Positive periodic solutions of discrete three-level food-chain model of Holling type II

Y.G. Sun a,*, S.H. Saker b

a Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China
b Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract

With the help of differential equations with piecewise constant arguments, we first derive a discrete analogy of continuous three level food-chain model of Holling type II, which is governed by difference equations with periodic coefficients. A set of sufficient conditions is derived for the existence of positive periodic solutions with strictly positive components by using the continuation theorem in coincidence degree theory. Particularly, the upper and lower bounds of the periodic solutions are also established.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Periodic; Food-chain model; Difference equations

1. Introduction

The dynamic relationship between predators and preys has long been and will continue to be one of the dominant subjects in mathematical ecology due to its universal existence and importance. At first sight, these problems may appear to be simple mathematically. However, in fact, they are often very challenging and complicated. For more details of different types of predator–prey system we refer to [2] and the references cited therein.

The functional response is a key element in all prey interactions. In population dynamics, the functional response refers to the numbers eaten per predator per unit time as a function of prey density. Holing assumed three types of predation function in [9–11] and studied predation of small mammals on pine sawflies and found that the predation rates increased with increasing prey population density. This results from two effects: (1) each predator increased its consumption rate when exposed to a higher density, and (2) predator density increased with increasing prey density. Holling suggested three kinds of functions response as follows:

\begin{align*}
(1) \quad p_1(x) &= ax, \\
(2) \quad p_2(x) &= \frac{ax}{a+x}, \\
(3) \quad p_3(x) &= \frac{ax^2}{a+x^2},
\end{align*}

* Corresponding author.

E-mail address: yugsun@pku.edu.cn (Y.G. Sun).

0096-3003/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
where $x$ represents the density of prey. Functions $p_1(x), p_2(x)$, and $p_3(x)$ referred to as Holling types I, II and III, respectively, and $a > 0$ is the predation rate of the predator.

In recent years, the ideas of the predator–prey systems extended to the more general models of different types of species with interacting or without interacting. The interacting two-prey one-predator model of Holling type I have been considered by many authors. Azar et al. [1] performed the numerical analysis of an interacting two-prey one-predator model where the predator is harvested. Kumar et al. [5] considered the model that has been studied by Azar et al. in [1] with harvesting of the predator at a constant rate and used the harvest rate as a control parameter. They have found that the periodic solutions arise from stable stationary states when the harvest rate exceeds a certain limit. Also, the stability of these periodic solutions is investigated with the variation of this control parameter. Recently, Bhattacharya and Karan [4] considered the non-interacting two-preys $y_1$ and $y_2$ in the presence of their common natural enemy (predator) $y_3$ in the form

$$\frac{dy_1(t)}{dt} = y_1(t)(a_1 - b_1y_3(t)),$$

$$\frac{dy_2(t)}{dt} = y_2(t)(a_2 - c_2y_3(t) - b_2y_3(t)),$$

$$\frac{dy_3(t)}{dt} = y_3(t)(-a_3 + b_3y_1(t) + b_4y_2(t)),$$

where $a_i$ are the natural growth rate of $y_i$ ($i = 1, 2$); $b_i$ are the predation coefficients for $y_i$, where $y_1$ is density-independent but $y_2$ is density-dependent with intra-specific coefficients $c_2$ and $y_2$ grows logistically with growth rate $a_2$ and carrying capacity $a_2/c_2$; $a_3$ is the natural death rate for the predator in the absence of preys; $b_3/b_1$ and $b_4/b_2$ are the conversion factors. The authors in [4] established some sufficient conditions for the global stability and discussed the control biologically by release of additional predators and chemically by using the non-selective non-residual pesticide.

In [2] Elabbasy and Saker, investigated the dynamics of the mathematical model of two non-interacting preys in presence of their common natural enemy (predator) based on the non-autonomous differential equations, (with periodic coefficients). They have established sufficient conditions for the permanence, extinction and global stability in the general non-autonomous case. In the periodic case, by means of the continuation theorem in coincidence degree theory, they have established a set of sufficient conditions for the existence of positive periodic solutions with strictly positive components. Also, some sufficient conditions for the global asymptotic stability of the positive periodic solution are obtained.

In this paper, we concentrate our work to the discrete analogy of different types of predator–prey models, namely three-level food-chain model with functional response of Holling type II (referred also as Michaelis–Menten type)

$$\frac{dy_1}{dt} = y_1 \left( a_1 - b_1y_1 - \frac{c_1y_2}{y_1 + 1} \right),$$

$$\frac{dy_2}{dt} = y_2 \left( -a_2 + \frac{c_2y_1}{y_1 + 1} - \frac{c_3y_3}{y_2 + 1} \right),$$

$$\frac{dy_3}{dt} = y_3 \left( -a_3 + \frac{c_4y_2}{y_2 + 1} \right),$$

where $a_i, b_1, c_j > 0$, $i = 1, 2, j = 1, 2, 3, 4$. The above equations represent an ecological situation where a prey population $y_1$ is predated by individuals of population $y_2$. This population, in turns, serves as a favorite food for individuals of population $y_3$ (top-predator). The common property of the above system is that $y_2$ and $y_3$ are specialist and generalist, respectively. The predator (top-predator) consumes the prey (predator) with functional response of Holling type II (Michaelis–Menten type), $a_2, a_3$ are the death rates for predator, top-predator, respectively.

The variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolution theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus,
the assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In fact, any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this it is realistic to assume that the parameters in the models are periodic functions of period $\omega$. Thus, the modification of (1.1) according to the environmental variation is the non-autonomous differential equations

$$\frac{dy_1(t)}{dt} = y_1(t) \left( a_1(t) - b_1(t)y_1(t) \right) - \frac{c_1(t)y_2(t)}{y_1(t) + 1},$$

$$\frac{dy_2(t)}{dt} = y_2(t) \left( -a_2(t) + \frac{c_2(t)y_1(t)}{y_1(t) + 1} - \frac{c_3(t)y_3(t)}{y_2(t) + 1} \right),$$

$$\frac{dy_3(t)}{dt} = y_3(t) \left( -a_3(t) + \frac{c_4(t)y_2(t)}{y_2(t) + 1} \right).$$

(1.2)

Analytically, non-linear differential equations are difficult to manage and therefore many articles have examined the models as the difference equations. In practice, one can formulate a discrete model directly from experiments and observations. Some times, for numerical purpose one wants to purpose a finite-difference scheme to numerically solve a given differential model, especially when the differential equation cannot be solved explicitly.

For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points. But unless, we can explicitly solve both equations, it is impossible to satisfy this requirement. Most of the time, it is desirable that a difference equation, when derived from a differential equation, preserves the dynamical features of the corresponding continuous time model such as equilibria, oscillation, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous models, then the discrete time models can be used without loss of any functional similarity to the continuous-time models and it will preserve the considered realities; such discrete time models can be called “Dynamically consistent” with the continuous time models.

There is no unique way of deriving discrete time version of dynamical systems corresponding to continuous time formulations. One of the ways of deriving difference equations modeling the dynamic of populations with non-overlapping generations is based on appropriate modifications of models with overlapping generations. In this approach, differential equations with piecewise constant arguments have been useful, see for example the paper by Liu and Gopalsamy [6]. Recently the method that has been established by Liu and Gopalsamy has been used by some authors to find the discrete analogy of some mathematical modes, we refer the reader to [3,8,12] and the references cited therein.

Before stating our main results about the existence of positive periodic solutions, we will derive the discrete analogy of Eq. (1.2). Thinks to differential equations with piecewise constant arguments, we can go on with the discrete analogy of Eq. (1.2). Let us assume that the average growth rate in (1.2) changes at regular intervals of time, then we can incorporate this aspect in (1.2) and obtain the following modified equation:

$$\frac{1}{y_1(t)} \frac{dy_1(t)}{dt} = a_1([t]) - b_1([t])y_1([t]) - \frac{c_1([t])y_2([t])}{y_1([t]) + 1},$$

$$\frac{1}{y_2(t)} \frac{dy_2(t)}{dt} = -a_2([t]) + \frac{c_2([t])y_1([t])}{y_1([t]) + 1} - \frac{c_3([t])y_3([t])}{y_2([t]) + 1},$$

$$\frac{1}{y_3(t)} \frac{dy_3(t)}{dt} = -a_3([t]) + \frac{c_4([t])y_2([t])}{y_2([t]) + 1},$$

(1.3)

where $[t]$ denotes the integer part of $t$, $t \in (0, \infty)$. Equation of type (1.3) is known as differential equation with piecewise with constant argument and this equation occupy a position midway between differential and difference equation. By a solution of (1.3), we mean a function $Y(t) = (y_1(t), y_2(t), y_3(t))^T$, which is defined for $t \in [0, \infty)$, and satisfy the properties:
(a) \( Y \) is continuous on \([0, \infty)\).
(b) The derivative \( \frac{dY}{dt} = \left( \frac{dY_1(t)}{dt}, \frac{dY_2(t)}{dt}, \frac{dY_3(t)}{dt} \right)^T \) exists at each point \( t \in [0, \infty) \) with the possible exception of the points \( t \in \{0, 1, 2, \ldots \} \), where left side derivative exists.
(c) Eq. (1.3) is satisfied on each interval \([n, n + 1)\) with \( n = 0, 1, 2, \ldots \)

By integrating (1.3) on any interval of the form \([n, n + 1)\), \( n = 0, 1, 2, \ldots \) we obtain
\[
\begin{align*}
y_1(t) &= y_1(n) \exp \left( \left[ a_1(n) - b_1(n) y_1(n) - \frac{c_1(n) y_2(n)}{y_1(n) + 1} \right] (t - n) \right), \\
y_2(t) &= y_2(n) \exp \left( \left[ -a_2(n) + \frac{c_2(n) y_1(n)}{y_1(n) + 1} - \frac{c_3(n) y_3(n)}{y_2(n) + 1} \right] (t - n) \right), \\
y_3(t) &= y_3(n) \exp \left( \left[ -a_3(n) + \frac{c_4(n) y_2(n)}{y_2(n) + 1} \right] (t - n) \right).
\end{align*}
\]
Letting \( t \to n + 1 \), we have
\[
\begin{align*}
y_1(n + 1) &= y_1(n) \exp \left( a_1(n) - b_1(n) y_1(n) - \frac{c_1(n) y_2(n)}{y_1(n) + 1} \right), \\
y_2(n + 1) &= y_2(n) \exp \left( -a_2(n) + \frac{c_2(n) y_1(n)}{y_1(n) + 1} - \frac{c_3(n) y_3(n)}{y_2(n) + 1} \right), \\
y_3(n + 1) &= y_3(n) \exp \left( -a_3(n) + \frac{c_4(n) y_2(n)}{y_2(n) + 1} \right),
\end{align*}
\]
which is the discrete time analogy of (1.3).

By a solution of Eq. (1.4), we mean a sequences \( \{y_1(n), y_2(n), y_3(n)\} \) which are defined for \( n \geq 0 \) and which satisfies (1.4) for \( n \geq 0 \). Considering the biological significance of (1.4), we specify
\[
y_1(0), \quad y_2(0), \quad y_3(0) > 0.
\]
The exponential forms of (1.4) assure that the solution \( (y_1(n), y_2(n), y_3(n))^T \) with respect to any initial condition (1.5) remains positive. We remark that in recent years periodic population dynamics has become a very popular subject. In fact, several different models have been studied in [8,12–17].

In this paper, we derive a set of sufficient conditions for existence of positive periodic solutions with strictly positive components for the discrete three-level chain-food model (1.4). Such an existence problem is highly non-trivial and to the best of our knowledge, no work has been done for the discrete model (1.4) of Holling type II. The method used here will depend on the continuation theorem in coincidence degree theory proposed by Gaines and Mawhin [7], which has been widely used in the study of ordinary differential equations, and recently some authors applied it for studying the existence of periodic solutions for some continuous and discrete mathematical models, see for example the papers [2,3,8,12].

2. Existence of positive periodic solutions

In this section, we will assume that the parameters in (1.4) are positive periodic sequences of period \( \omega \). A very basic and important ecological problem in the study of dynamics of population in a periodic environment is the existence of a positive periodic solution, which plays a similar role played by the equilibrium of the autonomous models. Thus, it is reasonable to ask for a condition under which the resulting periodic non-autonomous equation have a positive periodic solution.

For the reader’s convenience, we now recall some basic tools in the frame of Mawhin’s coincidence degree theorem that will be used to prove the existence of periodic solution of (1.4), borrowing notations and terminology from [7].

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Banach spaces, let \( L : \text{Dom} L \subset \mathcal{X} \to \mathcal{Y} \) be a linear mapping, and let \( N : \mathcal{X} \to \mathcal{Y} \) be a continuous mapping.
The mapping \( L \) will be called a Fredholm mapping of index zero if the following three conditions hold:

(i) \( \ker L \) has a finite dimension.
(ii) \( \text{Im} L \) is closed and has a finite codimension.
(iii) \( \dim \ker L = \text{codim} \text{Im} L < \infty \).

If \( L \) is a Fredholm mapping of index zero and there exist continuous projectors
\[
P : \mathbb{X} \to \mathbb{X} \quad \text{and} \quad Q : \mathbb{Y} \to \mathbb{Y},
\]
such that \( \text{Im} P = \ker L \), \( \text{Im} L = \ker Q = \text{Im}(I - Q) \), it follows that
\[
L|\text{Dom} L \cap \ker P : (I - P)\mathbb{X} \to \text{Im} L,
\]
is invertible. We denote the inverse of that map by \( K_p \).

If \( \Omega \) is an open bounded subset of \( \mathbb{X} \), the mapping \( N \) will be called \( L \)-compact on \( \overline{\Omega} \) if the mapping \( QN : \overline{\Omega} \to \mathbb{Y} \) is continuous, \( QN(\overline{\Omega}) \) is bounded, and \( K_p(I - Q)N : \overline{\Omega} \to \mathbb{X} \) is compact, i.e., it is continuous and \( K_p(I - Q)N(\overline{\Omega}) \) is relatively compact, where \( K_p : \text{Im} L \to \text{Dom} L \cap \ker P \) is the inverse of the restriction \( L_p \) of \( L \) to \( \text{Dom} L \cap \ker P \), so that \( LK_p = I \) and \( K_pL = I - P \). Since \( Q \) is isomorphic to \( \ker L \), there exists an isomorphic \( J : \text{Im} Q \to \ker L \).

Now we are ready to cite the continuation theorem [7, p. 40].

**Lemma 2.1** (Continuation Theorem). Let \( \mathbb{X} \) and \( \mathbb{Y} \) be two Banach spaces and \( L \) a Fredholm mapping of index zero. Assume that \( N : \overline{\Omega} \to \mathbb{Y} \) is \( L \)-compact on \( \overline{\Omega} \) with \( \overline{\Omega} \) is open and bounded in \( \mathbb{X} \). Furthermore assume:

\[(h_1) \text{ for each } \lambda \in (0, 1), \text{ every solution of } Lx = \lambda Nx \text{ is such that } x \notin \partial \Omega,
\[(h_2) \quad QNx \neq 0 \text{ for each } x \in \partial \Omega \cap \ker L, \text{ and}
\[
\deg\{QNx, \partial \Omega \cap \ker L, 0\} \neq 0.
\]

Then the operator equation \( Lx = Nx \) has at least one solution in \( \text{Dom} L \cap \overline{\Omega} \).

Let \( \mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{R}, \mathbb{R}^+ \) denote the sets of all integers, non-negative integers, natural numbers, real numbers and non-negative real numbers, respectively. For convenience in what follows we shall let
\[
I_\omega = \{0, 1, 2, \ldots, \omega - 1\}, \quad \bar{f} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} f(n),
\]
where \( f(n) \) is an \( \omega \)-periodic sequence of real numbers defined for all \( n \in \mathbb{Z} \).

We need the following lemma in the proof of our main results.

**Lemma 2.2** [18]. Let \( f : \mathbb{Z} \to \mathbb{R} \) be periodic, i.e., \( f(n + \omega) = f(n) \). Then for any fixed \( n_1, n_2 \in I_\omega \) and for any \( n \in \mathbb{Z} \), one has
\[
f(n) \leq f(n_1) + \sum_{s=0}^{\omega-1} |f(s + 1) - f(s)|,
\]
\[
f(n) \geq f(n_2) - \sum_{s=0}^{\omega-1} |f(s + 1) - f(s)|.
\]

Our main results on the existence and the estimation of lower and upper bounds of a positive \( \omega \)-periodic solution of (1.4) are given in the following theorem.

**Theorem 2.1.** Assume that the following conditions are satisfied

(a) \( \bar{a}_2 > \bar{c}_2 \),
(b) \( \bar{a}_1 > \left( \frac{\bar{c}_3 \bar{a}_2}{\gamma + \bar{a}_3} \right) \exp\{2\bar{a}_2 \omega\} \),
(c) \( \bar{c}_4 > \bar{a}_3 \).
Then Eq. (1.4) has a positive $\omega$-periodic solution $\bar{y}(n) = (\bar{y}_1(n), \bar{y}_2(n), \bar{y}_3(n))^T$ for sufficiently large or small $\bar{b}_1$ such that $\bar{y}_i \leq \bar{y}_i(n) \leq \bar{b}_i$, $i = 1, 2, 3$, where

\[
\begin{align*}
x_1 &= \frac{\bar{a}_1 \exp\{-2\bar{a}_2 \omega\} - \bar{c}_1 (\frac{\bar{a}_1}{\bar{c}_1})}{\bar{b}_1}, \quad \bar{b}_1 = \frac{\bar{a}_1}{\bar{b}_1} \exp\{2\bar{a}_1 \omega\}, \\
x_2 &= \frac{\bar{a}_3 \exp\{-2\bar{a}_2 \omega\}}{\bar{c}_4}, \quad \bar{b}_2 = \frac{\bar{a}_3}{\bar{c}_4 - \bar{a}_3} \exp\{2\bar{a}_2 \omega\}, \\
x_3 &= \frac{\bar{a}_2 - \bar{c}_2}{\bar{c}_3}, \quad \bar{b}_3 = \frac{(\frac{\bar{a}_1}{\bar{c}_1}) \exp\{2\bar{a}_2 \omega\} + 1}{\bar{c}_3} \exp\{2\bar{a}_3 \omega\}.
\end{align*}
\]

Proof. Define

$l_3 = \{ y = y(n) : y(n) \in \mathbb{R}^3, \quad n \in \mathbb{N} \}$.

Let $l^\omega \subseteq l_3$ denotes the subspace of all $\omega$-periodic sequences equipped with the usual norm $\|\cdot\|$, i.e.,

\[\|y\| = \| (y_1(n), y_2(n), y_3(n))^T \| = \max_{n \in I_\omega} |y_1(n)| + \max_{n \in I_\omega} |y_2(n)| + \max_{n \in I_\omega} |y_3(n)|, \quad y \in \mathbb{X} (or \mathbb{Z}).\]

It is easy to see that $l^\omega$ is a finite-dimensional Banach space. Let the linear operator $S : l^\omega \to \mathbb{R}$ be defined by

\[S(y) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} y(n), \quad y = \{y(n)\} \in l^\omega.\]

Then, we obtain two subspaces $l_{0}^\omega$ and $l_{c}^\omega$ defined by

\[l_{0}^\omega = \left\{ y = y(n) \in l^\omega : S(y) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} y(n) = 0 \right\}\]

and

\[l_{c}^\omega = \{ y = y(n) \in l^\omega : y(n) = \beta, \quad \text{for some } \beta \in \mathbb{R}^3 \text{ and for all } n \in \mathbb{N} \},\]

respectively. Then from Lemma 2.1 in [18], we find that $l_{0}^\omega$ and $l_{c}^\omega$ are closed linear subspaces of $l^\omega$ and $l^\omega = l_{0}^\omega \oplus l_{c}^\omega$, $\dim l^\omega = 3$.

To prove our main result, we let

\[y_i(n) = \exp\{x_i(n)\}, \quad i = 1, 2, 3.\] (2.1)

On substituting (2.1) into (1.4), we have

\[
\begin{align*}
x_1(n+1) - x_1(n) &= a_1(n) - b_1(n) \exp\{x_1(n)\} - \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1}, \\
x_2(n+1) - x_2(n) &= -a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_1(n)\} + 1} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1}, \\
x_3(n+1) - x_3(n) &= -a_3(n) + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1}. \quad (2.2)
\end{align*}
\]

In order to embed our problem into framework of continuation theorem, we define

\[l^\omega = \mathbb{X} = \mathbb{Z} = \left\{ x(n) = (x_1(n), x_2(n), x_3(n))^T \in \mathbb{R}^3 : x(n + \omega) = x(n) \right\},\]

\[N_x := \begin{pmatrix}
a_1(n) - b_1(n) \exp\{x_1(n)\} - \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \\
- a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_1(n)\} + 1} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1} \\
- a_3(n) + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1}
\end{pmatrix}, \quad x \in \mathbb{X}.
\]
and

\[ Lx := \Delta x(n), \]

where \( \Delta x(n) = (\Delta x_1(n), \Delta x_2(n), \Delta x_3(n))^T \). Then, from Lemma 2.1 in [18], we have

\[ \text{Ker} L = \mathbb{R}^3, \quad \text{Im} L = \left\{ z \in \mathbb{Z} : \sum_{n=0}^{\alpha-1} z(n) = 0 \right\} \text{ is closed in } \mathbb{Z}, \]

\[ \dim \text{Ker} L = 3 = \text{codim Im} L. \]

Therefore, \( L \) is a Fredholm mapping of index zero. Now, define

\[ Px = \frac{1}{\omega} \sum_{s=0}^{\omega-1} x(s), \quad x \in \mathbb{X} \quad \text{and} \quad Qz = \frac{1}{\omega} \sum_{s=0}^{\omega-1} z(s), \quad z \in \mathbb{Y}. \]

It follows that \( P \) and \( Q \) are continuous projectors such that

\[ \text{Im} P = \text{Ker} L \quad \text{and} \quad \text{Im} L = \text{Ker} Q = \text{Im} (I - Q). \]

Furthermore, the generalized inverse (of \( L \)) \( K_P : \text{Im} L \to \text{Ker} P \cap \text{Dom} L \) exists which is given by

\[ K_P(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) z(s). \]

Then \( QN : \mathbb{X} \to \mathbb{Z} \) and \( K_P(I - Q) \text{Im} L \to \text{Ker} P \cap \text{Dom} L \) read

\[ QNx = \left( \frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta x_1(n), \frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta x_2(n), \frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta x_3(n) \right)^T, \]

that is

\[ QNx = \left( \frac{1}{\omega} \sum_{n=0}^{\omega-1} \left( a_1(n) - b_1(n) \exp\{x_1(n)\} - \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right) \right) \]

\[ \left( \frac{1}{\omega} \sum_{n=0}^{\omega-1} \left( -a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_2(n)\} + 1} - \frac{c_1(n) \exp\{x_3(n)\}}{\exp\{x_1(n)\} + 1} \right) \right) \]

\[ \left( \frac{1}{\omega} \sum_{n=0}^{\omega-1} \left( -a_3(n) + \frac{c_3(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1} \right) \right) \]

and

\[ K_P(I - Q)Nx = (\Phi_1, \Phi_2, \Phi_3)^T, \]

where

\[ \Phi_i(x_i(s)) = \sum_{s=0}^{n-1} \Delta x_i(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) \Delta x_i(s) - \left( \frac{n}{\omega} - \frac{\omega + 1}{2\omega} \right) \sum_{s=0}^{\omega-1} \Delta x_i(s), \quad i = 1, 2, 3, \]

that is

\[ K_P(I - Q)Nx = \left( \sum_{s=0}^{n-1} \left( a_1(s) - b_1(s) \exp\{x_1(s)\} - \frac{c_1(s) \exp\{x_2(s)\}}{\exp\{x_1(s)\} + 1} \right) \right) \]

\[ \left( \sum_{s=0}^{n-1} \left( -a_2(s) + \frac{c_2(s) \exp\{x_1(s)\}}{\exp\{x_2(s)\} + 1} - \frac{c_1(s) \exp\{x_3(s)\}}{\exp\{x_2(s)\} + 1} \right) \right) \]

\[ \left( \sum_{s=0}^{n-1} \left( -a_3(s) + \frac{c_3(s) \exp\{x_2(s)\}}{\exp\{x_2(s)\} + 1} \right) \right) \]
Obviously, $QN : \mathbb{X} \rightarrow \mathbb{Y}$ and $K_{P}(I - Q)N : \mathbb{X} \rightarrow \mathbb{X}$ are continuous with respect to $s$ and they are mapping bounded continuous functions to bounded continuous functions. Since $\mathbb{X}$ is a finite dimensional Banach space, using the Ascoli–Arzela theorem, we see that $QN(\overline{\Omega})$ and $K_{P}(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, $N$ is $L$-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

Now, we search for an appropriate open and bounded subset $\Omega$ for the application of Lemma 2.1. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

\[
\begin{align*}
x_1(n + 1) - x_1(n) &= \lambda \left\{ a_1(n) - b_1(n) \exp\{x_1(n)\} - \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right\}, \\
x_2(n + 1) - x_2(n) &= \lambda \left\{ -a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_1(n)\} + 1} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1} \right\}, \\
x_3(n + 1) - x_3(n) &= \lambda \left\{ -a_3(n) + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1} \right\}.
\end{align*}
\]

(2.3)

Suppose that $x = x(n) \in \mathbb{X}$ is an arbitrary solution of the system (2.3) for a certain $\lambda \in (0, 1)$. Summing (2.3) over the interval $[0, \omega]$ and using the fact that $\sum_{s=0}^{\omega-1} \Delta x_i(s) = 0$, $i = 1, 2, 3$, we have

\[
\begin{align*}
\frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) \left\{ a_1(n) - b_1(n) \exp\{x_1(n)\} - \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right\} &= 0, \\
\frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) \left\{ -a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_1(n)\} + 1} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1} \right\} &= 0, \\
\frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) \left\{ -a_3(n) + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1} \right\} &= 0,
\end{align*}
\]

that is

\[
\begin{align*}
\sum_{n=0}^{\omega-1} \left\{ b_1(n) \exp\{x_1(n)\} + \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right\} &= \sum_{n=0}^{\omega-1} a_1(n) = \bar{a}_1 \omega, \\
\sum_{n=0}^{\omega-1} \left\{ -a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_1(n)\} + 1} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1} \right\} &= \omega \bar{a}_2, \\
\sum_{n=0}^{\omega-1} \left\{ -a_3(n) + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1} \right\} &= \omega \bar{a}_3.
\end{align*}
\]

(2.4) (2.5) (2.6)
From (2.3)–(2.6), it follows that
\[
\sum_{n=0}^{\omega-1} |x_1(n + 1) - x_1(n)| = \lambda \sum_{n=0}^{\omega-1} \left| a_1(n) - b_1(n) \exp\{x_1(n)\} - \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right|
\]
\[
< \sum_{n=0}^{\omega-1} a_1(n) + \sum_{n=0}^{\omega-1} \left( b_1(n) \exp\{x_1(n)\} + \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right) = (2\bar{a}_1)\omega,
\]
\[
\sum_{n=0}^{\omega-1} |x_2(n + 1) - x_2(n)| = \lambda \sum_{n=0}^{\omega-1} \left\{ -a_2(n) + \frac{c_2(n) \exp\{x_1(n)\}}{\exp\{x_1(n)\} + 1} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1} \right\}
\]
\[
< \sum_{n=0}^{\omega-1} a_2(n) + \sum_{n=0}^{\omega-1} \left\{ b_2(n) \exp\{x_1(n)\} + \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_2(n)\} + 1} \right\} = (2\bar{a}_2)\omega,
\]
\[
\sum_{n=0}^{\omega-1} |x_3(n + 1) - x_3(n)| = \lambda \sum_{n=0}^{\omega-1} \left\{ -a_3(n) + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1} \right\}
\]
\[
< \sum_{n=0}^{\omega-1} a_3(n) + \sum_{n=0}^{\omega-1} \left\{ b_3(n) \exp\{x_2(n)\} + \frac{c_4(n) \exp\{x_2(n)\}}{\exp\{x_2(n)\} + 1} \right\} = (2\bar{a}_3)\omega,
\]
that is
\[
\sum_{n=0}^{\omega-1} |x_1(n + 1) - x_1(n)| \leq (2\bar{a}_1)\omega,
\]
\[
\sum_{n=0}^{\omega-1} |x_2(n + 1) - x_2(n)| \leq (2\bar{a}_2)\omega,
\]
\[
\sum_{n=0}^{\omega-1} |x_3(n + 1) - x_3(n)| \leq (2\bar{a}_3)\omega.
\]

Since \(x(n) = (x_1(n), x_2(n), x_3(n))^n \in \mathbb{X}\), there exist \(\zeta_i, \eta_i \in [0, \omega]\) such that
\[
x_i(\zeta_i) = \min_{n \in [0, \omega]} x_i(n), \quad x_i(\eta_i) = \max_{n \in [0, \omega]} x_i(n), \quad i = 1, 2, 3.
\]

From (2.4) and (2.11), we see
\[
\bar{a}_1\omega = \sum_{n=0}^{\omega-1} \left\{ b_1(n) \exp\{x_1(n)\} + \frac{c_1(n) \exp\{x_2(n)\}}{\exp\{x_1(n)\} + 1} \right\} \geq \sum_{n=0}^{\omega-1} b_1(n) \exp\{x_1(\zeta_1)\} = \exp\{x_1(\zeta_1)\} b_1\omega,
\]
that is
\[
x_1(\zeta_1) \leq \ln \left( \frac{\bar{a}_1}{b_1} \right).
\]

Then from Lemma 2.2, (2.8) and (2.12), we have
\[
x_1(n) \leq x_1(\zeta_1) + \sum_{n=0}^{\omega-1} |x_1(n + 1) - x_1(n)| < \ln \left( \frac{\bar{a}_1}{b_1} \right) + (2\bar{a}_1)\omega := M_1.
\]

Also from (2.6) and (2.11), assumption (c) and noting that the function \(\frac{x}{x+1}\) is not decreasing, we find that
\[
x_2(\zeta_2) \leq \ln \left( \frac{\bar{a}_3}{c_4 - \bar{a}_3} \right).
\]

Then from Lemma 2.2, (2.9) and (2.14), we have
\[
x_2(n) \leq x_2(\zeta_2) + \sum_{n=0}^{\omega-1} |x_2(n + 1) - x_2(n)| < \ln \left( \frac{\bar{a}_3}{c_4 - \bar{a}_3} \right) + (2\bar{a}_2)\omega := M_2.
\]
Also from (2.5), (2.11) and (2.15), we obtain

\[
\omega \tilde{a}_2 = \sum_{n=0}^{\omega-1} \left\{ c_2(n) \exp\{x_1(n)\} - \frac{c_3(n) \exp\{x_3(n)\}}{\exp\{x_1(n)\} + 1} \right\} \leq \sum_{n=0}^{\omega-1} \left\{ c_2(n) - \frac{c_3(n) \exp\{x_3(\zeta_3)\}}{\exp\{M_2\} + 1} \right\}
\]

\[
= \omega \tilde{c}_2 - \frac{\omega \tilde{c}_3}{\exp\{M_2\} + 1} \exp\{x_3(\zeta_3)\},
\]

it follows that

\[
x_3(\zeta_3) \leq \ln \left( \frac{\exp\{M_2\} + 1)(\tilde{c}_2 + \tilde{a}_2)}{\tilde{c}_3} \right)
\]

and then by Lemma 2.2, (2.10), we have

\[
x_3(n) \leq x_3(\zeta_3) + \sum_{n=0}^{\omega-1} |x_3(n + 1) - x_3(n)| \leq \ln \left( \frac{\exp\{M_2\} + 1)(\tilde{c}_2 + \tilde{a}_2)}{\tilde{c}_3} \right) + (2\tilde{a}_3)\omega := M_3.
\]

From (2.6) and (2.11), we have

\[
x_2(\eta_2) \geq \ln \left( \frac{\tilde{a}_3}{\tilde{c}_4} \right).
\]

Then by Lemma 2.2, we have

\[
x_2(n) \geq x_2(\eta_2) - \sum_{n=0}^{\omega-1} |x_2(n + 1) - x_2(n)| \geq \ln \left( \frac{\tilde{a}_3}{\tilde{c}_4} \right) - (2\tilde{a}_2)\omega := M_4.
\]

From (2.4) and (2.11), we have

\[
\bar{b}_1\omega \exp\{x_1(\eta_1)\} \geq \tilde{a}_1\omega - \tilde{c}_1\omega \left( \frac{\tilde{a}_3}{\tilde{c}_4 - \tilde{a}_3} \right) \exp\{(2\tilde{a}_2)\omega\}.
\]

Then the assumption (b) implies that

\[
x_1(\eta_1) \geq \ln \left( \frac{\tilde{a}_1 - \tilde{c}_1(\tilde{a}_3 - \tilde{a}_3)}{\bar{b}_1} \exp\{(2\tilde{a}_2)\omega\} \right) := M_5.
\]

Then by Lemma 2.2, we have

\[
x_1(n) \geq x_1(\eta_1) - \sum_{n=0}^{\omega-1} |x_1(n + 1) - x_1(n)| \geq M_5 - (2\tilde{a}_2)\omega := M_6.
\]

From (2.5), we have

\[
\omega \tilde{a}_2 \leq \sum_{n=0}^{\omega-1} [c_2(n) + c_3(n) \exp\{x_3(\zeta_3)\}],
\]

i.e.,

\[
\tilde{c}_3 \exp\{x_3(\eta_3)\} \geq \tilde{a}_2 - \tilde{c}_2.
\]

Then the assumption (a) implies that

\[
x_3(\eta_3) \geq \ln \left( \frac{\tilde{a}_2 - \tilde{c}_2}{\tilde{c}_3} \right) := M_7.
\]
Then by Lemma 2.2, we have
\[ x_3(n) \geq x_3(\eta_3) - \sum_{n=0}^{\omega-1} |x_3(n+1) - x_3(n)| \geq M_7 - (2\bar{a}_3)\omega := M_8. \tag{2.19} \]

Eqs. (2.13) and (2.18) imply that
\[ \max_{n \in I_2} |x_1(n)| \leq \max \{|M_1|, |M_6|\} := M_1. \]

Eqs. (2.15) and (2.17) imply that
\[ \max_{n \in I_2} |x_2(n)| \leq \max \{|M_2|, |M_4|\} := M_2. \]

Eqs. (2.16) and (2.19) imply
\[ \max_{n \in I_2} |x_3(n)| \leq \max \{|M_3|, |M_8|\} := M_3. \]

Clearly, \( M_i, M_j (i = 1, 2, 3, 4, 5, 6, 7, 8 \) and \( j = 1, 2, 3 \) are independent of \( \lambda \). Under assumptions (a), (b) and (c), we see that the system of algebraic equations
\[
\begin{align*}
\bar{a}_1 - \bar{b}_1u_1 - \frac{\bar{c}_1u_2}{u_1 + 1} & = 0, \\
- \bar{a}_2 + \frac{\bar{c}_2u_1}{u_1 + 1} - \frac{\bar{c}_3u_3}{u_2 + 1} & = 0, \\
- \bar{a}_3 + \frac{\bar{c}_4u_2}{u_2 + 1} & = 0,
\end{align*}
\]
has two solutions \((u_1^*, u_2^*, u_3^*) \in \mathbb{R}_+^3\) with
\[
\begin{align*}
u_1^* & = \lambda_i \neq 0, \\
u_2^* & = \frac{\bar{a}_3}{\bar{c}_4 - \bar{a}_3} > 0, \\
u_3^* & = \frac{u_2^* + 1}{\bar{c}_3} \left( -\bar{a}_2 + \frac{\bar{c}_2u_1^*}{u_1^* + 1} \right) \neq 0,
\end{align*}
\]
where \( \lambda_i = \frac{1}{2\bar{b}_i} (\bar{a}_1 - \bar{b}_1 + (-1)^i \sqrt{\delta}) \), \( \delta = (\bar{b}_1 + \bar{a}_1)^2 - 4\bar{b}_1\bar{c}_1u_2^2 \) for \( i = 1, 2 \). Denote \( M = M_1 + M_2 + M_3 + M_4 \), where \( M_4 \) is taken sufficiently large such that \( |\ln\{u_1^*\}| + |\ln\{u_2^*\}| + |\ln\{u_3^*\}| \leq M_4 \) for \( i = 1 \) or \( i = 2 \). Take
\[
\Omega := \{ x(n) = (x_1(n), x_2(n), x_3(n))^T \in \mathbb{K} : ||x|| < M \}.
\]

It is clear that \( \Omega \) verifies requirement (h1) of Lemma 2.1. When \( x \in \partial\Omega \cap \text{Ker} L = \partial\Omega \cap \mathbb{R}^3 \), \( x \) is a constant with \( ||x|| = M \). Then, we have
\[
\text{deg}\{JQx, \Omega \cap \text{Ker} L, 0\} = \text{deg}\{JQx, \Omega \cap \mathbb{R}^3, 0\}
\]
Furthermore, direct calculation leads to
\[
\text{deg}\{JQx, \Omega \cap \text{Ker} L, 0\} = \text{deg}\{JQx, \Omega \cap \mathbb{R}^3, 0\}
\]
where $J$ be the identity mapping, since $\text{Im} P = \text{Ker} L$. From (2.20), it is easy to see that $\text{deg}[JQ Nx, \Omega \cap \text{Ker} L, 0] \neq 0$ for sufficiently large or small $b_1$ since $|x_1| + |x_2| + |x_3| \leq M$. By now we have proved that $\Omega$ verifies all the requirements of Lemma 2.1. Hence (2.2) has at least one $\omega$-periodic solution $\bar{x}(n) = (\bar{x}_1(n), \bar{x}_2(n), \bar{x}_3(n))^T$ in $\Omega$. Set $\bar{y}_i(n) = \exp(\bar{x}(n))$, then $\bar{y}(n) = (\bar{y}_1(n), \bar{y}_2(n), \bar{y}_3(n))^T$ is a positive $\omega$-periodic solution of (1.4) with strictly positive components. The boundedness of the components of $\bar{x}(n)$ implies that the existence of positive constants $\alpha$, $\beta$ defined as in Theorem 2.1 such that $\alpha \leq \bar{y}(n) \leq \beta$. The proof is complete. \hfill $\Box$

3. Conclusion

In this paper, we established some sufficient conditions for existence of positive periodic solutions of the non-autonomous discrete chain food-model and the lower and the upper bounds of the periodic solution are also investigated. The periodicity of the solution is the periodicity of the death rates, birth rates and the predation rates. In Theorem 2.1, we have seen that if the average of the death rates for predator is greater than the average of the predation rate of the predator, the average of the birth rate of the prey is greater than the multiplication of averages of the predation rate of the predator and the death rate of the top predator, and the average of the predation rate of the top predator is grater the average of its death rate, then there exists a positive periodic solution. This is biologically true. First, in this case there is no extinction since the source of the food of the predator exists and the birth rate is greater than the average of the death of the top predator and the average of the predation rate. Second, the source of the food of the top-predator also exist, the average of the predation rate for the top-predator is greater than the death rate of the predator and this leads to the conservation of the number of the prey, predator and the top-predator.

Also, one can see that in the system (1.4), if some or all terms in the predation functions involve discrete time delay (finite or infinite) or deviating arguments, Theorem 2.1 remains valid, i.e., time delay or deviating argument is harmless to the existence of periodic solutions, and if the functions of Holling type II replaced by Holling type III one can also use the above technique to prove the existence of positive periodic solutions.

For the biological significance it is important to study the global attractiveness of the periodic solution of (1.4) and this will be of our interest in future work.

References