials for the numerical evaluation of HT is proposed and analyzed. This is for the first time we have tested the stability of the proposed algorithm and applied it to solve the heat equation in cylindrical coordinates inside an infinitely long cylinder of radius unity. As the basis functions used to construct the hybrid functions are orthogonal and have compact supports, it makes them more useful and simple in actual computations. Our choice of hybrid functions make it more attractive in their application in the applied physical problems as they eliminate the problems connected with the Gibbs phenomenon taking place in [32,33]. The error associated with Filon quadrature philosophy [6,21,22] is appreciable for small \( p < 1 \) compared to our algorithm. It is obvious from the illustrative examples that the proposed algorithm is superior to the algorithms proposed in [34,35].

References


