Learning Control Algorithms for Tracking “Slowly” Varying Trajectories

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Abstract—To date, most of the available results in learning control have been utilized in applications where a robot is required to execute the same motion over and over again, with a certain periodicity. This is due to the requirement that all learning algorithms assume that a desired output is given a priori over the time duration \( t \in [0, T] \). For applications where the desired outputs are assumed to change “slowly,” we present a D-type, PD-type, and PID-type learning algorithms. At each iteration we assume that the system outputs and desired trajectories are contaminated with measurement noise, the system state contains disturbances, and errors are present during reinitialization. These algorithms are shown to be robust and convergent under certain conditions. In theory, the uniform convergence of learning algorithms is achieved as the number of iterations tends to infinity. However, in practice we desire to stop the process after a minimum number of iterations such that the trajectory errors are less than a desired tolerance bound. We present a methodology which is devoted to alleviate the difficulty of determining a priori the controller parameters such that the speed of convergence is improved. In particular, for systems with the property that the product matrix of the input and output coupling matrices, \( CB \), is not full rank. Numerical examples are given to illustrate the results.

I. INTRODUCTION

The idea of using practice to improve performance is one of the essential characteristics of human beings. For example, athletes exercise repeatedly to achieve a desired form of motion. In the same way, it should be possible to improve the performance of a system on a desired trajectory by learning through practice. During the past few years, interest in learning control has increased, as evidenced by the large number of publications appearing from 1984 to 1995. The IEEE Conference on Decision and Control, the American Control Conference, and the IEEE International Conference on Robotics and Automation included special sessions on learning control. The majority of the publications related to learning control is motivated from the results given by Arimoto [1].

To date, most of the papers available have been utilized in applications where the robot is required to execute the same motion over and over again, with a certain periodicity. This is because all learning algorithms assume that a desired output is given a priori (fixed) over the time duration or period \( t \in [0, T] \). Since learning control algorithms are iterative schemes, the robustness of such algorithms is critical in the presence of disturbances, measurement noise and perturbed errors of initialization. Recently there has been a number of efforts toward the robustness of learning algorithms. Arimoto [2] dealt with time-invariant mechanical systems and demonstrated robustness to initial state error and differentiable state disturbances, with the initial trajectory in a small neighborhood. Bondi et al. [3] proved the uniform boundedness of trajectories throughout the repetition of training in a local sense under the assumption that the given initial trajectory lies close to the desired one, again for time-invariant mechanical systems. Heinzinger et al. [4] has studied the robustness problem, using D-type algorithms, for a class of nonlinear system considered in [5], where the convergence of the algorithm was investigated. Arimoto [6]–[8] has proved robustness of P-type learning controls based on the passivity analysis of robot dynamics. Hac [9] examined the properties of the P-type and D-type learning algorithms for linear time-invariant systems in the presence of measurement noise. Saab [10] recently proved the convergence and global robustness for a class of nonlinear time varying systems for the P-type learning control, and examined the implication of his results to time varying linear systems. Also, in [11] and [12], Saab proposed a discrete-time learning control algorithm which was applied for a class of discrete-time systems. It was shown [11] that all the state errors will converge uniformly to zero if and only if the product of the output/input coupling matrices is full column rank. The same condition was shown to be sufficient for global boundedness of all trajectories. Note that none of the papers has included desired trajectory fluctuations.

The main goal of this paper is to study learning algorithms in the case where we allow the reference trajectory, in which the system is required to track, not to be fixed, but instead to vary “slowly” or to fluctuate at each iteration. The trajectories can be given a priori or they can be measured in real time. In this paper, we present learning algorithms applied to a class of nonlinear systems to track slowly varying trajectories. The boundedness of all trajectories is addressed, where we allow the desired trajectories to be contaminated with measurement noise in addition to the system state disturbances, measurement noise at the system output, and to the reinitialization errors at each iteration. Thus, specifically this allows us to apply learning algorithms to a different class of applications, for example, car retarders in classification yards. An estimated 300 million freight cars are classified (i.e., sorted into different
tracks) in railroad classification yards each year. In hump yards freight car classification is performed by pushing a large group of cars up a slight hill, uncoupling the cars at the crest of the hump, and then switching the cars into the appropriate classification tracks as they roll freely down the other side of the hump. The free-rolling cars are controlled in one to four short track sections, where mechanical retarders can slow down the cars within a specified speed limit at the coupling between cars to allow the switches to be thrown safely. So far no solution is obtained, using the same mechanical systems, to control the smaller error margin in the coupling speed restricted by General Motors. One solution is proposed to use learning algorithms to control the retarder in order to achieve the desired coupling speed. The slow variation of trajectories, discussed in the latter, come about the slow variation of the classification track. The track space between the last retarder and the last coupled car (measured by car space equipment) decreases as the car accumulation increases in the same track. In this case, the reference trajectories are measured and not specified a priori. If the speed error is required to ensure boundedness of trajectories and convergence, then a Doppler radar can be employed to measure the speed. Thus, if certain system assumptions are met, then the learning algorithms proposed in this paper can be applied. Another example, is controlling a camera driven by a motor to track targets such as airplanes landing or taking off from the same runway.

We consider the following time-varying, nonlinear systems with linear input action

\[ \dot{x}(t) = f(x(t), t) + B(x(t), t)u(t), \]
\[ y(t) = g(x(t), t). \]

(1)

For this, we consider the learning operator of the following form

\[ u_{k+1}(t) = (1 - \gamma)u_k(t) + \gamma u_0(t) + L(y_k(t), t) \]
\[ + \frac{1}{\gamma} \left[ \int_0^t (y_{d,k+1}(\tau) - y_k(\tau))d\tau \right] \quad 0 \leq \gamma < 1 \]

(2)

where the subscript \( k \) denotes the iteration number of operation, for example, \( y_k(t) \) is the value of the system output at time \( t \), \( 0 \leq t \leq T \), at the \( k \)th operation, etc. Similarly, subscript \((d,k)\) denotes the iteration number of operation for the desired variable. Thus, the error is the difference between the present referenced trajectory \( y_{d,k+1} \), which may be specified a priori or measured in real time, and the output \( y_k \) which corresponds to \( u_k \). \( L(\cdot, \cdot), P(\cdot, \cdot), \) and \( Q(\cdot, \cdot) \) are memoryless linear maps which operate on the derivative of the output error, the output error, and its integral respectively at each iteration. It is called the PID-type learning algorithm. \( \gamma \) is a forgetting factor.

Since the desired trajectory is not fixed anymore, then a set of additional assumptions are considered:

\[ \text{(B1)} \quad \text{Each trajectory is to be considered for a fixed finite time } T > 0. \]

\[ \text{(B2)} \quad \text{The desired trajectory } y_{d,k} \text{ is differentiable } \forall k \text{ and for } [0, T]. \]

\[ \text{(B3)} \quad \text{In each iteration the desired trajectory differs from the previous one by a small deviation level, i.e.,} \]
\[ y_{d,k}(t) = y_{d,k-1}(t) + \eta_k(t) \quad \forall k > 2 \]
\[ \text{where } \eta_k(t) \text{ must satisfy} \]
\[ ||\eta_k||_\infty \leq \varepsilon_1 \quad t \in [0, T] \]

(4)

for small \( \varepsilon_1 > 0 \). Note that \( y_{d,k}(t) \) and \( y_{d,k+n}(t) \) may be totally different for large \( n \).

\[ \text{(B4)} \quad \text{In each iteration the desired trajectory } y_{d,k}(t), \text{ if not specified, is allowed to be measured within a small specified noise level, i.e.,} \]
\[ y_{d,k}(t) = g(x_{d,k}(t)) + \xi_k(t) \]
\[ \text{where } \xi_k(t) \text{ must satisfy} \]
\[ ||\xi_k||_\infty \leq \varepsilon_2 \quad t \in [0, T] \]

(6)

for small \( \varepsilon_2 > 0 \). \( \xi_k \) is assumed to be differentiable and

\[ \left| \frac{d\xi_k}{dt} \right|_\infty \leq \varepsilon_3 \quad t \in [0, T]. \]

(7)

Assumption (B1) is valid for most tracking problems using learning algorithms. (B2) is reasonable. (B3) is restrictive in general, but for “learning” systems, it is not unreasonable because one does not expect a system to learn random trajectories. (B4) requires the measurement noise to be differentiable. Theoretically, this is a fair assumption in time domain.

The main target is to find learning algorithms in which each one guarantees that the motion trajectories converge to a neighborhood of the desired one and remain in it in the presence of all disturbances. Moreover, the motion trajectory converges uniformly to the desired one whenever the state disturbances, fluctuation of the desired trajectories, measurement noise at the output, measurement noise on the desired trajectories, and reinitialization errors tend to zero.

In Section II, we state the problem formally and present sufficient conditions for robustness and convergence of the proposed algorithms. In Section III, we give a numerical example to illustrate the results. Section IV presents a methodology to accelerate the speed of convergence of the proposed algorithms. Moreover, it examines the convergence and boundedness of trajectories in the case where the product of the system output/input coupling matrices is not full column rank. Another example is added to show “optimality” of the proposed method. Finally, in Section V we give a conclusion.

II. ROBUST LEARNING ALGORITHMS

In this paper we present the problem and proposed algorithms. In addition, sufficient conditions for tracking convergence and global robustness of D-type, PD-type, and PID-type learning algorithms for a class of nonlinear systems are given. In our problem we include state disturbances, fluctuation of the desired trajectories, measurement noise at the output, measurement noise on the desired trajectories, and reinitialization
errors at each iteration. Consider the time varying, nonlinear systems with linear input action described by the state space equations

\[ \dot{x}_k(t) = f(x_k(t), t) + B(x_k(t), t)u_k(t) + w_k(t) \]
\[ y_k(t) = g(x_k(t), t) + \theta_k(t) \]  

where \( k \) is the iterative index. For all \( t \in [0, T] \) and for all \( k, x_k(t) \in \mathbb{R}^n, u_k(t) \in \mathbb{R}^m, y_k(t) \in \mathbb{R}^m, \) and \( w_k(t) \in \mathbb{R}^m \). The functions \( f : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n, B : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times r} \) are piecewise continuous in \( t \), and \( g : \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}^m \) is absolutely continuous in \( x \) (which implies differentiability in \( x \) almost everywhere), and differentiable in \( t \) with partial derivatives \( \theta_k(t) \) and \( \theta_k(t) \). Moreover, we let the inputs\( u_k : [0, T] \rightarrow \mathbb{R}^r \) to be piecewise continuous in \( t \). Let \( R \) denote the state mapping from \( C([0, T], \mathbb{R}^n) \times \mathbb{R}^m \rightarrow C([0, T], \mathbb{R}^m) \), and \( S \) denote the output mapping from \( C([0, T], \mathbb{R}^m) \times \mathbb{R}^m \rightarrow C([0, T], \mathbb{R}^m) \) for all \( t \in [0, T] \), for each fixed \( x_k(0) \) with \( u_k(t) = 0 \) and \( \theta_k(t) = 0 \), i.e., \( x_k(t) = R(u_k(\cdot), x_k(0)) \) and \( y_k(\cdot) = S(u_k(\cdot), x_k(0)) \). We denote \( \| \cdot \| \) to be the Euclidean norm.

**Assumptions:**

(A1) All functions are measurable and integrable.

(A2) All mapping \( R \) and \( S \) are one-to-one.

(A3) The disturbance \( w_k \) is such that \( \|w_k(t)\| \leq b_w \), and the measurement noise of the output error \( \eta_k(t) = \theta_k(t) - u_k(t) \), where \( u_k(t) \) is the measurement noise on the desired trajectory, and \( \theta_k(t) \) is the measurement noise at the output at each \( k \) is differentiable and such that \( \|\eta_k(t)\| \leq b_\eta \) and \( \|\theta_k(t)\| \leq b_\theta \) on \([0, T]\) and for all \( k \).

(A4) The functions \( f(\cdot), B(\cdot), g(\cdot, \cdot) \) are uniformly globally Lipschitz in \( x \) on \([0, T]\).

(A5) The operators \( g_x(\cdot, \cdot) \) and \( B(\cdot) \) are bounded on \( \mathbb{R}^n \times [0, T] \). Moreover, \( \exists \epsilon_{g_x,B} > 0 \) such that \( \min_{1 \leq i \leq n} (|B_j((x(t), t)B(x(t), t))|g_x(x, t)B(x, t)) \geq \epsilon_{g_x,B} \).

(A6) Repeatability of the initial setting is satisfied within an admissible deviation level, i.e., \( \|x_d, x_k(0) - x_k(0)\| \leq b_d \).

(A7) The desired trajectories \( y_{d,k}(t) \) are differentiable and are in the same neighborhood on \([0, T]\) and for all \( k \).

Assumption (A1) is not unreasonable for piecewise continuous functions on a compact interval. (A2) and (A5) require that both \( B(x(t), t), g_x(x(t), t) \cdot B(x(t), t) \) to have full column rank for any \( x(t), t \in \mathbb{R}^n \times [0, T] \) that can be achieved by the system [8], it guarantees that given a realizable trajectory and an appropriate initial condition, there is a unique control that will generate the trajectory. This restriction on the system is equivalent to the sufficient condition given in Theorems 1–3. (A3) restricts the disturbances which present random and deterministic disturbances of the system to be bounded, but they may be discontinuous such as nonreproducible friction and other physical phenomena. (A5) is not unreasonable for physical systems. (A5) forces the learning gain matrix to remain bounded. Having \( g(\cdot, \cdot) \) absolutely continuous and from (A5), \( g_x(\cdot, \cdot) \) is bounded, implies that \( g_x(\cdot) \) is uniformly globally Lipschitz in \( x \) on \([0, T]\). Note that (A3) and (A7) imply (B1)–(B4).

### A. D-Type Algorithm

Consider the update law given by

\[ u_{k+1}(t) = (1 - \gamma)u_k(t) + \gamma w_k(t) + L(y_k(t), t) \times [y_{d,k+1}(t) - y_k(t)] 0 \leq \gamma < 1. \]  

\( \gamma \) is a forgetting factor which was first introduced by Heinzinger et al. [4] \( L : \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}^{m \times m} \) is piecewise continuous and bounded. This learning operator is similar to most of the papers in learning control which updates the system in a linear input action and the control in the pointwise fashion. Note that if the initial input \( u_0(t) \) is piecewise continuous, then our control is piecewise continuous as well, for all \( k \). In the following, we present the major results. The proof of Theorems 1–3 are given in the Appendix.

**Theorem 1:** Suppose the system of (8) satisfies assumptions (A1)–(A7) and the trajectories \( y_{d,k}(t) \) are realizable. Then \( \exists L_k \) such that the learning operator given by (9) will generate a sequence of inputs \( u_k(t), t \in [0, T] \), such that the error between \( y_{d,k}(t) \) and \( u_k(t), x_{d,k}(t) \) and \( x_k(t) \), and the error between \( y_{d,k} \) and \( y_k \) are all bounded \( \forall k \) and \( \forall t \in [0, T] \). Moreover, \( \|y_{d,k} - y_k\| \rightarrow 0; \|x_{d,k} - x_k\| \rightarrow 0; \) and \( \|y_{d,k} - y_k\| \rightarrow \infty \) whenever \( \gamma \rightarrow 0, \beta \rightarrow 0, \beta \rightarrow 0, \) and \( \beta \rightarrow 0 \) tend to zero.

### B. PD-Type Algorithm

**Theorem 2:** If the update law is given by

\[ u_{k+1}(t) = (1 - \gamma)u_k(t) + \gamma w_k(t) + L(y_k(t), t) \times [y_{d,k+1}(t) - y_k(t)] + P(y_k(t), t) \times [y_{d,k+1}(t) - y_k(t)] 0 \leq \gamma < 1 \]  

with \( P(\cdot) \) being bounded, then we have the same results as in Theorem 1 using its hypothesis.

### C. PID-Type Algorithm

**Theorem 3:** If the update law is given by

\[ u_{k+1}(t) = (1 - \gamma)u_k(t) + \gamma w_k(t) + L(y_k(t), t) \times [y_{d,k+1}(t) - y_k(t)] + P(y_k(t), t) \times [y_{d,k+1}(t) - y_k(t)] + Q(y_k(t), t) \times \int_0^t (y_{d,k+1}(t) - y_k(t)) dt 0 \leq \gamma < 1 \]  

with \( Q(\cdot) \) being bounded, then we have the same results as in Theorem 2 using its hypothesis.

Note that whenever we modify the D-type algorithm by employing a proportional gain of the output error (PD-type) and the integral error (PID-type), one should expect a slower convergence. This can be observed whenever we compare the contraction in the input error sequence in the proof of Theorems 1–3.
III. EXAMPLE

In this section we apply the proposed D-type algorithm to a nonlinear system to illustrate the tracking performance, in particular, robustness and uniform convergence. Consider the following system:

\[
\dot{x}_1(t) = x_2(t) + 0.01 \text{ rand}
\]

\[
\dot{x}_2(t) = 0.3x_2(t) + \frac{1}{|x_2(t)| + 1} + \sin(x_2(t))u(t) + 0.01 \text{ randn}
\]

\[y(t) = x_2(t) + 0.01 \text{ randn}\]

where \text{randn} is a scalar random generator, supported by MATLAB, with normal distribution, mean = 0 and variance = 1 (white Gaussian noise). Note that \text{randn} generates a random scalar every time it is called which depends on discrete-time sample, iteration and variable. Euler integration is used with integration frequency = 100 Hz. In order to approximate the derivative of the output error, we use the difference between two consecutive discrete-time samples and divide by the integration period. Before we apply the learning algorithm, one should check if all the assumptions are met. For (A1)–(A3) and (A5), these conditions can be readily verified. Except for the lower bound of \sin(x) which is not met for singularities in the sine function. An addition of a scalar (e.g., 2) overcomes this exception and actually increases the speed of convergence. But for the Lipschitz condition (A4), we use three of the Lipschitz properties: (P1)—The addition of Lipschitz functions (Lf) is an Lf. (P2)—Scalar multiplication with an Lf is an Lf (multiplication of Lfs is generally not an Lf). (P3)—If a function has continuous and bounded partial derivatives (in x), then it is Lipschitz. Therefore, from (P3) we may conclude that \dot{x}_2(t) \text{ is Lipschitz}. It remains to show that \dot{g}(\cdot) is Lipschitz. From (P1) and (P2), it will be sufficient to show that \dot{g}(x) = \frac{1}{|x|+1} is an Lf. Note that

\[|\dot{g}(x) - \dot{g}(y)| = \frac{|x| - |y|}{(|x|+1)(|y|+1)} \leq |g| - |y| \leq |x - y|
\]

so \dot{g} is an LF (more precisely, it is a contraction function). For the sufficient condition given in Theorems 1–3, this can also be met by fixing the controller gain \rho < 1, and the sign of \dot{L} would be the same as the sign of \sin(x_2(t)). An “optimum” choice of \rho (\rho is used in the proofs of Theorems 1–3) is zero. On the other hand, this choice of \rho can increase the fluctuation due to random disturbances. More analysis in choosing the control gain matrix are given in the next section of this paper. For now, we set \dot{L}(t) = \frac{\alpha}{\sin(x_2(t))}, 0 < \alpha \leq 1 where \alpha is chosen to be close to zero if the size due to random disturbances is large, and vise versa. In this example \alpha = 0.5 in presence of disturbances, and \alpha = 1 in absence of all disturbances. The desired trajectories for \dot{t} \in [0, 5] (s) are given to be

\[x_{\dot{t},k}(t) = \frac{a_k}{100} t^{\rho_k} + 0.1a_k \]

\[\dot{x}_{\dot{t},k}(t) = x_{\dot{t},k}(t)\]

where \text{ where } a_k = 2 + 0.01k (k is the iterative index), \text{ x}_{\dot{t},k}(0) = x_{\dot{t},k}(0) + 0.01 \text{ randn. } j = 1, 2, \text{ and the initial input, for } t \in [0, 5], \text{ is set to zero. Note that } \text{ x}_{\dot{t},k}(t) \text{ given in (B3) of Section 1, is the difference between the last two consecutive trajectories. Obviously, the restrictions on the desired trajectories for all } k \text{ are satisfied.}

Figs. 1 and 2 show the change in the desired trajectories. Note that for 4 \leq t \leq 5, the desired trajectory varies significantly with iterations. Next, in Fig. 2, we show the robustness of the trajectories in presence of all disturbances, in particular, \text{ max } x_{\dot{t},k}(t) - x_{\dot{t},k}(t) \text{ in } t \in [0, 5], \text{ for } \alpha = 1, 2. \text{ The maximum absolute errors } \dot{t} \text{ for } k = 2, 5 \text{ are 1.0 and 0.57 for } x_1 \text{ and } x_2 \text{ respectively. It is not expected from the system to learn totally unpredicted disturbances, such as white noise. Fig. 3 shows trajectory tracking, for the last iterate, in presence of disturbances. In Figs. 4 and 5, we show the uniform convergence of the algorithm in absence of state disturbance, reinitialization errors and measurement noise, where the variation of trajectories with iterations re-}
mains unchanged, i.e., trajectories are changing with iterations. Maximum absolute errors \( \forall t \) and for \( k = 2; 75 \) are 0.08 and 0.05 for \( x_1 \) and \( x_2 \), respectively.

### IV. IMPROVING IN THE CONTRACTION OF THE INPUT SEQUENCE

In practice we desire to stop the process in a finite number of iterations where the error is smaller than some given tolerance \( \varepsilon \). The proofs of Theorems 1–3 are based on contracting the input sequence implying the convergence results, i.e., if all disturbances are zero, then \( \|u_d - u_{k+1}\| \leq \overline{p}\|u_d - u_k\| \ \forall k \), where \( 0 \leq \overline{p} < 1 \). To improve the speed of convergence, it is sufficient to minimize the size of \( \overline{p} \). A feasible parameter which decreases the size of \( \overline{p} \) is the controller gain \( L_k \). There are another two common parameters that can be chosen appropriately to help in improving the speed of convergence are the initial input \( u_0(t) \) and the forgetting factor \( \gamma \). The initial input may be obtained by applying the inverse of the system model to the desired trajectories. Varying the forgetting factor \( \gamma \) as the iterations progress may improve the performance of the algorithm. The bias term \( \gamma u_0(t) \) is used to keep the input from wandering initially, and with time we want to decrease its influence by decreasing the size of \( \gamma \). The last two alternatives are more cosmetic than basic.

All the results, discussed in this paper so far, have the following common condition:

\[
\|(1 - \gamma)I - L_k(g_k(x, t)B(x, t))\| \\
\leq \rho < 1 \ \forall (x, t) \in \mathbb{R}^n \times [0, T].
\]

In the proof of Theorems 1–3, it is shown that

\[
\|u_d - u_{k+1}\| \leq \overline{p}\|u_d - u_k\| + \varepsilon
\]

where

\[
\overline{p} \equiv \rho + \frac{M}{\lambda - M} (1 - e^{(M-M)\gamma T}),
\]

\( M \) and \( \varepsilon \) combine Lipschitz constants and norm bounds of the disturbances. If all disturbances and \( \gamma = 0 \), then \( \varepsilon = 0 \). These terms are included to show that when minimizing the value
of \(\rho\), the speed of convergence can be improved in presence of all disturbances. Since \(\rho < 1\), we can find \(\lambda\) large enough such that \(\overline{\lambda} < 1\).

Note that if \(\rho\) gets smaller, \(\overline{\lambda}\) decreases and (12) implies that the speed of convergence increases to a certain extent. Thus, to speed up the convergence, one way will be to minimize \(\rho\), i.e.,

\[
\min_{L_k} \| (1 - \gamma)I - L_k (g_k(x,t)B(x,t)) \|.
\]

Without loss of generality we assume that \(g_k(x,t)B(x,t)\) is given, if not, either use the model values of \(g_k(x,t)B(x,t)\) or one may estimate \(g_k(x,t)B(x,t)\), e.g., employing a recursive least-squares method in which the recursion formula works in the domain of iteration sequence of operations with time frozen (on line estimation).

Recall that in our assumptions, it was implicitly assumed that \(g_k(x,t)B(x,t) \neq 0\). Set \(M \equiv g_k(x,t)B(x,t)\) and \(N \equiv L_k\). The problem becomes a minimization problem, i.e.,

\[
\min_N \| (1 - \gamma)I - NM \| \tag{13}
\]

where \(M \in \mathbb{R}^{m \times r}, N \in \mathbb{R}^{r \times m}\), and \(\| \cdot \|\) is the Euclidean norm.

If \(M\) is a nonsingular square matrix, the \(N = (1 - \gamma)M^{-1}\).

If \(M(m > r)\) is full column rank matrix, then \(N\) will be right inverse of \(M\), i.e., \(N = (1 - \gamma)(MTM)^{-1}M^T\).

Else, rank \(M = p < \min(m,r)\). For this case the output mapping

\[
C([0,T], \mathbb{R}^r) \times \mathbb{R}^m \to C([0,T], \mathbb{R}^r) \forall t \in [0,T]
\]

for each fixed \(x_k(t)\) is not one-to-one, i.e., there exists a realizable trajectory with some initial condition where the control that will generate this particular trajectory is not unique, or may not be reached. In order to ensure that the learning gain matrix \(N\) or \(L_k\) will not diverge to overcome the singularity of \(MTM\), one should fix \(N\) in the “iterative” space. In the latter, for the case where rank \(M = p < \min(m,r)\), we propose a methodology which is based on intuitive analysis motivated from the contraction input sequence presented in the proves of Theorems 1–3. We remind the reader that it is assumed that the desired or prespecified trajectories are assumed to be realizable, i.e., there exist an input vector which drives the system to track the desired trajectories. This methodology will force the “controllable” modes, which are directly related to the \(p\)-inputs, to converge in an “optimal” speed. The remaining modes will converge in a bounded neighborhood.

Note that (13) is a convex function which is generally not differentiable. Therefore, the minimum exists but unfortunately, in the case where rank \(M < \min(m,r)\), the cost will be very high to reach the minimum numerically. One approach is to minimize the square of the Frobenius norm, i.e.,

\[
\min_N \| (1 - \gamma)I - NM \|^2 \tag{14}
\]

where \(\| M \|^2 = \text{trace}(MTM)\). Note the fact that

\[
\| M \|^2 = \int_{i=1}^{m} \| m_i \|^2
\]

where \(m_i\) are the columns of \(M\).

**Proposition 1:** The solution of (14) is given by

\[
N = (1 - \gamma)V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T
\]

where the unitary matrices \(U\) and \(V\) are such that

\[
U^TMV \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}
\]

and \(\Sigma = \text{diag} (\sigma_1, \ldots, \sigma_p)\) and \(\sigma_1 > \sigma_2 > \cdots > \sigma_p > 0\).

The numbers \(\sigma_1, \ldots, \sigma_p\) are called singular values of the matrix \(M\).

**Proof:**

\[
\| (1 - \gamma)I - NM \|^2 = \| (1 - \gamma)I - M^TN^T \|^2
\]

\[
= \| (1 - \gamma)I - M^T(c_1, \ldots, c_m) \|^2
\]

\[
= \sum_{i=1}^{m} \| (1 - \gamma)c_i - M^Tc_i \|^2
\]

where \(c_i\) and \(c_i\) are the columns of \(I\) and \(N^T\) respectively.

Employing the Singular Value Decomposition Theorem, \(\exists\) unitary matrices \(U\) and \(V\) such that

\[
V^TM^TU = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}
\]

where \(\Sigma\) is defined as in the proposition above and

\[
c_i = (1 - \gamma)U \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^Tc_i
\]

are vectors which minimize \(\| c_i - M^Tc_i \|^2 \forall i\). Moreover, if \(\| c_i - M^Tc_i \|^2\) is minimal and \(c_i \neq c_i^\prime\), then \(\| c_i \|^2 < \| c_i^\prime \|^2\). The above implies that

\[
N^T = (1 - \gamma)U \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^T
\]

which implies that

\[
N = (1 - \gamma)V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T
\]

Using the results of Proposition 1 and the singular presentation of \(M\), the product \(NM\) will be

\[
NM = V \begin{bmatrix} (1 - \gamma)I_p & 0 \\ 0 & 0 \end{bmatrix} V^T
\]

Since \(p < \min(m,r)\), then \(\exists\) some entries of the matrix \(N, Ni_i = 0\), for \(i \in [p+1, \min(m,r) - p]\) which can be replaced by some constants without changing the product \(NM\), i.e., the norm in (13) and (14) remains the same. Note that if these entries \(Ni_i \neq 0\), then the update law will be using the extra controllers which may be helpful and without any degradation of the tracking error as long as the size of \(Ni_i\)’s are chosen properly.

Given a matrix \(M\), recall that \(\| M \|_2 = \| VMV^T \|_2 = \rho(M)\) (special radius) since \(V\) is a unitary matrix and the 2-norm is an operator norm. Applying Position 1, we find that

\[
\| (1 - \gamma)I - NM \|_2 = 1 - \gamma
\]

From the equality above, it will be reasonable to modify \(N\) by choosing \(Ni_i = 1 - \gamma\). It remains to choose an appropriate
value for the forgetting factor $\gamma$. In absence of disturbances, for the first $p$-rows of $NM$, one should set $\gamma = 0$ to guarantee “ideal” convergence. For the remaining rows, one should have, $0 < \gamma < 1$. The theory behind this, is to ensure that $\rho < 1$, hence

$$\lim_{k \to \infty} \|u_{d,k} - u_k\|_\lambda \leq \frac{\varepsilon}{1 - \rho}$$

where for this case, $\varepsilon = \gamma\|u_{d,k} - u_k\|_\lambda$. Recall that the update law has the form

$$u_{k+1}(t) = (1 - \gamma)u_k(t) + \gamma u_{d}(T) + L_k e_k(t) \quad 0 < \gamma < 1$$

Note that if $\gamma$ is increased than the controller will have as a dominant term the initial input $u_{d}(t)$. Therefore, the choice of $\gamma$ depends on the quality of the model.

### A. Example

The following numerical example shows the effectiveness of the above results. This example is an application to linear time invariant system where the product of the output coupling matrix and the input coupling matrix is not full column rank. Note that readily, one can check that the assumptions considered for the class of systems considered in this paper are satisfied for linear time invariant systems. For this particular example, we show that the results given in the previous section, are optimum in some sense.

Consider the following linear time invariant system which is represented by

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t)$$

$$y_k(t) = Cx_k(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1.2 & -1.6 & -3.6 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$  

In order to focus closer on the performance of the convergence speed of the proposed algorithm, we fix the desired trajectories. For $t \in [0, 1]$, these trajectories are given by

$$y_{d1}(t) = \sin(t)$$

$$y_{d2}(t) = \cos(t).$$

The update law is the D-type which is presented by

$$u_{k+1}(t) = (1 - \gamma)u_k(t) + \gamma y_{d}(T) + L_k e_k(t) \quad 0 < \gamma < 1$$

where the error $e_k(t) = y_k(t) - y_{d}(t)$ and the initial input $u_{d}(t)$ is obtained by applying the inverse of the model on the desired trajectory, i.e.,

$$u_{d}(t) = (B^T B)^{-1}B^T (\dot{x}_d(t) - Am x_{d}(t))$$

where

$$Am = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}.$$  

Consider the product $CB$

$$CB = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$  

Thus rank$(CB) = 1$. For this we apply the proposition above with the modification, we get

$$L = \begin{bmatrix} 0.2(1 - \gamma) & 0 \\ 0 & 1 - \gamma \end{bmatrix}.$$  

In order to show the performance of this methodology, we set $\gamma = 0.5$ in the learning algorithm. But ideally, since no disturbances are included in this example, one should then set all $\gamma$ which appear in the first row of the learning algorithm to zero.

$$L = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix}.$$  

A step size of 0.01 s is used for (Euler) numerical integration. max$(\|u_{d}(t) - y_{d}(t)\|)$ decreased 81 times (from 0.0081 to 0.0001) and max$(\|y_{d}(t) - y_{d}(t)\|)$ decreased 3.87 times (from 0.0786 to 0.0203) in 11 iterations, for $t \in [0, 1]$. The values for $L$ were altered to many different values, the results were not compatible to our results, i.e., when the error did not diverge, it would decrease much slower than the algorithm which uses the proposed controller gain. For this, $\gamma$ was fixed to the same value $\gamma = 0.5$, and the value of the first diagonal entry of $L$ was altered to different values, so the closer the value was chosen (to 0.1), better results were obtained. To illustrate the last statement, we present the results for different values of $L$ after one iteration and then ten iterations which are presented in Table I. We denote by $L_{11} \equiv L_{11}, L_{22} \equiv L_{22}$, $Error(i) \equiv \text{sup}_{t \in [0, 1]} \|y_{d}(t) - y_{d}(t)\|$ for $i = 1, 2$.

#### Remarks:

1) The initial input is held fixed for all trials for consistency, which results to Error $1 = 0.0081$, and Error $2 = 0.0786$.

<table>
<thead>
<tr>
<th>$\gamma$, $L_{11}$, $L_{22}$</th>
<th>Error1</th>
<th>Error2</th>
<th>Error1</th>
<th>Error2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5,0.1,0.5</td>
<td>0.0064</td>
<td>0.0387</td>
<td>0.0001</td>
<td>0.0203</td>
</tr>
<tr>
<td>0.5,0.09,0.5</td>
<td>0.0061</td>
<td>0.0387</td>
<td>0.0012</td>
<td>0.0203</td>
</tr>
<tr>
<td>0.5,0.11,0.5</td>
<td>0.0067</td>
<td>0.0386</td>
<td>0.0011</td>
<td>0.0204</td>
</tr>
<tr>
<td>0.5,0.1,0.6</td>
<td>0.0085</td>
<td>0.0307</td>
<td>0.0002</td>
<td>0.0232</td>
</tr>
<tr>
<td>0.5,0.1,0.4</td>
<td>0.0047</td>
<td>0.0466</td>
<td>0.0001</td>
<td>0.0204</td>
</tr>
<tr>
<td>0.5,0.1,0.3</td>
<td>0.0037</td>
<td>0.0546</td>
<td>0.0001</td>
<td>0.0337</td>
</tr>
</tbody>
</table>
2) Whenever $L_{11}$ is altered by just $\pm 10\%$, the second and third rows show that the convergence of Error 1 get slower and Error 2 remains almost unchanged.

3) Whenever $L_{22}$ is changed, the fourth, fifth and sixth rows show that the convergence of Error 2 become slower.

4) If the entry of $L_{21}$, $L_{12}$ or $L_{22}$ are set to values different than zero, then the results will not be compatible, i.e., if $L = [0.1, 0.1; 0.1, 0.5]$ and $\gamma = 0.5$, then Error 1 increases to 0.3729 and remains the same $\forall k > 15$.

The above example shows that our results are very close to the “optimum” results, where the cost of choosing a good gain matrix is negligible.

Figs. 6 and 7 compare the error of both outputs with respect to the first eleven iterations whenever $L_{11}$ entry is altered by 10% from the implied result. Figs. 8 and 9 compare the error of both outputs with respect to the first 11 iterations whenever the $L_{22}$ entry is altered by 20% from the implied result. Note in Fig. 7, the dashed line is not visible due to overlap with the solid line.

V. DISCUSSION

In the above work, it was shown that choosing the controller parameters appropriately, the speed of convergence can be improved. That was for learning algorithms which employ the derivative of the error. Unfortunately, this is not the case for the P-type algorithms which are shown to be robust and convergent [10]. Unlike the D-type learning algorithms where the convergence was based on contracting the error at every iteration, the proof of convergence of the P-type learning algorithms is based on the convergence of the error pointwise as the number of iterations tends to infinity and then using the Ascoli–Arzelà’s Theorem to achieve the uniform convergence. Therefore, it can not be concluded that at each iteration we have a contraction. If one changes the controller parameters with every set of iterations in order to extract a convergent subsequence, the original sequence will lose the
track of convergence. Hence, the extracted subsequence will be another convergent sequence in which the rate of convergence is not necessarily faster than the original one. From some simulation results using the P-type algorithm, it is concluded that the larger the controller gain matrix is employed, the faster convergence is obtained. The disadvantage is that the fluctuations of the error are going to be larger which may result in saturation of one or more of the system states. Arimoto [8] proposed a new method called “selective learning” which updates the input $u_0$ in the long-term memory by selecting the best command among the past several trials where it is claimed that this method accelerates the speed of convergence, but there was not any application to show the performance of the proposed method.

VI. CONCLUSION

The main contribution of this paper is that we showed that learning algorithms can be applied to a wider set of problems where we allowed the desired trajectories to change “slowly” with each iteration and to be contaminated with measurement noise, in addition to the system state disturbances, measurement noise at the output, and reinitialization errors at each iteration. It has been shown without any linearization that if certain conditions are met, the motion trajectories converge to a neighborhood of the desired one and eventually remain in it. Furthermore, the motion trajectory converges uniformly to the desired one whenever all disturbances and fluctuation of the desired trajectories tend to zero. This was proven for the D-type, PD-type, and PID-type learning algorithms. A numerical example was given in which the applicability of learning algorithms to “slowly” varying referenced trajectories is demonstrated. A methodology was also presented to alleviate the difficulty of determining a priori the controller parameters in particular the controller gain matrix for systems with the property that the product matrix of the input and output coupling matrices, $CB$, is not full rank such that the speed of convergence is improved. Finally, a numerical example of a linear time-invariant system where the matrix $CB$ is not full rank was given. It was shown that the application of the proposed results are optimum in some sense.

APPENDIX

We denote by the $\lambda$ norm for a function $f : [0,T] \rightarrow \mathbb{R}^n$ to be defined as

$$\|f(t)\|_{\lambda} = \sup_{t \in [0,T]} e^{\lambda t} \|f(t)\|$$

where $\lambda > 0$.

Note that, $\|f(t)\|_{\lambda} \leq \sup_{t \in [0,T]} e^{\lambda t} \|f(t)\| = e^{\lambda T} \|f(t)\|$, Notations to be used for simplification are as follows:

$$f_k \equiv f(x_k(t), t), \quad f_{dk} \equiv f(x_{dk}(t), t)$$

$$B_k \equiv B(x_k(t), t), \quad B_{dk} \equiv B(x_{dk}(t), t)$$

$$g_k \equiv g(x_k(t), t), \quad g_{dk} \equiv g(x_{dk}(t), t)$$

$$g_{xk} \equiv \frac{\partial(g(x, t))}{\partial t} |_{x=x_k(t)} \quad g_{xdk} \equiv \frac{\partial(g(x, t))}{\partial t} |_{x=x_{dk}(t)}$$

$$u_k \equiv u_k(t), \quad L_k \equiv L(y_k(t), t), \quad \nu_k \equiv \nu_k(t) - u_k(t) \quad P_k \equiv P(y_k(t), t), \quad Q_k \equiv Q(y_k(t), t).$$

$M_{px}, M_{px}, M_f, M_B$, and $M_g$ are the Lipschitz constants for $g_k(\cdot), g_k(\cdot), f(\cdot), B(\cdot), g_k(\cdot)$ respectively. $b_L, b_{px}, b_{px}, b_p, b_R$ are the norm bounds for $L(\cdot), g_k(\cdot), B(\cdot), P(\cdot)$ and $Q(\cdot)$ respectively.

$$b_d \equiv \sup_k \|f_{dk} + B_{dk} u_{dk}\|_\infty \quad b_{ud} \equiv \sup_k \|u_{dk}\|_\infty$$

Again, $\|z\|_{\infty} \equiv \sup_{(x,t)} \|z(t)\|$.

Lemma 1: If the system of (8) satisfies assumption (A5), then $\exists$ bounded $L_k$ such that $\|(1-\gamma)I - L_k g_{zk} B_k\| \leq \rho < 1 \forall (x,t) \in \mathbb{R}^n \times [0,T]$.

Proof:

Existence:

Set $M_k \equiv g_{zk} B_k$. From (A5), one may conclude that $M_k^T M_k$ is nonsingular. If the learning gain matrix is given by $L_k \equiv \alpha (M_k^T M_k)^{-1} M_k^T$, where $1-\gamma - \rho \leq \alpha \leq 1-\gamma + \rho$, then the inequality is always satisfied.

Boundedness:

Now $L_k \equiv \alpha (M_k^T M_k)^{-1} M_k^T$, where $1-\gamma - \rho \leq \alpha \leq 1-\gamma + \rho$ ($\alpha > 0$), thus,

$$\|L_k\| = \alpha \|(M_k^T M_k)^{-1} M_k^T\| \leq \alpha \|(M_k^T M_k)^{-1}\| \|M_k^T\|$$

$$= \frac{\alpha \|M_k^T\|}{\lambda_{\text{min}}(M_k^T M_k)}.$$ 

Using (A5) one can conclude boundedness of the learning gain; i.e., $\|L_k\| \leq b_L \forall (x,t) \in \mathbb{R}^n \times [0,T]$. □

Proof of Theorem 1: From (8), the error between the $k$th iterate and the $(k+1)$th iterate of desired state

$$x_{dk+1} - x_k = x_{dk+1}(0) - x_k(0)$$

$$+ \int_0^t \{(f_{dk+1} + B_{dk+1} u_{dk+1}) - (f_k + B_k u_k + u_k)\}dt.$$

(15)

Taking norms, and using their properties

$$\|x_{dk+1} - x_k\| \leq \|x_{dk+1}(0) - x_k(0)\|$$

$$+ \int_0^t \{\|f_{dk+1} - f_k\| + \|B_{dk+1} - B_k\| \|u_{dk+1}\| + \|B_k\| \|u_{dk+1} - u_k\| + \|u_k\|\}dt. \quad (16)$$

Using the Lipschitz condition

$$\|x_{dk+1} - x_k\| \leq \|x_{dk+1}(0) - x_k(0)\|$$

$$+ \int_0^t (M_f + MB \|u_{dk+1} - u_k\| + b_{ud}) dt. \quad (17)$$
Set $M_1 \equiv M_f + M_B b_{ud}$, and using an integral inequality [13]

$$
||x_{d,k+1} - x_k||
\leq ||x_{d,k+1}(0) - x_k(0)||e^{M_1 t}
+ \int_0^t e^{M_1(t-\tau)} (b_B||x_{d,k+1}(\tau) - u_k(\tau)|| + b_w) d\tau.
$$

(18)

From (8) and (9), the error for the $(k+1)^{th}$ iterate

$$
\begin{align*}
\eta_{d,k+1} - u_{k+1} \\
= & (1 - \gamma)(\eta_{d,k} - u_k - \gamma u_0 - L_k(\eta_{d,k+1} - \eta_k)) \\
= & (1 - \gamma)(\eta_{d,k} - u_k) + \gamma(u_{d,k+1} - u_0) \\
- & L_k\{g_{x}(d_{k+1}) + B_{d,k+1}u_{d,k+1} + g_{a}(d_{k+1})\} \\
- & g_{x} (f_k + B_{k} u_{k} + u_{k}) - g_{x} - \hat{v}_k \\
= & \gamma(u_{d,k+1} - u_0) + [(1 - \gamma)I - L_k g_{x} B_k] (u_{d,k+1} - u_k) \\
- & L_k\{g_{x}(d_{k+1}) + B_{d,k+1}u_{d,k+1} + B_{d,k+1}u_{d,k+1} + g_{a}(d_{k+1})\} \\
- & g_{x} (f_k + B_{k} u_{k} + u_{k}) - g_{x} - \hat{v}_k \\
= & \gamma(u_{d,k+1} - u_0) + [(1 - \gamma)I - L_k g_{x} B_k] (u_{d,k+1} - u_k) \\
\end{align*}
$$

(19)

Taking norms, and using their properties (see (22) at the bottom of the page). Using Lemma 1, Lipschitz conditions, and the bounds

$$
\begin{align*}
||u_{d,k+1} - u_{k+1}|| \\
\leq & \gamma||u_{d,k+1} - u_0|| + \rho||u_{d,k} - u_k|| \\
+ & b_L(M_f||x_{d,k+1} - x_k|| + b_w) \\
+ & b_{g,x}||x_{d,k+1} - x_k|| + M_b||x_{d,k+1} - x_k|| + b_{w} \\
+ & M_g||x_{d,k+1} - x_k|| + b_{w_2},
\end{align*}
$$

(20)

Set $M_2 \equiv b_L M_g + b_{g,x} (M_f + M_B b_{ud}) + M_g I$

$$
||u_{d,k+1} - u_{k+1}|| \leq \rho||u_{d,k+1} - u_0|| + \gamma||u_{d,k+1} - u_0|| \\
+ M_2||x_{d,k+1} - x_k|| + b_L b_x b_w + b_L b_{w_2}.
$$

(21)

Using (18) and (20)

$$
\begin{align*}
||u_{d,k+1} - u_{k+1}|| \\
\leq & \rho||u_{d,k+1} - u_0|| + \gamma||u_{d,k+1} - u_0|| \\
+ & M_2||x_{d,k+1} - x_k|| + b_L b_x b_w + b_L b_{w_2}.
\end{align*}
$$

(22)

Using (22) and (24)

$$
\begin{align*}
||u_{d,k+1} - u_{k+1}|| \\
\leq & \rho||u_{d,k+1} - u_0|| + \gamma||u_{d,k+1} - u_0|| \\
+ & M_2||x_{d,k+1} - x_k|| + b_L b_x b_w + b_L b_{w_2}.
\end{align*}
$$

(23)

$$
\begin{align*}
\sup_{t \in [0,T]} & \int_0^t e^{-\lambda_T} e^{(M_1 - \lambda_T)(t-\tau)} d\tau \\
\leq & \sup_{t \in [0,T]} \int_0^t e^{(M_1 - \lambda_T)(t-\tau)} d\tau \\
= & \frac{1}{\lambda - M_1} \left(1 - e^{(M_1 - \lambda_T)}\right).
\end{align*}
$$

Using (18) and (22)

$$
\begin{align*}
\sup_{t \in [0,T]} & \int_0^t e^{-\lambda_T} ||u_{d,k+1}(\tau) - u_k(\tau)|| e^{(M_1 - \lambda_T)(t-\tau)} d\tau \\
\leq & \sup_{t \in [0,T]} \int_0^t e^{(M_1 - \lambda_T)(t-\tau)} ||u_{d,k+1} - u_k|| d\tau \\
= & \frac{1}{\lambda - M_1} \left(1 - e^{(M_1 - \lambda_T)}\right) ||u_{d,k+1} - u_k||
\end{align*}
$$

(24)
then, (26) implies
\[
\|u_{k+1} - u_k\|_\lambda \\
\leq \left[\rho + \frac{M}{\lambda - M} (1 - e^{(M-N)T})\right]\|u_{k+1} - u_k\|_\lambda \\
+ \gamma\|u_{k+1} - u_k\|_\lambda + M_2\|x_{d,k+1}(0) - x_k(0)\| \\
+ b_w\left[\frac{M_2}{\lambda - M} (1 - e^{(M-N)T}) + b_Lb_{gy}\right] + b_Lb_{s2}.
\] (27)

Set \(\bar{\rho} \equiv \rho + \frac{M}{\lambda - M} (1 - e^{(M-N)T})\), and
\[
\varepsilon \equiv \left\{
\gamma\|u_{d,k} - u_k\|_\lambda + M_2\|x_{d,k+1}(0) - x_k(0)\| \\
+ b_w\left[\frac{M_2}{\lambda - M} (1 - e^{(M-N)T}) + b_Lb_{gy}\right] + b_Lb_{s2}\right\}.
\]

Since \(\rho < 1\), we can find a finite \(\lambda\) large enough such that
\[
\bar{\rho} < 1.
\]

Iterating \(k\), we find
\[
\|u_{d,k} - u_k\|_\lambda \leq \bar{\rho}^k\|u_{d,1} - u_0\|_\lambda + \left(\frac{1 - \bar{\rho}^k}{1 - \bar{\rho}}\right)\varepsilon.
\] (28)

Since \(\bar{\rho} < 1\), then
\[
\lim_{k \to \infty} \|u_{d,k} - u_k\|_\lambda \leq \frac{\varepsilon}{1 - \bar{\rho}}.
\] (30)

From (8), the error for the \(k\)th iterate
\[
x_{d,k} - x_k = x_{d,k}(0) - x_k(0) \\
+ \int_0^t \{(f_{d,k} + B_{d,k}u_{d,k}) - (f_k + B_ku_k + w_k)\}\,dt.
\]

Taking norms, and using their properties
\[
\|x_{d,k} - x_k\| \leq \|x_{d,k}(0) - x_k(0)\| \\
+ \int_0^t \|f_{d,k} - f_k\| + \|B_{d,k} - B_k\|\|u_{d,k}\| \\
+ \|B_k\|\|u_{d,k} - u_k\| + \|w_k\|\,dt.
\]

Using the Lipschitz condition
\[
\|x_{d,k} - x_k\| \leq \|x_{d,k}(0) - x_k(0)\| \\
+ \int_0^t (M_I + M_gb_{rud})\|x_{d,k} - x_k\|\,dt \\
+ \int_0^t (b_{B_2}\|u_{d,k} - u_k\| + b_{w_2})\,dt.
\]

Set \(M_I \equiv M_I + M_gb_{rud}\), and using an integral inequality [13, p. 96],
\[
\|x_{d,k} - x_k\| \leq \|x_{d,k}(0) - x_k(0)\| e^{M_I t} \\
+ \int_0^t \sum_{i=1}^n \left\{L_k\|u_{d,k}(\tau) - u_k(\tau)\| + b_{w_i}\right\}\,d\tau.
\]

Note that the last four equations are similar to (15)–(18).

Multiply both sides of the last inequality by \(e^{-\lambda t}\) and take the supremum for \(t \in [0, T]\)
\[
\|x_{d,k} - x_k\|_\lambda \\
\leq \|x_{d,k}(0) - x_k(0)\| \\
+ \sup_{\tau \in [0,T]} b_{B_2} \int_0^t e^{-\lambda \tau}\|u_{d,k}(\tau) - u_k(\tau)\|\,d\tau \\
\times e^{(M_I - \lambda)T} + \sup_{\tau \in [0,T]} b_{w_2} \int_0^t e^{-\lambda \tau} e^{(M_I - \lambda)T}\,d\tau \\
\times \int_0^t e^{-\lambda \tau} e^{(M_I - \lambda)T}\,d\tau \\
\leq \|x_{d,k}(0) - x_k(0)\| + \frac{b_{B_2}}{\lambda - M_I} (1 - e^{(M_I - \lambda)T}) \\
\times \|u_{d,k} - u_k\|_\lambda + \frac{b_{w_2}}{\lambda - M_I} (1 - e^{(M_I - \lambda)T}) \\
+ \frac{1}{\lambda - M_I} (1 - e^{(M_I - \lambda)T})(b_{B_2}\|u_{d,k} - u_k\|_\lambda + b_{w_2}).
\]

Taking the limit, and using (30)
\[
\lim_{k \to \infty} \|x_{d,k} - x_k\|_\lambda \leq b_{a_0} + \frac{1}{\lambda - M_I} (1 - e^{(M_I - \lambda)T}) \\
\times \left(\frac{b_{B_2}\varepsilon}{1 - \bar{\rho}} + b_{w_2}\right).
\] (33)

From (8) and using Lipschitz condition, the error of the \(k\)th iterate implies
\[
\|y_{d,k} - y_k\| \leq \|y_{d,k} - y_k\| + \|y_k\| \\
\leq M_g\|x_{d,k} - x_k\|_\lambda + \|y_k\| \\
\leq M_g\|x_{d,k} - x_k\|_\lambda + \|y_k\| \\
\leq M_g\sqrt{\varepsilon}\|x_{d,k} - x_k\|_\lambda + b_{w_2}.
\]

Taking the limit, and using (33)
\[
\lim_{k \to \infty} \|y_{d,k} - y_k\| \\
\leq M_g\sqrt{\varepsilon}\|
\]

and since \(\lambda\) is finite, as \(\gamma, \varepsilon, b_{a_0}, b_{w_2}, b_{w_2}\), and \(b_{w_2}\) tend to zero, this implies that \(\varepsilon \to 0\); hence, (30) implies that \(u_k \to u_{d,k}\).

Then, (33) gives \(x_k \to x_{d,k}\), and finally (38) gives \(y_k \to y_{d,k}\). \(\Box\)

**Proof of Theorem 2** From (8) and (10), the error for the \((k + 1)\)th iterate
\[
u_{d,k+1} - u_{k+1} \\
= u_{d,k+1} - (1 - \gamma)u_k - \gamma u_0 \\
- L_k(\hat{g}_{d,k+1} - \hat{g}_k) - P_k(\hat{g}_{d,k+1} - y_k) \\
= (1 - \gamma)(u_{d,k+1} - u_k) + \gamma(u_{d,k+1} - u_0) \\
- L_k(\hat{g}_{d,k+1} + \hat{g}_{d,k} + u_k) + P_k(\hat{g}_{d,k+1} - y_k).
\]

(39)
Taking norms, and using their properties
\[
\|u_{d,k+1} - u_{k+1}\| \leq \gamma \|u_{d,k+1} - u_0\| + \|I - L_k g_{dk} B_k\| \|u_{d,k+1} - u_k\| + P_k \|g_{d,k+1} - g_k\| + \|v_k\|,
\]
(41)

Using Lipschitz condition, and the bounds
\[
\|u_{d,k+1} - u_{k+1}\| \leq \rho \|u_{d,k+1} - u_k\| + \gamma \|u_{d,k+1} - u_0\| + M_2 \|x_{d,k+1} - x_k\| + b_L \|b_{dx} b_w + b_{lv} b_{v2} + b_{rv} b_{v1}\|,
\]
(42)

Set \(M_2 \equiv b_L \{M_{gb} b_d + b_{gx} (M_f + M_B b_{ud}) + M_{gt}\} + M_g b_P\)
\[
\|u_{d,k+1} - u_{k+1}\| \leq \rho \|u_{d,k+1} - u_k\| + \gamma \|u_{d,k+1} - u_0\| + M_2 \|x_{d,k+1} - x_k\| + b_L \|b_{dx} b_w + b_{lv} b_{v2} + b_{rv} b_{v1}\| + \int_0^t e^{M_2 (t-\tau)} (b_{dx} b_{w} + b_{lv} b_{v2} + b_{rv} b_{v1}) d\tau.
\]
(43)

Using (18) and (44)
\[
\|u_{d,k+1} - u_{k+1}\| \leq \rho \|u_{d,k+1} - u_k\| + \gamma \|u_{d,k+1} - u_0\| + M_2 \|x_{d,k+1} - x_k\| + b_L \|b_{dx} b_w + b_{lv} b_{v2} + b_{rv} b_{v1}\| + \int_0^t e^{M_2 (t-\tau)} (b_{dx} b_w + b_{lv} b_{v2} + b_{rv} b_{v1}) d\tau.
\]
(44)

Set \(M \equiv \max\{M_{gb} b_d, M_g, M_2\}\), and multiply both sides by \(e^{-\gamma t}\) where \(\gamma > M\), then
\[
e^{-\gamma t}\|u_{d,k+1} - u_{k+1}\| \leq \rho e^{-\gamma t} \|u_{d,k+1} - u_k\| + \gamma e^{-\gamma t} \|u_{d,k+1} - u_0\| + M_2 \|x_{d,k+1} - x_k\| e^{M_2 t} + \int_0^t e^{M_2 (t-\tau)} (b_{dx} b_w + b_{kv} b_{v2} + b_{rv} b_{v1}) d\tau.
\]
(45)

Taking norms, and using their properties
\[
\|u_{d,k+1} - u_{k+1}\| \leq \gamma \|u_{d,k+1} - u_k\| + \|I - L_k g_{dk} B_k\| \|u_{d,k+1} - u_k\| + \|P_k\| \|g_{d,k+1} - g_k\| + \|v_k\| + \int_0^t e^{-\gamma (t-\tau)} (b_{dx} b_w + b_{lv} b_{v2} + b_{rv} b_{v1}) d\tau.
\]
(46)
Using Lipschitz condition, and the bounds
\[
\begin{align*}
\|u_{d,k+1} - u_{k+1}\| \\
&\leq \gamma (\|u_{d,k+1} - u_0\| + M_2 \|x_{d,k+1} - x_k\| + b_1 b_2 + b_Q b_{12} T) \\
&+ b_Q M_2 \int_0^t \|x_{d,k+1} - x_k\| d\tau \\
&+ b_1 M_2 b_2 + b_2 b_1 b_2 + b_Q b_{12} + M_3 \|x_{d,k+1} - x_k\|.
\end{align*}
\]
(52)

Set
\[
M_3 \equiv b_L (M_3 b_2 + b_2 (M_f + M_M b_2 d + M_f) + M_f b_P) \\
\|u_{d,k+1} - u_{k+1}\| \\
\leq M_3 \|x_{d,k+1} - x_k\| + b_1 b_2 b_3 + b_1 b_2 + b_P b_{12} + b_Q b_{12} T.
\]
(53)

Note the fact that the right hand side of (18) is a non decreasing function of \(t\), i.e., (18) has the form of \(\psi(t) \leq u(t)\) where \(u(t)\) is the nondecreasing function of \(t\). This implies that
\[
\int_0^t \psi(\tau) d\tau \leq \int_0^t u(\tau) d\tau \leq T u(T) \quad 0 \leq t \leq T.
\]
Now, using (18) and (53)
\[
\begin{align*}
\|u_{d,k+1} - u_{k+1}\| \\
&\leq \rho (\|u_{d,k+1} - u_{k+1}\| + M_3 \|x_{d,k+1} - x_k\|) + b_1 b_2 b_3 + b_1 b_2 + b_P b_{12} + b_Q b_{12} T.
\end{align*}
\]
(54)

Set
\[
M_2 \equiv M_3 + b_2 b_1 T, \quad M = \max\{M_2, M_1\}, \quad \text{and multiply both sides by} \quad e^{-\lambda t} \quad \text{where} \quad \lambda > M,
\]
\[
e^{-\lambda t} (\|u_{d,k+1} - u_{k+1}\|) \\
\leq \rho e^{-\lambda t} (\|u_{d,k+1} - u_{k+1}\|) + M e^{-\lambda t} (\|x_{d,k+1} - x_k\|) + b_1 b_2 b_3 + b_1 b_2 + b_P b_{12} + b_Q b_{12} T.
\]
(55)

Take the supremum for \(t \in [0, T]\), then (55) implies
\[
\begin{align*}
\|u_{d,k+1} - u_{k+1}\| &\leq \rho \frac{M}{\lambda - M_1} (1 - e^{(M-\lambda)T}) \|u_{d,k+1} - u_{k+1}\| + M e^{-\lambda T} (\|x_{d,k+1} - x_k\|) + b_1 b_2 b_3 + b_1 b_2 + b_P b_{12} + b_Q b_{12} T.
\end{align*}
\]
(56)

The rest of the proof is exactly the same as in the proof of Theorem 1.

REFERENCES

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