A Sequent Calculus for Dynamic Topological Logic

Samuel Reid*

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Abstract

We introduce a sequent calculus for the temporal-over-topological fragment \(\text{DTL}^{\circ/\Box}_0\) of dynamic topological logic \(\text{DTL}\), prove soundness semantically, and prove completeness syntactically using the axiomatization of \(\text{DTL}^{\circ/\Box}_0\) given in [3]. A cut-free sequent calculus for \(\text{DTL}^{\circ/\Box}_0\) is obtained as the union of the propositional fragment of Gentzen’s classical sequent calculus, two \(\Box\) structural rules for the modal extension, and nine \(\circ\) (next) and \(\ast\) (henceforth) structural rules for the temporal extension. Future research will focus on the construction of a hypersequent calculus for dynamic topological \(\text{S5}\) logic in order to prove Kremer’s Next Removal Conjecture for the logic of homeomorphisms on almost discrete spaces \(\text{S5H}\).

1 Introduction

Within recent years there has been an outburst of research activity in spatial-temporal reasoning, leading to important advances in computer science and logic. The Handbook of Spatial Logics [1], and references therein, summarize the main recent achievements such as mereotopology, spatial constraint calculi, modal logics of space, connections between topology and epistemic logic, logics for fragments of elementary geometry, mathematical morphology, logics of space-time and relativity theory, and dynamic topological logic. The present paper focuses on extending the research area of dynamic topological logic by means of a Gentzen-style proof calculus for the temporal-over-topological fragment of dynamic topological logic \(\text{DTL}^{\circ/\Box}_0\). We achieve this by extending the sequent calculi developed for the fragments of dynamic topological logic defined in [2] as the logic of control action \(\text{S4F}\) and the logic of continuous control action \(\text{S4C}\). The main motivation for this work is to combine structural proof theory and dynamic topological logic. Thus laying the groundwork for defining hypersequent calculi for dynamic topological \(\text{S5}\) logic and the conservative axiomatizable extensions \(\text{S5C, S5H, S5Ct, and S5Ht}\) known as the logics of continuous functions on almost discrete spaces, homeomorphisms on almost discrete spaces, functions on trivial spaces, and homeomorphisms on trivial spaces, respectively [4]. These logics have been defined quite recently and the development of a sequent calculus for them would lead to applications such as proof search and automated theorem proving in dynamic topological logic. We identify a main conjecture of this recent research area and propose that future research regarding hypersequent calculi for dynamic topological \(\text{S5}\) will provide a positive solution to this conjecture.

Conjecture 1 (Kremer’s Next Removal Conjecture). The logic of homeomorphisms on almost discrete spaces can be axiomatized without next removal by

\[
\text{S5H} = \text{S5} + \text{LTL} + (\circ \Box A \Leftrightarrow \Box \circ A).
\]

Dynamic topological logic provides the foundation for breakthroughs in topics ranging from control theory and robot-motion planning to dynamical systems and eschatological cosmology, as statements regarding the possibility and necessity of spatio-temporal properties can be understood with systematic logical precision.

*University of Calgary, Department of Mathematics and Statistics, Calgary, AB, Canada. e-mail: smrei@ucalgary.ca.
2 Dynamic Topological Logic

We now present an introduction to dynamic topological logic, freely citing from the seminal papers on dynamic topological logic [2], [3], [4], [5].

The main idea of dynamic topological logic is to combine of topological semantics in logic, temporal logic, and topological dynamics (asymptotic properties of continuous maps on topological spaces). Interestingly predating the well-known Kripke semantics for modal logic, the McKinsey-Tarki topological semantics interprets the purely temporal modal propositional language

\[ L^2 = \land \land \neg \diamond p_1 \cdots p_n \cdots \]

in terms of topological spaces with the interpretation of \( \diamond \) given by topological interior. Then the propositional letters denote subsets of \( X \); \( \lor, \land, \neg \) express union, intersection, and complement, respectively, with \( \diamond := \neg \diamond \neg \) interpreted as closure. We then have

\[
\bigwedge_{i=1}^n p_i \to k \bigvee_{j=1}^k q_j \iff \bigwedge_{i=1}^n p_i \subseteq \bigcup_{j=1}^k q_j
\]

thus providing us with a semantic interpretation for sequents and language to prove the soundness of structural rules.

A topological model is an ordered pair \( \mathcal{M} = \langle X, V \rangle \), where \( X \) is a topological space and \( V : \text{Var} \to \mathcal{P}(X) \). The function \( V \) is extended to formulas of \( L^2 \) by

\[
V(\neg A) = X - V(A) \\
V(A \land B) = V(A) \cap V(B) \\
V(\diamond A) = \text{int}(V(A))
\]

with four validity relations, where \( \mathcal{T} \) is a class of topological spaces.

\[
\mathcal{M} \models A \text{ iff } V(A) = X \\
X \models A \text{ iff } \langle X, V \rangle \models A, \forall V : \text{Var} \to \mathcal{P}(X) \\
\mathcal{T} \models A \text{ iff } X \models A, \forall X \in \mathcal{T} \\
\models A \text{ iff } X \models A, \forall X
\]

A dynamic topological system is an ordered pair \( \langle X, f \rangle \), where \( X \) is a topological space and \( f \) is a continuous function on \( X \). We interpret the temporal connectives of the modal-temporal language \( L \) by means of the function \( f \):

- \( \diamond A \) is true at \( x \) iff \( A \) is true at \( fx \).
- \( \ast A \) is true at \( x \) iff \( A \) is true at \( \bigwedge_{i=1}^\infty f^i x, \forall i \in \mathbb{N} \).

A dynamic topological model is an ordered triple \( \mathfrak{M} = \langle X, f, V \rangle \) where \( \langle X, f \rangle \) is a dynamic topological system and \( V : \text{Var} \to \mathcal{P}(X) \) is extended to all formulas of \( L \) by

\[
V(\diamond A) = f^{-1}(V(A)) \\
V(\ast A) = \bigcap_{i=1}^\infty f^{-i}(V(A))
\]

with five validity relations, where \( \mathcal{F}_X \) is a class of continuous functions associated with the topological space.
$X$ and $F$ is an arbitrary class of continuous functions.

$\forall R \vdash A$ if $V(A) = X$

$(X,f) \vdash A$ if $\langle X,f,V \rangle \vdash A, \forall V : \text{Var} \rightarrow P(X)$

$X \vdash A$ if $\langle X,f \rangle \vdash A, \forall f \in F_X$

$\Gamma \vdash A$ if $X \vdash A, \forall X \in \Gamma$

$\Gamma, F \vdash A$ if $\langle X,f \rangle \vdash A, \forall X \in \Gamma, \forall f \in F$

$\vdash A$ if $X \vdash A, \forall X$

From this we define $DTL_0 = \{ A : \vdash A \}$ to be the logic of all dynamic topological systems. We now provide the axiomatization of linear time logic ($LTL$).

**S4 axioms for $\ast$:**

* $A \supset \ast A$

* $A \supset \ast \ast A$

$\circ$ commutes with $\neg, \lor, \ast : \circ A \iff \ast \circ A$

$\circ (A \lor B) \iff \circ A \lor \circ B$

$\circ \ast A \iff \ast \circ A$

henceforth implies next: $\ast (A \supset \circ A) \supset \ast A$

induction axiom: $A \land \ast (A \supset \circ A) \supset \ast A$

We can then provide the axiomatization given in [3] for the temporal-over-topological fragment of dynamic topological logic as follows:

$$DTL_0^{\ast/\circ} = S4|_{\mathcal{L}^0} + LTL|_{\mathcal{L}^{\ast/\circ}}$$

where $S4|_{\mathcal{L}^0}$ is the S4 axioms for $\square$ with $A, B \in \mathcal{L}^0$ and $LTL|_{\mathcal{L}^{\ast/\circ}}$ is $LTL$ where the scope of $\square$ in any subformula is $\circ$-free and $\ast$-free. This defines $DTL_0^{\ast/\circ}$ with the temporal-over-topological language $\mathcal{L}^{\ast/\circ}$.

We now define the sequent calculus for $DTL_0^{\ast/\circ}$ by means of six weak structural rules: weakening in the antecedent (WA), weakening in the succedent (WS), contraction in the antecedent (CA), contraction in the succedent (CS), interchange in the antecedent (IA), and interchange in the succedent (IS); eight logical rules: $\land R, \lor R, \lor L, \supset R, \supset L, \neg R, \neg L$; the strong structural rule of cut; two modal rules for the $\square$-modality: $\square R, \square L$; and nine temporal rules: $\ast L, \ast R, \ast \circ L, \ast \circ R, \ast \neg L, \ast \circ A, IND, L \circ R$. The 22 unary inference rules and 4 binary inference rules are as follows:
3 Soundness of \(\text{DTL}_0^{\circ/\Box}\)

We show soundness for the temporal-over-topological fragment of dynamic topological logic by giving semantic proofs of the soundness of the rules of the temporal-over-topological sequent calculus.

**Proposition 1.** The \(\Box L\) rule is sound.

\[
\frac{\Gamma, A \rightarrow \Theta}{\Gamma, \Box A \rightarrow \Box \Theta} \quad \Box L
\]

Proof. Assume \(\Gamma \cap A \subseteq \Theta\). Then \(\Gamma \cap \text{int}(A) \subseteq \Theta\). \(\Box\)

**Proposition 2.** The \(\Box R\) rule is sound.

\[
\frac{\Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} \quad \Box R
\]

Proof. Assume \(\text{int}(\Gamma) \subseteq A\). Then \(\text{int}(\Gamma) \subseteq \text{int}(A)\). \(\Box\)

**Proposition 3.** The \(* L\) rule is sound.

\[
\frac{\Gamma, A \rightarrow \Theta}{\Gamma, * A \rightarrow \Theta} \quad * L
\]

Proof. Assume \(\Gamma \cap A \subseteq \Theta\). Then

\[
\Gamma \cap \bigcap_{i=1}^{\infty} f^{-i}(A) \subseteq \Theta.
\]

\(\Box\)

**Proposition 4.** The \(* R\) rule is sound.

\[
\frac{* \Gamma \rightarrow A}{* \Gamma \rightarrow * A} \quad * R
\]

Proof. Assume

\[
\bigcap_{i=1}^{\infty} f^{-i}(\Gamma) \subseteq A.
\]

Then

\[
\bigcap_{i=1}^{\infty} f^{-i}(\Gamma) \subseteq \bigcap_{i=1}^{\infty} f^{-i}(A).
\]

\(\Box\)

**Proposition 5.** The \(\circ * R\) rule is sound.

\[
\frac{\Gamma \rightarrow * \circ A, \Theta}{\Gamma \rightarrow \circ * A, \Theta} \quad \circ * R
\]

Proof. Assume

\[
\Gamma \subseteq \bigcap_{i=1}^{\infty} f^{-i}(f^{-1}(A)) \cup \Theta.
\]

Then

\[
\Gamma \subseteq f^{-1}\left(\bigcap_{i=1}^{\infty} f^{-i}(A)\right) \cup \Theta.
\]

\(\Box\)
Proposition 6. The $\circ * L$ rule is sound.

$$
\frac{\Gamma, \circ A \rightarrow \Theta}{\Gamma, * A \rightarrow \Theta} \circ L
$$

Proof. Assume

$$
\Gamma \cap \bigcap_{i=1}^{\infty} f^{-i}(f^{-1}(A)) \subseteq \Theta.
$$

Then

$$
\Gamma \cap f^{-1} \left( \bigcap_{i=1}^{\infty} f^{-i}(A) \right) \subseteq \Theta.
$$

Proposition 7. The $\circ \neg R$ rule is sound.

$$
\frac{\Gamma \rightarrow \neg \circ A, \Theta}{\Gamma \rightarrow \neg A, \Theta} \neg R
$$

Proof. Assume $\Gamma \subseteq (X - f^{-1}(A)) \cup \Theta$. Then $\Gamma \subseteq f^{-1}(X - A) \cup \Theta$.

Proposition 8. The $\circ \neg L$ rule is sound.

$$
\frac{\Gamma, \neg \circ A \rightarrow \Theta}{\Gamma, \neg A \rightarrow \Theta} \circ \neg L
$$

Proof. Assume $\Gamma \cap (X - f^{-1}(A)) \subseteq \Theta$. Then $\Gamma \cap f^{-1}(X - A) \subseteq \Theta$.

Proposition 9. The $\circ CA$ rule is sound.

$$
\frac{A, \circ A, \Gamma \rightarrow \Theta}{\circ \circ A, \Gamma \rightarrow \Theta} \circ CA
$$

Proof. Assume

$$
A \cap f^{-1} \left( \bigcap_{i=1}^{\infty} f^{-i}(A) \right) \cap \Gamma \subseteq \Theta.
$$

Then

$$
\bigcap_{i=1}^{\infty} f^{-i}(A) \cap \Gamma \subseteq \Theta.
$$

Proposition 10. The IND rule is sound.

$$
\frac{\circ A, \circ (A \circ A) \rightarrow \circ A}{A, \circ (A \circ A) \rightarrow \circ A} IND
$$

Proof. Assume

$$
\bigcap_{i=1}^{\infty} f^{-i}(A) \cap \bigcap_{i=1}^{\infty} f^{-i}(X - A) \cup f^{-1}(X - A) \subseteq f^{-1} \left( \bigcap_{i=1}^{\infty} f^{-i}(A) \right).
$$

Then

$$
A \cap \bigcap_{i=1}^{\infty} f^{-i}(X - A) \cup f^{-1}(A) \subseteq \bigcap_{i=1}^{\infty} f^{-i}(A).
$$
Proposition 11. The $L \circ R$ rule is sound.

$$ \frac{\Gamma \rightarrow \Theta}{\circ \Gamma \rightarrow \circ \Theta} L \circ R $$

Proof. Assume $\Gamma \subseteq \Theta$. Then $f^{-1}(\Gamma) \subseteq f^{-1}(\Theta)$. \qed

4 Completeness for $\text{DTL}^\circ_0/\Box$

We show completeness for the temporal-over-topological fragment of dynamic topological logic by giving sequent calculus derivations of the axioms of $\text{DTL}^\circ_0/\Box$ using the soundness of the sequent calculus rules.

Proposition 12. The sequent calculus for $\text{DTL}^\circ_0/\Box$ proves $\Box A \supset A$.

Proof. $\frac{A \rightarrow A}{\Box A \rightarrow A} \Box L$

$$ \frac{\Box A \rightarrow A}{\Box A \supset A} \Box \supset R $$

Proposition 13. The sequent calculus for $\text{DTL}^\circ_0/\Box$ proves $\Box A \supset \Box \Box A$.

Proof. $\frac{\Box A \rightarrow \Box A}{\Box A \rightarrow \Box \Box A} \Box R$

$$ \frac{\Box A \rightarrow \Box \Box A}{\Box A \supset \Box \Box A} \Box \supset R $$

Proposition 14. The sequent calculus for $\text{DTL}^\circ_0/\Box$ proves $\Box (A \supset B) \supset (\Box A \supset \Box B)$.

Proof. $\frac{A \rightarrow A}{\Box A \supset A} \Box L$

$$ \frac{\Box A \supset A \rightarrow B}{\Box A \supset B} \Box L $$

$$ \frac{\Box A \supset A \rightarrow \Box B}{\Box A \supset \Box B} \Box L $$

$$ \frac{\Box A \supset \Box B \rightarrow \Box \Box A \supset \Box \Box B}{\Box A \supset \Box \Box A \supset \Box \Box B} \Box \supset R $$

Proposition 15. The sequent calculus for $\text{DTL}^\circ_0/\Box$ proves $* A \supset A$.

Proof. $\frac{A \rightarrow A}{* A \rightarrow A} * L$

$$ \frac{* A \rightarrow A}{* A \supset A} * \supset R $$

Proposition 16. The sequent calculus for $\text{DTL}^\circ_0/\Box$ proves $* A \supset * * A$.

Proof. $\frac{* A \rightarrow * A}{* A \rightarrow * * A} * R$

$$ \frac{* A \rightarrow * * A}{* A \supset * * A} * \supset R $$

Proposition 17. The sequent calculus for $\text{DTL}^\circ_0/\Box$ proves $(A \supset B) \supset (* A \supset B)$. 

6
Proposition 20. The sequent calculus for $\mathcal{DTL}_0^{\ast/\triangleright}$ proves $\ast A \supset \circ A$.

Proof. 
\[
\begin{align*}
A \rightarrow A & \quad B \rightarrow B \quad \supset \ L \\
A \supset B, A \rightarrow B & \quad \ast L \\
A \supset B, \ast A \rightarrow B & \quad IA \\
\ast A, A \supset B & \rightarrow B \quad \ast L \\
\ast A, (A \supset B) & \rightarrow \ast B \quad \ast R \\
\ast A, (A \supset B) & \rightarrow A \supset \ast B \quad \supset R \\
(A \supset B) & \rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow \supset (A \supset B) \supset (\ast A \supset \ast B) \supset R
\end{align*}
\]

Proposition 19. The sequent calculus for $\mathcal{DTL}_0^{\ast/\triangleright}$ proves $\circ A \equiv \ast \circ A$.

Proof. 
\[
\begin{align*}
\ast \circ A & \rightarrow \ast \circ A \quad \ast L \\
\circ A \rightarrow \ast \circ A & \rightarrow \ast \circ A \quad \ast R \\
\rightarrow \circ \circ A \supset \circ \circ A & \supset R \\
\rightarrow \circ \circ A \supset \circ \circ A \quad \ast R \\
\rightarrow \supset \circ \circ A \supset \circ \circ A \quad \supset R \\
\rightarrow (\circ \circ A \supset \circ \circ A) \supset \circ \circ A \quad \supset R \\
\rightarrow (\circ \circ A \supset \circ \circ A) \supset \circ \circ A \quad \supset R
\end{align*}
\]

Proposition 21. The sequent calculus for $\mathcal{DTL}_0^{\ast/\triangleright}$ proves $\circ (A \supset B) \equiv (\circ A \cup \circ B)$.

Proof. 
\[
\begin{align*}
A \rightarrow A & \quad B \rightarrow B \quad \supset \ L \\
A \supset A, A \rightarrow B & \quad \ast L \\
A \supset B, \ast A \rightarrow B & \quad IA \\
\ast A, A \supset B & \rightarrow B \quad \ast L \\
\ast A, (A \supset B) & \rightarrow \ast B \quad \ast R \\
\ast A, (A \supset B) & \rightarrow A \supset \ast B \quad \supset R \\
(A \supset B) & \rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R
\end{align*}
\]

\[
\begin{align*}
A \rightarrow A & \quad B \rightarrow B \quad \supset \ L \\
A \supset A, A \rightarrow B & \quad \ast L \\
A \supset B, \ast A \rightarrow B & \quad IA \\
\ast A, A \supset B & \rightarrow B \quad \ast L \\
\ast A, (A \supset B) & \rightarrow \ast B \quad \ast R \\
\ast A, (A \supset B) & \rightarrow A \supset \ast B \quad \supset R \\
(A \supset B) & \rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R
\end{align*}
\]

\[
\begin{align*}
A \rightarrow A & \quad B \rightarrow B \quad \supset \ L \\
A \supset A, A \rightarrow B & \quad \ast L \\
A \supset B, \ast A \rightarrow B & \quad IA \\
\ast A, A \supset B & \rightarrow B \quad \ast L \\
\ast A, (A \supset B) & \rightarrow \ast B \quad \ast R \\
\ast A, (A \supset B) & \rightarrow A \supset \ast B \quad \supset R \\
(A \supset B) & \rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R
\end{align*}
\]

\[
\begin{align*}
A \rightarrow A & \quad B \rightarrow B \quad \supset \ L \\
A \supset A, A \rightarrow B & \quad \ast L \\
A \supset B, \ast A \rightarrow B & \quad IA \\
\ast A, A \supset B & \rightarrow B \quad \ast L \\
\ast A, (A \supset B) & \rightarrow \ast B \quad \ast R \\
\ast A, (A \supset B) & \rightarrow A \supset \ast B \quad \supset R \\
(A \supset B) & \rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R
\end{align*}
\]

\[
\begin{align*}
A \rightarrow A & \quad B \rightarrow B \quad \supset \ L \\
A \supset A, A \rightarrow B & \quad \ast L \\
A \supset B, \ast A \rightarrow B & \quad IA \\
\ast A, A \supset B & \rightarrow B \quad \ast L \\
\ast A, (A \supset B) & \rightarrow \ast B \quad \ast R \\
\ast A, (A \supset B) & \rightarrow A \supset \ast B \quad \supset R \\
(A \supset B) & \rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R \\
\rightarrow (A \supset B) \supset (\ast A \supset \ast B) \supset R
\end{align*}
\]
Proposition 22. The sequent calculus for $\mathsf{DTL}_{0}^{\circ \land / \square}$ proves $(A \land * (A \supset A)) \supset * A$.

Proof.

\[
\begin{array}{c}
A \rightarrow A \\
A \supset \circ A, A \rightarrow \circ A \\
(A \supset \circ A), * A \rightarrow \circ A \\
* A, (A \supset \circ A) \rightarrow \circ A \\
* A, (A \supset \circ A) \rightarrow \circ A \\
* A, (A \supset \circ A) \rightarrow \circ A \\
A, (A \supset \circ A) \rightarrow * A \\
A \land *(A \supset \circ A), *(A \supset \circ A) \rightarrow * A \\
(A \land *(A \supset \circ A), A \land *(A \supset \circ A) \rightarrow * A \\
A \land *(A \supset \circ A) \rightarrow * A \\
\rightarrow (A \land *(A \supset \circ A)) \supset * A \supset R
\end{array}
\]

Lastly, we remark that all derivations in the sequent calculus for $\mathsf{DTL}_{0}^{\circ \land / \square}$ are cut-free and harmonious.

5 Admissibility of $\mathsf{DTL}_{0}^{\circ \land / \square}$ Rules of Inference

We follow the strategy for cut-elimination and recall the definitions of descendents, ancestors, and depth of a formula from \cite{6}. All of the inferences of $\mathsf{DTL}_{0}^{\circ \land / \square}$ (with the exception of the cut rule) have a principal formula which is, by definition, the formula occurring in the lower sequent of the inference which is not in the cedents $\Gamma$ or $\Delta$ (or $\Theta$ or $\Lambda$). The exchange inferences have two principle formulas, as do $\circ \land CA$, $IND$, and $L \circ R$. Every inference, except weakenings, has one or more auxiliary formulas which are the formulas $A$ and $B$, occurring in the upper sequent(s) of the inference. The formulas which occur in the cedents $\Gamma, \Delta, \Theta, \Lambda$ are called side formulas of the inference. The two auxiliary formulas of a cut inference are called the cut formulas. If $C$ is a side formula in an upper sequent of an inference then the immediate descendent of $C$ is the corresponding occurrence of the same formula in the same position in the same cedent in the lower sequent of the inference. If $C$ is an auxiliary formula of any inference except an exchange or cut inference, then the principal formula of the inference is the immediate descendent of $C$. For an exchange inference, the immediate descendent of the $A$ or $B$ in the upper sequent is the $A$ or $B$, respectively, in the lower sequent. The cut formulas of a cut inference do not have immediate descendents. We say that $C$ is an immediate ancestor of $D$ if and only if $D$ is an immediate descendent of $C$. The ancestor relation is defined to be the reflexive, transitive closure of the immediate ancestor relation; thus, $C$ is an ancestor of $D$ if and only if there is a chain of zero or more immediate ancestors from $D$ to $C$. A direct ancestor of $D$ is an ancestor of $C$ of $D$ such that $C$ is the same formula as $D$; descendent and direct descendent are defined conversely. The depth of a formula $A$ is the height of the tree representation of the formula, with the depth of a cut inference the depth of its cut formula:

- $\text{depth}(A) = 0$ for $A$ atomic.
- $\text{depth}(A \land B) = \text{depth}(A \lor B) = \text{depth}(A \supset B) = 1 + \max\{\text{depth}(A), \text{depth}(B)\}$.
- $\text{depth}(\neg A) = \text{depth}(\square A) = \text{depth}(\circ A) = \text{depth}(\ast A) = 1 + \text{depth}(A)$.

The superexponentiation function $2^{x}_{i}$, for $i, x \geq 0$, is defined inductively by $2^{x}_{0} = x$ and $2^{x}_{i+1} = 2^{2^{x}_{i}}$. 

8
Theorem 1. Let \( \Pi \) be a \( \text{DTL}_0^{\alpha/\Box} \)-proof and suppose every cut formula in \( \Pi \) has depth less than or equal to \( d \). Then there is a cut-free \( \text{DTL}_0^{\alpha/\Box} \)-proof \( \Pi^* \) with the same endsequent as \( \Pi \), with size
\[
\|\Pi^*\| < 2^{|\Pi|/d+2}.
\]

Proof. Lemma 1 shows how to replace a single cut by lower depth cut inference. Iteration of this construction removes all cuts of maximum depth \( d \) in a proof, which is stated as Lemma 2, from which the theorem is a consequence.

Lemma 1. Let \( \Pi \) be a \( \text{DTL}_0^{\alpha/\Box} \)-proof with final inference a cut of depth \( d \) such that every other cut in \( \Pi \) has depth strictly less than \( d \). Then there is a \( \text{DTL}_0^{\alpha/\Box} \)-proof \( \Pi^* \) with all cuts in \( \Pi^* \) of depth less than \( d \) with
\[
\|\Pi^*\| < \|\Pi\|^2.
\]

We closely follow the proof of Lemma 2.4.2.1 in [6] with case (e) and (f) omitted, and consider the additional cases of the cut formula as \( \Box B, \circ B, \) and \( \ast B \).

Proof. The proof \( \Pi \) ends with a cut inference
\[
\alpha \quad \beta \\
\vdots \\
\Gamma \rightarrow \Theta, A \quad A, \Gamma \rightarrow \Theta \\
\Gamma, \Gamma \rightarrow \Theta, \Theta \\
\Gamma \rightarrow \Theta
\]
CUT
where the depth of the cut formula \( A \) equals \( d \) and where all cuts in the subproofs \( \alpha \) and \( \beta \) have depth strictly less than \( d \). The lemma is proved by cases, based on the outermost logical connective of the cut formula \( A \). We can assume without loss of generality that both \( \alpha \) and \( \beta \) contain at least one strong inference; since otherwise, we must have \( A \in \Gamma \) or \( \Theta \), or have a formula which occurs in both \( \Gamma \) and \( \Theta \), and in the former case, the sequent \( \Gamma \rightarrow \Theta \) is obtainable by weak inferences from one of the upper sequents and the cut can therefore be eliminated. In the latter case, \( \Gamma \rightarrow \Theta \) can be inferred with no cut inference at all.

1. Suppose \( A \) is a formula of the form \( \neg B \). We shall form new proofs \( \alpha^* \) and \( \beta^* \) of the sequents \( B, \Gamma \rightarrow \Theta \) and \( \Gamma \rightarrow \Theta, B \), which can then be combined with a cut inference of depth \( d-1 \) to give the proof \( \Pi^* \) of \( \Gamma \rightarrow \Theta \). To form \( \alpha^* \), first form \( \alpha' \) by replacing every sequent \( \Delta \rightarrow \Lambda \) in \( \alpha \) with the sequent \( \Delta, B \rightarrow \Lambda^- \), where \( \Lambda^- \) is obtained from \( \Lambda \) by removing all direct ancestors of the cut formula \( A \). Of course, \( \alpha' \) is not a valid proof; e.g., a \( \neg R \) inference in \( \alpha \) of the form \( \Delta, \Lambda \rightarrow \Delta \rightarrow \Lambda, \neg B \) could become in \( \alpha' \), \( \Delta, B \rightarrow \Lambda^- \). This is not a valid inference, but can be modified to become a valid inference by inserting some exchanges and a contraction. In this manner, \( \alpha' \) can be modified so that it becomes a valid proof \( \alpha^* \) by removing some \( \neg L \) inferences and inserting some weak inferences. The proof \( \beta^* \) of \( \Gamma \rightarrow \Theta, B \) is formed in a similar manner from \( \beta \). No new cuts are introduced by this process and, since we do not count weak inferences, \( \|\alpha^*\| \leq \|\alpha\| \) and \( \|\beta^*\| \leq \|\beta\| \); thus \( \Pi^* \) has only cuts of depth < \( d \) and has \( \|\Pi^*\| \leq \|\Pi\| \).

2. Now suppose the cut formula \( A \) is of the form \( B \lor C \). We defined \( \alpha' \) as a tree of sequents, with root labelled \( \Gamma \rightarrow \Theta, B, C \), by replacing every sequent \( \Delta \rightarrow \Lambda \) in \( \alpha \) with the sequent \( \Delta \rightarrow \Lambda^- \), \( B, C \), where \( \Lambda^- \) is \( \Lambda \) minus all occurrences of direct ancestors of the cut formula. By removing some formerly \( \lor R \) inferences from \( \alpha' \) and by adding some weak inferences, \( \alpha' \) can be transformed into a valid proof \( \alpha^* \). Now construct \( \beta_B \) from \( \beta \) by replacing every occurrence in \( \beta \) of \( B \lor C \) as a direct ancestor of the cut formula with just the formula \( B \). One way that \( \beta_B \) can fail to be valid is that an \( \lor L \) inference \( B, \Delta \rightarrow \Lambda \lor C, \Delta \rightarrow \Lambda \lor \Delta \rightarrow \Lambda \lor B \rightarrow \Lambda \) may become just \( B, \Delta \rightarrow \Lambda \lor C, \Delta \rightarrow \Lambda \lor \Delta \rightarrow \Lambda \) in \( \beta_B \). This is no longer
a valid inference, but it can be fixed up by discarding the inference and its upper right hypothesis, including discarding the entire subproof of the upper right hypothesis. The only other changes needed to make \( \beta_B \) valid are the addition of weak inferences, and in this way, a valid proof \( \beta_B \) of \( B, \Gamma \rightarrow \Theta \) is formed. A similar process forms a valid proof \( \beta_C \) of \( C, \Gamma \rightarrow \Theta \). The proof \( \Pi^* \) can now be defined to be

\[
\begin{align*}
\alpha^* & \\
\vdots & \\
\Gamma \rightarrow \Theta, B, C & \quad \vdots \quad C, \Gamma \rightarrow \Theta \quad \text{CUT} \\
\Gamma, \Gamma \rightarrow \Theta, B & \quad \vdots \\
\Gamma \rightarrow \Theta, B & \quad \text{CUT} \\
\Gamma, \Gamma \rightarrow \Theta, \Theta & \\
\Gamma \rightarrow \Theta 
\end{align*}
\]

The process of forming \( \alpha^*, \beta_B, \) and \( \beta_C \) did not introduce any new cuts or any new strong inferences. Thus, we clearly have that every cut in \( \Pi^* \) has depth \( < d \), and that \( \| \Pi^* \| \leq \| \alpha^* \| + \| \beta_B \| + 2 \). Since \( \| \Pi \| = \| \alpha \| + \| \beta \| + 1 \) and \( \| \alpha^* \|, \| \beta \| \geq 1 \), this suffices to prove the lemma for this case.

3. The cases where \( A \) has outermost connective \( \land \) or \( \supset \) are very similar to the previous case and are omitted.

4. Now suppose \( A \) is of the form \( \Box B \); we remark that the case where \( A \) is of the form \( \ast B \) is identical. Then we transform the proof \( \Pi \) with a cut of depth \( d \):

\[
\begin{align*}
\vdots & \\
\Box \Delta \rightarrow B & \quad \Box \Delta \rightarrow \Box B \quad \Box R \\
\Box \Delta \rightarrow \Box B & \quad \Box \Delta \rightarrow \Box B \\
\Gamma \rightarrow \Theta, \Box B & \\
\Gamma, \Gamma \rightarrow \Theta, \Theta & \quad \text{CUT} \\
\Gamma \rightarrow \Theta 
\end{align*}
\]

into the proof \( \Pi^* \) with a cut of depth \( d - 1 \):

\[
\begin{align*}
\vdots & \\
\Box \Delta \rightarrow B & \quad \Box \Delta \rightarrow \Box B \\
\Box \Delta, \Box B \rightarrow \Delta & \quad \Box \Delta \rightarrow \Box B \\
\Box \Delta, \Box B \rightarrow \Delta & \quad \Box \Delta \rightarrow \Box B \\
\Gamma \rightarrow \Theta, \Box B & \\
\Gamma, \Gamma \rightarrow \Theta, \Theta & \quad \text{CUT} \\
\Gamma \rightarrow \Theta 
\end{align*}
\]

where clearly \( \| \Pi^* \| < \| \Pi \|^2 \).

5. Now suppose \( A \) is of the form \( \Diamond B \). Then we transform the proof \( \Pi \) with a cut of depth \( d \):

\[
\begin{align*}
\vdots & \\
\Delta \rightarrow B & \quad \Diamond \rightarrow \Diamond R \\
\Diamond \rightarrow \Diamond B & \quad \Diamond \rightarrow \Diamond R \\
\Gamma \rightarrow \Theta, \Diamond B & \\
\Gamma, \Gamma \rightarrow \Theta, \Theta & \quad \text{CUT} \\
\Gamma \rightarrow \Theta 
\end{align*}
\]
into the proof $\Pi^*$ with a cut of depth $d - 1$:

\[
\begin{array}{c}
\Delta \rightarrow B \\
\Delta \rightarrow A \\
\circ \Delta \rightarrow \circ A \\
\Gamma \rightarrow \Theta
\end{array}
\ 
CUT
\]

\[
\begin{array}{c}
\Delta \rightarrow A \\
B \rightarrow \Lambda
\end{array}
\]

\[
L \circ R
\]

where clearly $\|\Pi^*\| < \|\Pi\|^2$.

6. Finally, consider the case where $A$ is atomic. Form $\beta'$ from $\beta$ by replacing every sequent $\Delta \rightarrow \Lambda$ in $\beta$ with the sequent $\Delta^-, \Gamma \rightarrow \Theta, \Lambda$, where $\Delta^-$ is $\Delta$ minus all occurrences of direct ancestors of $A$. $\beta'$ will end with the sequent $\Gamma, \Gamma \rightarrow \Theta, \Theta$ and will be valid as a proof, except for its initial sequents. The initial sequents $B \rightarrow B$ in $\beta$, with $B$ not a direct ancestor of the cut formula $A$, become $B, \Gamma \rightarrow \Theta, \Theta$ in $\beta'$; these are readily inferred from the initial sequent $B \rightarrow B$ with only weak inferences. On the other hand, the other initial sequents $A \rightarrow A$ in $\beta$ become $\Gamma \rightarrow \Theta, A$ which is just the endsequent of $\alpha$. The desired proof $\Pi^*$ of $\Gamma \rightarrow \Theta$ is thus formed from $\beta'$ by adding some weak inferences and adding some copies of the subproof $\alpha$ to the leaves of $\beta'$, and by adding some exchanges and contractions to the end of $\beta'$. Since $\alpha$ and $\beta$ have only cuts of degree $< d$ (i.e., have no cuts, since $d = 0$), $\Pi^*$ likewise has only cuts of degree $< d$. Also, since the number of initial sequents in $\beta'$ is bounded by $\|\beta\| + 1$, the size of $\Pi^*$ can be bounded by

\[
\|\Pi^*\| < \|\beta\| + \|\alpha\|(\|\beta\| + 1) < (\|\alpha\| + 1)(\|\beta\| + 1) < \|\Pi\|^2.
\]

This completes the proof of Lemma 1. \hfill \Box

**Lemma 2.** If $\Pi$ is a $\text{DTL}_0^{\alpha/\circ}$-proof with all cuts of depth at most $d$, there is a $\text{DTL}_0^{\alpha/\circ}$-proof $\Pi^*$ with the same endsequent which has all cuts of depth strictly less than $d$ and with size

\[
\|\Pi^*\| < 2^{2^{\|\Pi\|}}.
\]

We exactly follow Lemma 2.4.2.2 of [6] to prove the result for $\text{DTL}_0^{\alpha/\circ}$ as follows.

**Proof.** Lemma 2 will be proved by induction on the number of depth $d$ cuts in $\Pi$. The base case with no depth $d$ cuts is trivial as $\|\Pi\| < 2^{2^{\|\Pi\|}}$. For the induction, we prove the lemma in the case where $\Pi$ ends with a cut inference

\[
\begin{array}{c}
\alpha \\
\beta
\end{array}
\ 
CUT
\]

\[
\begin{array}{c}
\Gamma \rightarrow \Theta, A \\
\Gamma, \Gamma \rightarrow \Theta, \Theta
\end{array}
\ 
CUT
\]

where $\alpha$ and $\beta$ are proofs of $\Gamma \rightarrow \Theta, A$ and $A, \Gamma \rightarrow \Theta$, respectively, and the cut formula $A$ is of depth $d$.

First suppose that one of the subproofs, say $\beta$, does not have any strong inferences; i.e., $\|\beta\| = 0$. Therefore, $\beta$ must contain either the axiom $A \rightarrow A$, or must have direct ancestors of the cut formula $A$ introduced only by weakenings. In the former case, $A$ must appear in $\Theta$, and the desired proof $\Pi^*$ can be obtained from $\alpha$ by adding some exchanges and a contraction to the end of $\alpha$. In the second case, $\Pi^*$ can be obtained from $\beta$ by removing all the $\text{WA}$ inferences that introduce direct ancestors of the cut formula $A$ (and possibly removing some exchanges and contractions involving these $A$'s). A similar argument works for the case $\|\alpha\| = 0$. In both cases, $\|\Pi^*\| < \|\Pi\| < 2^{2^{\|\Pi\|}}$. Second, suppose that $\|\alpha\|$ and $\|\beta\|$ are both nonzero. By the induction hypothesis, there are proofs $\alpha^*$ and $\beta^*$ of the same endsequeants with all cuts of depth less than $d$, and with $\|\alpha^*\| < 2^{2^{\|\alpha\|}}$ and $\|\beta^*\| < 2^{2^{\|\beta\|}}$. Applying Lemma 1 to the proof

11
\[
\begin{array}{c}
\alpha^* \\
\vdots \\
\Gamma \rightarrow \Theta, A \\
\vdots \\
\beta^* \\
\end{array}
\]

\[
\frac{A, \Gamma \rightarrow \Theta}{\Gamma, \Gamma \rightarrow \Theta, \Theta}
\]

\[
\frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta}
\]

\(CUT\)

gives a proof \(\Pi^*\) of \(\Gamma \rightarrow \Theta\) with all cuts of depth \(< d\), so that

\[
\|\Pi^*\| < (\|\alpha^*\| + \|\beta^*\| + 1)^2 \leq \left(2^{2\|\alpha\|} + 2^{2\|\beta\|} - 1\right)^2 < 2^{2\|\alpha\|+\|\beta\|+1} = 2^\|\Pi\|.
\]

The final inequality holds since \(\|\alpha\|, \|\beta\| \geq 1\). This completes the proof of Lemma 2 and the Cut-Elimination Theorem for \(\text{DTL}_{0^*/\Box}\).

As a corollary, we cite a depth independent bound on \(\|\Pi^*\|\) from page 42 of [6].

**Corollary 1.** Suppose \(\Pi\) is a \(\text{DTL}_{0^*/\Box}\)-proof of the sequent \(\Gamma \rightarrow \Theta\). Then there is a cut-free proof \(\Pi^*\) of the same sequent with size

\[
\|\Pi^*\| < 2^\|\Pi\|/2.
\]

In conclusion, every theorem provable from the axiomatization of \(\text{DTL}_{0^*/\Box}\) in [3] has a \(\text{DTL}_{0^*/\Box}\) sequent calculus derivation, from which completeness follows. This holds due to cut elimination as the four rules of inference are admissible in the \(\text{DTL}_{0^*/\Box}\)-sequent calculus; namely the first three:

\[
\begin{array}{c}
\rightarrow A \\
\rightarrow \Box A \\
\rightarrow \circ A \\
\end{array}
\]

are easily proved to be admissible and the cut elimination theorem for \(\text{DTL}_{0^*/\Box}\) shows that

\[
\rightarrow (A \supset B) \\
\rightarrow B
\]

is admissible in the sequent calculus for \(\text{DTL}_{0^*/\Box}\).

**References**


