Research Article
Henstock-Kurzweil Integral Transforms

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Received 23 March 2012; Accepted 30 August 2012

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We show conditions for the existence, continuity, and differentiability of functions defined by $\Gamma(s) = \int_{-\infty}^{\infty} f(t)g(t, s)dt$, where $f$ is a function of bounded variation on $\mathbb{R}$ with $\lim_{|t| \to \infty} f(t) = 0$.

1. Introduction

Let $g$ be a complex function defined on a certain subset of $\mathbb{R}^2$. Many functions on functional analysis are integrals of the following form:

$$\Gamma(s) = \int_{-\infty}^{\infty} f(t)g(t, s)dt.$$ (1.1)

We discuss the above function $\Gamma$, where the integral that we use is that of Henstock-Kurzweil. This integral introduced independently by Kurzweil and Henstock in 1957-58 encompasses the Riemann and Lebesgue integrals, as well as the Riemann and Lebesgue improper integrals.

In Lebesgue theory, there are well-known results about the existence, continuity, and differentiability of $\Gamma$. For Henstock-Kurzweil integrals also there are results about this, for example, Theorems 12.12 and 12.13 of [1]. However, they all need the stronger condition: $f(t)g(t, s)$ is bounded by a Henstock-Kurzweil integrable function $r(t)$. We provide other conditions for the existence, continuity, and differentiability of $\Gamma$. 
2. Preliminaries

Let us begin by recalling the definition of Henstock-Kurzweil integral. For finite intervals in $\mathbb{R}$ it is defined in the following way.

**Definition 2.1.** Let $f : [a, b] \to \mathbb{R}$ be a function. One can say that $f$ is Henstock-Kurzweil (shortly, HK-) integrable, if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is a function $\gamma_\epsilon : [a, b] \to (0, \infty)$ (named a gauge) with the property that for any $\delta_\epsilon$-fine partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ of $[a, b]$ (i.e., for each $i$, $[x_{i-1}, x_i] \subset [t_i - \gamma_\epsilon(t_i), t_i + \gamma_\epsilon(t_i)]$), one has

$$\sum_{i=1}^nf(t_i)(x_i - x_{i-1}) - A < \epsilon.$$  

(2.1)

The number $A$ is the integral of $f$ over $[a, b]$ and it is denoted as $A = \int_a^b f$.

In the unbounded case, the Henstock-Kurzweil integral is defined as follows.

**Definition 2.2.** Given a gauge function $\gamma : [a, \infty] \to (0, \infty)$, one can say that a tagged partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ of $[a, \infty)$ is $\gamma$-fine, if

a) $a = x_0$, $x_{n+1} = \infty$,

b) $[x_{i-1}, x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$ for all $i = 1, 2, \ldots, n$,

c) $[x_n, \infty] \subseteq [1/\gamma(t_{n+1}), \infty]$.

**Definition 2.3.** A function $f : [a, \infty] \to \mathbb{R}$ is Henstock-Kurzweil integrable on $[a, \infty]$, if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is a gauge $\gamma_\epsilon : [a, \infty] \to (0, \infty)$ for which (2.1) is satisfied for all tagged partition $P$ which is $\delta_\epsilon$-fine according to Definition 2.2.

Let $f$ be a function defined on an infinite interval $[a, \infty)$, One can suppose that $f$ is defined on $[a, \infty]$ assuming that $f(\infty) = 0$. Thus, $f$ is Henstock-Kurzweil integrable on $[a, \infty)$ if $f$ extended on $[a, \infty]$ is HK-integrable. For functions defined over intervals $(-\infty, a]$ and $(-\infty, \infty)$ One can makes similar considerations.

Let $I$ be a finite or infinite interval. The space of all Henstock-Kurzweil integrable functions over $I$ is denoted by $\mathcal{HK}(I)$. This space will be considered with the Alexiewicz seminorm, which it is defined as follows:

$$\|f\|_I = \sup_{J \subseteq I} \left| \int_J f \right|,$$

(2.2)

where the supremum is being taken over all intervals $J$ contained in $I$.

**Definition 2.4.** Let $\varphi : I \to \mathbb{R}$ be a function, where $I \subseteq \mathbb{R}$ is a finite interval. The variation of $\varphi$ over the interval $I$ is defined as follows:

$$V_I\varphi = \sup \left\{ \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| : P \text{ is partition of } I \right\}.$$  

(2.3)

We say that the function $\varphi$ is of bounded variation on $I$ if $V_I\varphi < \infty$. Now if $\varphi$ is a function defined on an infinite interval $I$, then $\varphi$ is of bounded variation on $I$, if $\varphi$ is of
bounded variation on each finite subinterval of \( I \) and there is \( M > 0 \) such that \( V_{[a,b]} \varphi \leq M \) for all \( [a,b] \subseteq I \). The variation of \( \varphi \) on \( I \) is \( V \varphi = \sup \{ V_{[a,b]} \varphi \} \) for all \( [a,b] \subseteq I \).

Given an interval \( I \), the space of all bounded variation functions on \( I \) is denoted by \( BV(I) \). We set \( BV_0(\mathbb{R}) = \{ f \in BV(I) \mid \lim_{|t| \to \infty} f(t) = 0 \} \). The following are some classical theorems that are used throughout this paper. The first is given in \([2, \text{Lemma } 24]\) and is an immediate consequence of \([1, \text{Theorem } 10.12, \text{and Corollary } H.4]\).

**Theorem 2.5.** If \( g \) is a HK-integrable function on \( [a,b] \subseteq \mathbb{R} \) and \( f \) is a function of bounded variation on \( [a,b] \), then \( fg \) is HK-integrable on \( [a,b] \) and

\[
\left| \int_a^b fg \right| \leq \inf_{t \in [a,b]} |f(t)| \left| \int_a^b g(t) dt \right| + \| g \|_{L^1[a,b]} V_{[a,b]} f.
\]  

**Theorem 2.6** ([1] Chartier-Dirichlet’s test). Let \( f \) and \( g \) be functions defined on \( [a, \infty) \). Suppose that

(i) \( g \in A_K([a, c]) \) for every \( c \geq a \), and \( G \) defined by \( G(x) = \int_a^x g \) is bounded on \( [a, \infty) \);

(ii) \( f \) is of bounded variation on \( [a, \infty) \) and \( \lim_{x \to \infty} f(x) = 0 \).

Then \( fg \in A_K([a, \infty)) \).

**Definition 2.7** (see [3]). Let \( E \subseteq [a,b] \). A function \( f : [a,b] \to \mathbb{R} \) is \( AC_\delta \) on \( E \), if for every \( \varepsilon > 0 \), there exist \( \eta_\varepsilon > 0 \) and a gauge \( \delta_\varepsilon \) on \( E \) such that

\[
\sum_{i=1}^n |f(v_i) - f(u_i)| < \varepsilon,
\]  

whenever \( P = \{ ([u_i, v_i], t_i) \}_{i=1}^n \) is a \( (\delta_\varepsilon, E) \)-fine subpartition of \( [a,b] \) (i.e., \( P \) is \( \delta_\varepsilon \)-fine and the tags \( t_i \) belong to \( E \)) and \( \sum_{i=1}^n |v_i - u_i| < \eta_\varepsilon \).

We say that \( f \) is \( AC_\delta \) on \( [a,b] \), if \( [a,b] \) can be written as a countable union of sets on each of which the function \( f \) is \( AC_\delta \).

If \( h(t,s) \) is a function on \( \mathbb{R} \times \mathbb{R} \), then we use the notation \( D_2 h \) for the partial derivative of \( h \) with respect to the second component \( s \).

**Theorem 2.8** ([4, Theorem 4]). Let \( a, b \in \mathbb{R} \). If \( h : \mathbb{R} \times [a,b] \to \mathbb{C} \) is such that

(i) \( h(t, \cdot) \) is \( AC_\delta \) on \( [a,b] \) for almost all \( t \in \mathbb{R} \);

(ii) \( h(\cdot, s) \) is HK-integrable on \( \mathbb{R} \) for all \( s \in [a,b] \).

Then \( H := \int_{-\infty}^\infty h(t, \cdot) dt \) is \( AC_\delta \) on \( [a,b] \) and \( H'(s) = \int_{-\infty}^\infty D_2 h(t, s) dt \) for almost all \( s \in (a,b) \), if and only if,

\[
\int_s^t \int_{-\infty}^\infty D_2 h(t, s) dt ds = \int_{-\infty}^\infty \int_s^t D_2 h(t, s) ds dt,
\]

for all \( [s,t] \subseteq [a,b] \). In particular,

\[
H'(s_0) = \int_{-\infty}^\infty D_2 h(t, s_0) dt,
\]

when \( H_2 := \int_{-\infty}^\infty D_2 h(t, \cdot) dt \) is continuous at \( s_0 \).
3. Main Results

All results in this paper are based on functions in the vector space $BV_0(\mathbb{R})$. Note that $BV_0(\mathbb{R}) \not\subseteq L(\mathbb{R})$, where $L(\mathbb{R})$ is the space of Lebesgue integrable functions. Indeed, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 1), \\ \frac{1}{x} & \text{if } x \in [1, \infty), \end{cases} \quad (3.1)$$

is in $BV_0(\mathbb{R}) \setminus L(\mathbb{R})$. However, for bounded intervals $I$, functions in $BV(I)$ are Lebesgue integrable on $I$.

To facilitate the statement of these results, it seems appropriate to introduce some additional terminology. If $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a function and $s_0 \in \mathbb{R}$, we say that $s_0$ satisfies Hypothesis (H) relative to $g$ if

$$(H) \text{ there exist } \delta = \delta(s_0) > 0 \text{ and } M = M(s_0) > 0, \text{ such that, if } |s - s_0| < \delta \text{ then}$$

$$\left| \int_u^v g(t,s) \, dt \right| \leq M, \quad (3.2)$$

for all $[u,v] \subseteq \mathbb{R}$.

This type of condition plays a major role in the results of the present work.

**Theorem 3.1.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be functions. If $f \in BV_0(\mathbb{R})$, and $s_0 \in \mathbb{R}$ satisfies Hypothesis (H) relative to $g$, then

$$\Gamma(s) = \int_{-\infty}^{\infty} f(t) g(t,s) \, dt \quad (3.3)$$

exists for all $s$ in a neighborhood of $s_0$.

**Proof.** It follows by Theorem 2.6. \hfill \Box

**Theorem 3.2.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be functions such that

(i) $f \in BV_0(\mathbb{R})$, $g$ is bounded, and

(ii) $g(t, \cdot)$ is continuous for all $t \in \mathbb{R}$.

If $s_0 \in \mathbb{R}$ satisfies Hypothesis (H) relative to $g$, then the function $\Gamma$ defined in Theorem 3.1 is continuous at $s_0$.

**Proof.** There exist $\delta_1 > 0$ and $M > 0$, such that, if $|s - s_0| < \delta_1$ then

$$\left| \int_u^v g(t,s) \, dt \right| \leq M, \quad (3.4)$$

for all $[u,v] \subseteq \mathbb{R}$. From Theorem 3.1, $\Gamma(s)$ exists for all $s \in B_{\delta_1}(s_0)$. 
Let $\epsilon > 0$ be given. By Hake’s Theorem, there exists $K_1 > 0$ such that
\[
\left| \int_{|t| \geq u} f(t)g(t,s)dt \right| < \frac{\epsilon}{3},
\] (3.5)
for all $u \geq K_1$. On the other hand, as
\[
\lim_{t \to -\infty} V(-\infty,t)f = 0, \quad \lim_{t \to \infty} V(t,\infty)f = 0,
\] (3.6)
there is $K_2 > 0$ such that for each $t > K_2$,
\[
V(-\infty,-t)f + V(t,\infty)f < \frac{\epsilon}{3M}.
\] (3.7)

Let $K = \max\{K_1, K_2\}$. From Theorem 2.5, it follows that for every $v \geq K$ and every $s \in B_{\delta_1}(s_0)$,
\[
\left| \int_{-\infty}^{v} f(t)g(t,s)dt \right| \leq \|g(\cdot,s)\|_{[K,v]} \left[ \inf_{t \in [K,v]} |f(t)| + V_{[K,v]}f \right] 
\leq M\left[ |f(\nu)| + V_{[K,\infty]}f \right],
\] (3.8)
where the second inequality is true due to (3.4). This implies, since $\lim_{t \to \infty} |f(t)| = 0$, that
\[
\left| \int_{K}^{\infty} f(t)g(t,s)dt \right| \leq M \cdot V_{[K,\infty]}f.
\] (3.9)

Analogously we have that
\[
\left| \int_{-\infty}^{-K} f(t)g(t,s)dt \right| \leq M \cdot V_{[-\infty,-K]}f.
\] (3.10)

Therefore, for each $s \in B_{\delta_1}(s_0),
\[
\left| \int_{|t| \geq K} f(t)g(t,s)dt \right| \leq M\left[ V_{[-\infty,-K]}f + V_{[K,\infty]}f \right] < M\frac{\epsilon}{3M} = \frac{\epsilon}{3}.
\] (3.11)

By hypothesis, $f$ is Lebesgue integrable on $[-K,K]$, $g$ is bounded, and $g(t,\cdot)$ is continuous for all $t \in \mathbb{R}$. From this it is easy to see, for example using [1, Theorem 12.12], that $\Gamma_K: \mathbb{R} \to \mathbb{R}$ defined as
\[
\Gamma_K(s) = \int_{-K}^{K} f(t)g(t,s)dt, \quad s \in \mathbb{R},
\] (3.12)
Theorem 3.3. Let \( a, b \in \mathbb{R} \). If \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \times [a, b] \to \mathbb{C} \) are functions such that

(i) \( f \in \mathcal{BU}_0(\mathbb{R}) \), \( g \) is measurable, bounded, and

(ii) for all \( s \in [a, b] \), \( s \) satisfies hypothesis (H) relative to \( g \).

then

\[
\int_a^b \int_{-\infty}^{\infty} f(t) g(t, s)\, dt\, ds = \int_{-\infty}^{\infty} \int_a^b f(t) g(t, s)\, ds\, dt.
\] (3.15)

Proof. From condition (ii) and by the compactness of \([a, b]\), we claim that there exists \( M > 0 \) such that, for each \( s \in [a, b] \), \( \left| \int_u^v g(t, s)\, dt \right| \leq M \), for all \([u, v] \subseteq \mathbb{R}\).

For each \( r > 0 \) and \( s \in [a, b] \), let \( \Gamma_r(s) = \int_r^\infty f(t) g(t, s)\, dt \). Observe, by Theorem 2.5,

\[
|\Gamma_r(s)| = \left| \int_r^\infty f(t) g(t, s)\, dt \right|
\leq \|g(\cdot, s)\|_{[-r,r]} \left( \inf_{t \in [-r,r]} |f(t)| + V_{[-r,r]} f \right)
\leq M \left( |f(0)| + V f \right),
\] (3.16)

for all \( s \in [a, b] \).

So, for each \( r > 0 \), \( \Gamma_r \) is HK-integrable on \([a, b]\) and is bounded for a fixed constant. Moreover, by Theorem 3.1 and Hake’s Theorem,

\[
\lim_{r \to \infty} \Gamma_r(s) = \Gamma(s), \tag{3.17}
\]

for all \( s \in [a, b] \).
Therefore, by dominated convergence theorem, $\Gamma$ is HK-integrable on $[a, b]$ and

$$\int_{a}^{b} \Gamma(s) ds = \lim_{r \to \infty} \int_{a}^{b} \Gamma_{r}(s) ds. \quad (3.18)$$

Now, since $f$ is Lebesgue integrable on $[-r, r]$, $g$ is measurable and bounded; it follows by Fubini’s Theorem that

$$\int_{a}^{b} \int_{-r}^{r} f(t)g(t, s) dt \ ds = \int_{-r}^{r} \int_{a}^{b} f(t)g(t, s) ds \ dt. \quad (3.19)$$

Consequently,

$$\lim_{r \to \infty} \int_{-r}^{r} \int_{a}^{b} f(t)g(t, s) ds \ dt = \lim_{r \to \infty} \int_{a}^{b} \Gamma_{r}(s) ds = \int_{a}^{b} \Gamma(s) ds. \quad (3.20)$$

So by Hake’s Theorem,

$$\int_{-\infty}^{\infty} \int_{a}^{b} f(t)g(t, s) ds \ dt = \int_{a}^{b} \Gamma(s) ds = \int_{a}^{b} \int_{-\infty}^{\infty} f(t)g(t, s) dt \ ds. \quad (3.21)$$

**Theorem 3.4.** Consider $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ functions, where $f \in BV_0(\mathbb{R})$ and the partial derivative $D_2g$ exists on $\mathbb{R} \times \mathbb{R}$ and is bounded and continuous. If $s_0 \in \mathbb{R}$ such that

(i) there is $K > 0$ for which $\|g(\cdot, s_0)\|_{[u,v]} \leq K$ for all $[u, v] \subseteq \mathbb{R}$, and

(ii) $s_0$ satisfies Hypothesis (H) relative to $D_2g$,

then $\Gamma$ is differentiable at $s_0$, and

$$\Gamma'(s_0) = \int_{-\infty}^{\infty} f(t)D_2g(t, s_0) dt. \quad (3.22)$$

**Proof.** It is not difficult to prove, using conditions (i), (ii), and the Mean Value Theorem, that there exist $\delta > 0$ and $M > 0$ such that, for each $s \in (s_0 - \delta, s_0 + \delta)$,

$$\left| \int_{u}^{v} D_2g(t, s) dt \right| < M, \quad \left| \int_{u}^{v} g(t, s) dt \right| < M, \quad (3.23)$$

for all $[u, v] \subseteq \mathbb{R}$.

Consider $a, b \in \mathbb{R}$ with $s_0 - \delta < a < s_0 < b < s_0 + \delta$. In order to show (3.22), we use Theorem 2.8. The function $f(t)g(t, \cdot)$ is differentiable on $[a, b]$ for each $t \in \mathbb{R}$, so $f(t)g(t, \cdot)$ is $ACG_{\delta}$ on $[a, b]$ for all $t \in \mathbb{R}$. Also, by (3.23) and Theorem 2.6, $f(\cdot)g(\cdot, s)$ is HK-integrable on $\mathbb{R}$ for all $s \in [a, b]$. Then

$$\Gamma'(s_0) = \int_{-\infty}^{\infty} f(t)D_2g(t, s_0) dt, \quad (3.24)$$
if

$$\Gamma_2 := \int_{-\infty}^{\infty} f(t) D_2 g(t, \cdot) dt$$  \hspace{1cm} (3.25)

is continuous at $s_0$, and

$$\int_{s}^{t} \int_{-\infty}^{\infty} f(t) D_2 g(t, s) dt ds = \int_{-\infty}^{\infty} \int_{s}^{t} f(t) D_2 g(t, s) ds dt,$$  \hspace{1cm} (3.26)

for all $[s, t] \subseteq [a, b]$. The first affirmation is true by (3.23) and Theorem 3.2. The second affirmation is true due to (3.23) and Theorem 3.3. \hfill \Box

Remark 3.5. In the previous theorems the kernel $g(t, s)$ satisfies $|\int_{u}^{v} g(t, s) dt| \leq M$, for all $[u, v] \subseteq \mathbb{R}$. Moreover, if $g$ will satisfy

$$\left|\int_{u}^{v} g(t, s) dt\right| \leq \frac{M_0}{|s|},$$  \hspace{1cm} (3.27)

for all $[u, v] \subseteq \mathbb{R}$, then $\lim_{|s| \to \infty} \Gamma(s) = 0$, when $f \in B\mathcal{U}_0(\mathbb{R})$ (a version of Riemann-Lebesgue Lemma).

4. Applications

If $f : \mathbb{R} \to \mathbb{R}$, then its Fourier transform at $s \in \mathbb{R}$ is defined as follows:

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-it s} dt.$$  \hspace{1cm} (4.1)

Talvila in [2] has done an extensive work about the Fourier transform using the Henstock-Kurzweil integral: existence, continuity, inversion theorems and so forth. Nevertheless, there are some omissions in those results that use [2, Lemma 25(a)]. Also Mendoza Torres et al. in [5] have studied existence, continuity, and Riemann-Lebesgue Lemma about the Fourier transform of functions belonging to $\mathcal{H}K(\mathbb{R}) \cap B\mathcal{U}(\mathbb{R})$. Following the line of [5], in Theorem 4.2, we include some results from them as consequences of theorems in the above section.

Let $f$ and $g$ be real-valued functions on $\mathbb{R}$. The convolution of $f$ and $g$ is the function $f \ast g$ defined by

$$f \ast g(x) = \int_{-\infty}^{\infty} f(x - y) g(y) dy,$$  \hspace{1cm} (4.2)

for all $x$ such that the integral exists. Various conditions can be imposed on $f$ and $g$ to guarantee that $f \ast g$ is defined on $\mathbb{R}$, for example, if $f$ is HK-integrable and $g$ is of bounded variation.

Lemma 4.1. If $f \in \mathcal{H}K(\mathbb{R}) \cap B\mathcal{U}(\mathbb{R})$, then $\lim_{|x| \to \infty} f(x) = 0$. 

Proof. Since \( f \) is of bounded variation on \( \mathbb{R} \), then \( \lim_{x \to -\infty} f(x) \) and \( \lim_{x \to \infty} f(x) \) exist. Suppose that \( \lim_{x \to -\infty} f(x) = a \neq 0 \). If \( a > 0 \), there exists \( A > 0 \) such that \( a/2 < f(x) \), for all \( x > A \). If \( a < 0 \), there is \( B > 0 \) such that \(-a/2 < -f(x)\), for all \( x > B \). This shows that \( f \notin \mathcal{K}([A, \infty)) \) or \( -f \notin \mathcal{K}([B, \infty)) \), which contradicts \( f \in \mathcal{K}(\mathbb{R}) \), so \( \lim_{x \to -\infty} f(x) = 0 \). Using a similar argument, we show that \( \lim_{x \to \infty} f(x) = 0 \). \( \square \)

As a consequence of Lemma 4.1, the vector space \( \mathcal{K}(\mathbb{R}) \cap \mathcal{B}U(\mathbb{R}) \) is contained in \( \mathcal{B}U_0(\mathbb{R}) \). So the next theorem is an immediate consequence of the above section.

**Theorem 4.2.** If \( f \in \mathcal{K}(\mathbb{R}) \cap \mathcal{B}U(\mathbb{R}) \), then

(a) \( \hat{f} \) exists on \( \mathbb{R} \).

(b) \( \hat{f} \) is continuous on \( \mathbb{R} \setminus \{0\} \).

(c) \( \lim_{|s| \to \infty} \hat{f}(s) = 0 \).

(d) Define \( g(t) = tf(t) \) and suppose that \( g \in \mathcal{K}(\mathbb{R}) \cap \mathcal{B}U(\mathbb{R}) \), then \( \hat{f} \) is differentiable on \( \mathbb{R} \setminus \{0\} \), and

\[
\hat{f}'(s) = -ig(s), \quad \forall s \in \mathbb{R} \setminus \{0\}.
\]  

(4.3)

(e) If \( h \in L(\mathbb{R}) \cap \mathcal{B}U(\mathbb{R}) \), then \( \hat{f} \ast \hat{h}(s) = \hat{f}(s)\hat{h}(s) \) for all \( s \in \mathbb{R} \).

Proof. First observe that

\[
\left| \int_u^v e^{-its} dt \right| \leq \frac{2}{|s|},
\]  

for all \( [u, v] \subseteq \mathbb{R} \). Then, each \( s_0 \neq 0 \) satisfies Hypothesis (H) relative to \( e^{-its} \).

(a) Theorem 3.1 implies that \( \hat{f}(s_0) \) exists for all \( s_0 \neq 0 \) and, since \( f \in \mathcal{K}(\mathbb{R}) \), \( \hat{f}(0) \) exists. Thus, \( \hat{f} \) exists on \( \mathbb{R} \).

(b) From Theorem 3.2, \( \hat{f} \) is continuous at \( s_0 \), for all \( s_0 \neq 0 \).

(c) It follows by Remark 3.5 and (4.4).

(d) It follows by Theorem 2.8 in a similar way to the proof of Theorem 3.4.

(e) Take \( s \in \mathbb{R} \) and let \( k(x, y) = f(y-x)e^{-is} \). Then, for each \( y \in \mathbb{R} \) and all \( [u, v] \subseteq \mathbb{R} \),

\[
\left| \int_u^v k(x, y) dx \right| = \left| \int_u^v f(y-x) dx \right| = \left| \int_{y-u}^{y-v} f(z) dz \right| \leq \| f \|.
\]  

(4.5)

Thus, for every \( y \in \mathbb{R} \), \( y \) satisfies Hypothesis (H) relative to \( k \). Now, observe that \( h \in \mathcal{B}U_0(\mathbb{R}) \) and \( k \) is measurable and bounded. So by Theorem 3.3,

\[
\int_{-\infty}^{a} \int_{-\infty}^{\infty} h(x)k(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{a} h(x)k(x, y) dy dx,
\]  

(4.6)

for all \( a > 0 \).
On the other hand,

\[
\begin{align*}
\left| h(x) \int_{-a}^{a} f(y-x)e^{-isy} \, dy \right| & \leq |h(x)| \left| \int_{-a}^{a-x} f(z)e^{-isz} \, dz \right| \\
& \leq |h(x)||f(\cdot)e^{-i(\cdot)s}|.
\end{align*}
\] (4.7)

Thus, since \( h \in L(\mathbb{R}) \), dominated convergence theorem implies that

\[
\hat{f}(s)\hat{h}(s) = \int_{-\infty}^{\infty} h(x) \int_{-\infty}^{\infty} f(y-x)e^{-isy} \, dy \, dx \\
= \lim_{a \to \infty} \int_{-\infty}^{\infty} h(x) \int_{-a}^{a} f(y-x)e^{-isy} \, dy \, dx,
\] (4.8)

but from (4.6), we have that

\[
\hat{f}(s)\hat{h}(s) = \lim_{a \to \infty} \int_{-a}^{a} \int_{-\infty}^{\infty} h(x)f(y-x)e^{-isy} \, dx \, dy \\
= \lim_{a \to \infty} \int_{-a}^{a} (f \ast h)(y)e^{-isy} \, dy.
\] (4.9)

Therefore, by Hake’s Theorem,

\[
\widehat{f \ast h}(s) = \hat{f}(s)\hat{h}(s).
\] (4.10)

If \( f : [0, \infty) \to \mathbb{R} \), then its Laplace transform at \( z \in \mathbb{C} \) is defined as follows:

\[
L(f)(z) = \int_{0}^{\infty} f(t)e^{-zt} \, dt.
\] (4.11)

Here, also the Laplace transform is considered as Henstock-Kurzweil integral.
Theorem 4.3. If $f \in \mathcal{H}(0, \infty) \cap \mathcal{BV}(0, \infty)$, then

(a) $L(f)(z)$ exists for all $z \in \mathbb{C}$ with $\text{Re } z \geq 0$.

(b) If $F(x, y) = L(f)(x + iy)$, then $F(\cdot, y)$ is continuous on $\mathbb{R}^+ \cup \{0\}$ for all $y \neq 0$, and $F(x, \cdot)$ is continuous on $\mathbb{R}$ for all $x > 0$.

Proof. It is an easy consequence from Theorems 3.1 and 3.2, since $|\int_0^y e^{-(x+iy)t} dt| \leq 2/|x + iy|$ for all $u, v, x \in \mathbb{R}^+ \cup \{0\}$, $y \in \mathbb{R}$ with $x + iy \neq 0$.

Moreover, the Riemann-Lebesgue Lemma holds the following.

Theorem 4.4. If $f \in \mathcal{H}(0, \infty) \cap \mathcal{BV}(0, \infty)$ and $z = x + iy$, with $x \geq 0$, then $\lim_{y \to \infty} L(f)(z) = 0$.

Proof. It results by Remark 3.5 and (4.4), because $f(\cdot)e^{-x\cdot}$ is in $\mathcal{H}(0, \infty) \cap \mathcal{BV}(0, \infty)$.

References


