A blow-up result for nonlinear generalized heat equation

M. Kbiri Alaoui\textsuperscript{a,}\textsuperscript{*}, S.A. Messaoudi\textsuperscript{b}, H.B. Khenous\textsuperscript{a}

\textsuperscript{a} Department of Mathematics, King Khalid University, PO. Box 9004, Abha 61321, Saudi Arabia
\textsuperscript{b} Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia

\textbf{A R T I C L E I N F O}

Article history:
Received 17 February 2014
Received in revised form 2 September 2014
Accepted 18 October 2014
Available online xxxx

Keywords:
Nonlinear heat equation
Blow up
Sobolev spaces with variable exponents

\textbf{A B S T R A C T}

In this paper we consider a nonlinear heat equation with nonlinearities of variable-exponent type. We show that any solution with nontrivial initial datum blows up in finite time. We also give a two-dimension numerical example to illustrate our result.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

In this paper we are interested in the existence and the blow-up in finite time of solutions of the following problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{m(x)-2} \nabla u) &= |u|^{p(x)-2} u + f \\
&\text{in } Q = \Omega \times (0, T), \\
&\text{on } \partial Q = \partial \Omega \times [0, T), \\
&u(x, 0) = u_0(x) \\
&\text{in } \Omega,
\end{aligned}
\]

\[(P)\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), \(m(\cdot)\) and \(p(\cdot)\) are two continuous functions on \(\overline{\Omega}\) such that:

\[2 \leq m_- \leq m(x) \leq m_+ < p_- \leq p(x) \leq p_+ < m_*(x),\]

where \(m_*(x) = \left\{ \begin{array}{ll}
\frac{N m(x)}{(N - m(x))^+} & \text{if } m_+ < N, \\
\infty & \text{if } m_+ \geq N.
\end{array} \right.\) (1.1)

We also assume that

\[|m(x) - m(y)| \leq \frac{A}{\log \left( \frac{1}{|x-y|} \right)}, \quad \text{for all } x, y \in \Omega \text{ with } |x - y| < \delta\]

(1.2)

with \(A > 0, \ 0 < \delta < 1\) and

\[\inf_{x \in \Omega} (m^*(x) - p(x)) > 0.\] (1.3)

\textsuperscript{*} Corresponding author.

E-mail addresses: mka_la@yahoo.fr (M. Kbiri Alaoui), messaoud@kfupm.edu.sa (S.A. Messaoudi), khenous@kku.edu.sa (H.B. Khenous).

http://dx.doi.org/10.1016/j.camwa.2014.10.018
0898-1221/© 2014 Elsevier Ltd. All rights reserved.
When \( m(x) = m, p(x) = p \), existence and blow-up results have been established by many authors (see [1–6]). For instance, Fujita in [4] considered the case where
\[
\partial_t u = \Delta u + u^p, \quad \text{in } \mathbb{R}^N
\]
and proved the existence of a critical exponent \( p_* = 1 + \frac{2}{N} \) such that:

- If \( 1 < p < p_* \), then there is only one nonnegative global (in time) solution, which is \( u = 0 \).
- If \( p > p_* \), then there exists a global solution for sufficiently small initial values.

Levine et al. [7] considered the generalized mean curvature, when \( \Omega = \mathbb{R}^N \) and when the elliptic operator is of the form
\[
\text{div}(\psi(1 + |\nabla u|^{2})^{1/2} \nabla u)
\]
with \( \psi \) a real function satisfying some specific conditions, and established the following results:

- If \( 1 < p < p_* \), then there are no nontrivial positive solutions.
- If \( p > p_* \), then there exist both a positive global solution, and solutions which blow up in finite time; however no conclusion was obtained when \( p = p_* \).

Ötani [6] studied the existence and the asymptotic behavior of solutions of (P) and overcame the difficulties caused by the use of nonmonotone perturbation theory. The quasilinear case, with \( m \neq 2 \), requires a strong restriction on the growth of the forcing term \( |u|^{p-2} u \), which is caused by the loss of the elliptic estimate for the \( m \)-Laplacian operator defined by \( \Delta_m u = \text{div}(u |\nabla u|^{m-2} \nabla u) \) (see [2]). Recently in [1], Agaki proved an existence and blow up result for the initial datum \( u_0 \in L^1(\Omega) \).

In the case where \( m(\cdot) \) and \( p(\cdot) \) are two measurable functions, some different techniques must be adopted to study the existence and blow-up of the problem (P) and the traditional methods may fail unless some modifications are made.

Let us mention that the study of differential equations with nonstandard \( p(x) \)-growth is a new and interesting topic. It arises from the nonlinear elasticity theory, electrorheological fluids, etc. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. For a general account of the underlying physics, we refer the reader to [8] and for the mathematical theory see [9]. In this context, a series of papers by Diening and Ružička [10,11] related to problems in the so-called rheological and electrorheological fluids, which lead to spaces with variable exponents, have appeared lately. The results developed in those papers were collected in the books [12,13]. Many mathematical models in fluid mechanics, elasticity theory (recently in image processing, see for example [14]), etc. were shown to be naturally related to the problems with non-standard local growth.

Our objective in this paper is to study the blow up phenomenon of solutions of the problem (P) in the framework of the Lebesgue and Sobolev spaces with variable exponents. We will establish a blow up result and give a numerical example in 2D to illustrate our theoretical result.

2. Preliminaries

2.1. Functional framework

We list some well-known results about the Lebesgue and Sobolev spaces with variable exponents (see [13]). Let \( p : \Omega \to [1, \infty] \) be a measurable function, where \( \Omega \) is a domain of \( \mathbb{R}^N \) with \( N \geq 2 \). We denote by
\[
p_- = \text{ess inf}_{x \in \Omega} p(x), \quad p_+ = \text{ess sup}_{x \in \Omega} p(x).
\]
The \( p(.) \) modular of a measurable function \( u : \Omega \to \mathbb{R}^N \) is defined as
\[
\varphi_{p(\cdot)} = \int_{\Omega_{-\infty}} |u(x)|^{p(x)} \, dx + \text{ess sup}_{x \in \Omega_{\infty}} |u(x)|,
\]
where
\[
\Omega_{\infty} = \{ x \in \Omega : p(x) = \infty \}.
\]
The variable-exponent Lebesgue space \( L^{p(\cdot)}(\Omega) \) consists of all measurable functions \( u \) defined on \( \Omega \) for which
\[
\varphi_{p(\cdot)}(\mu u) < \infty,
\]
for some \( \lambda > 0 \).

The Luxembourg norm on this space is defined as
\[
\|u\|_{p(\cdot)} = \inf\{ \lambda > 0 : \varphi_{p(\cdot)}(u/\lambda) \leq 1 \}.
\]
Equipped with this norm, \( L^{p(\cdot)}(\Omega) \) is a Banach space (see [13,15]).

Remark 2.1. The variable exponent-Lebesgue space is a special case of more general Orlicz–Musielak spaces. For the constant function \( p(x) = p \), the variable exponent Lebesgue space coincides with the classical Lebesgue space.
The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient $\nabla u$ exists and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$. This space is a Banach space with respect to the norm
\[
\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.
\]
In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects. For the following assertions, see [15]:

- The Hölder inequality holds.
- They are reflexive if and only if $1 < p_- \leq p_+ < \infty$.
- Continuous functions are dense if and only if $p_+ < \infty$.
- If $\Omega$ has a finite measure and $p$, $q$ are variable exponents so that $p(x) \leq q(x)$ almost everywhere in $\Omega$, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ holds.

The spaces $W^{1,p(\cdot)}_0(\Omega)$ and $W^{-1,p(\cdot)}(\Omega)$ are defined by the same way as the usual Sobolev spaces where $p'(\cdot)$ is the function such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

2.2. Useful results

In this subsection, we present some useful results which will be used later. Let $p$ satisfy the following Zhikov–Fan condition:
\[
|p(x) - p(y)| \leq \frac{A}{\log \left(\frac{1}{|x-y|}\right)} \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{2.1}
\]
with $A > 0$ and $0 < g \delta < 1$.

**Lemma 2.1.** (1) If (2.1) holds, then $\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}$ for all $u \in W^{1,p(\cdot)}_0(\Omega)$, where $\Omega$ is bounded. In particular, the space $W^{1,p(\cdot)}_0(\Omega)$ has a norm given by $\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}$ for all $u \in W^{1,p(\cdot)}_0(\Omega)$.

(2) If $p \in C(\Omega)$, $q : \Omega \to [1, \infty)$ is a measurable function and
\[
\text{essinf}_{x \in \Omega}(p'(x) - q(x)) > 0
\]
with $p'(x) = \frac{np(x)}{(n-p(x))p(x)}$, then
\[
W^{1,p(\cdot)}_0(\Omega) \hookrightarrow \hookrightarrow L^q(\Omega).
\]

3. Main result

In this section, we present our main blow-up result. We start with a local existence result for the problem (P), which is a direct result of the existence theorem by Agaki and Ōtani [2].

**Theorem 3.1.** For all $u_0 \in W^{1,m(\cdot)}_0(\Omega)$, there exists a number $T_0 \in (0, T]$ such that the problem (P) has a strong solution $u$ on $[0, T_0]$ satisfying:
\[
u \in C_w([0, T_0]; W^{1,m(\cdot)}_0(\Omega)) \cap C([0, T_0]; L^{p(\cdot)}(\Omega)) \cap W^{1,2}(0, T_0; L^2(\Omega)).
\]

Next, we take $f \equiv 0$, state and prove our main theorem.

We multiply Eq. (P) by $u_t$ and integrate over $\Omega$, to get
\[
\int_\Omega u_t^2 \, dx + \frac{d}{dt} \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)} \, dx = \frac{d}{dt} \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx.
\]
We then define the energy by
\[
E(t) = \int_\Omega \left( \frac{1}{m(x)} |\nabla u|^{m(x)} - \frac{1}{p(x)} |u|^{p(x)} \right) \, dx.
\]
Clearly, we have
\[
E'(t) = -\int_\Omega u_t^2 \, dx \leq 0.
\]
Let $H(t) = -E(t)$. So, $H'(t) \geq 0$. 
Theorem 3.2. Let $u_0 \in W^{1,m(\cdot)}_0(\Omega)$ such that $\int_\Omega u_0^2 \, dx > 0$, $f \equiv 0$ and
\[
\int_\Omega \left( \frac{1}{p(x)} |u_0|^{p(x)} - \frac{1}{m(x)} |\nabla u_0|^{m(x)} \right) \, dx \geq 0.
\]
Then
\[
F(t) = \frac{1}{2} \int_\Omega u^2(x, t) \, dx
\]
blows up in finite time $t^* < +\infty$.

**Proof.** By differentiating $F$, we get
\[
F'(t) = \int_\Omega uu_t \, dx = \int_\Omega u (\text{div}(|\nabla u|^{m(x)} - 2 \nabla u) + |u|^{p(x)} - 2) \, dx
\]
\[
= \int_\Omega \left( |u|^{p(x)} - |\nabla u|^{m(x)} \right) \, dx
\]
\[
= \int_\Omega m_+ \left( \frac{1}{m_+} - \frac{1}{p(x)} \right) |u|^{p(x)} \, dx + \int_\Omega m_- \left( \frac{1}{p(x)} |u|^{p(x)} - \frac{1}{m_-} |\nabla u|^{m(x)} \right) \, dx
\]
\[
\geq \int_\Omega m_+ \left( \frac{1}{m_+} - \frac{1}{p(x)} \right) |u|^{p(x)} \, dx
\]
\[
\geq \int_\Omega m_+ \left( \frac{1}{m_+} - \frac{1}{p(x)} \right) |u|^{p(x)} \, dx = c_0 \int_\Omega |u|^{p(x)} \, dx
\]
since
\[
\int_\Omega m_+ \left( \frac{1}{p(x)} |u|^{p(x)} - \frac{1}{m_+} |\nabla u|^{m(x)} \right) \, dx \geq \int_\Omega m_+ \left( \frac{1}{p(x)} |u|^{p(x)} - \frac{1}{m_+} |\nabla u|^{m(x)} \right) \, dx
\]
\[
= m_+ \int_\Omega \left( \frac{1}{p(x)} |u|^{p(x)} - \frac{1}{m_+} |\nabla u|^{m(x)} \right) \, dx \geq 0.
\]
We define the sets $\Omega_+ = \{ x \in \Omega ||u| \geq 1 \}$ and $\Omega_- = \{ x \in \Omega ||u| < 1 \}$.

So,
\[
F' (t) \geq c_0 \left( \int_{\Omega_-} |u|^{p_+} + \int_{\Omega_+} |u|^{p_-} \right) \geq c_1 \left( \left( \int_{\Omega_-} |u|^2 \, dx \right)^{p_+/2} + \left( \int_{\Omega_+} |u|^2 \right)^{p_-/2} \right),
\]
using the fact that $||u||_2 \leq C ||u||_q$, for all $q \geq 2$.

This implies that $F'(t) \geq c_1 \left( \int_{\Omega_-} |u|^2 \, dx \right)^{p_+/2}$ which gives, in turn,
\[
(F'(t))^{2/p_+} \geq c_2 \int_{\Omega_-} |u|^2 \, dx
\]
and similarly,
\[
(F'(t))^{2/p_-} \geq c_3 \int_{\Omega_-} |u|^2 \, dx.
\]
Simple addition leads to
\[
(F'(t))^{2/p_-} + (F'(t))^{2/p_+} \geq C_4 F(t), \quad \forall t \geq 0
\]
(3.1)
or
\[
(F'(t))^{2/p_-} \left( 1 + (F'(t))^{2 \left( \frac{1}{p_-} - \frac{1}{p_+} \right)} \right) \geq C_4 F(t), \quad \forall t \geq 0.
\]
(3.2)
By (3.1) and the fact that $F(t) \geq F(0) > 0$, $(F'(t) \geq 0)$, we have, for each $t > 0$, either
\[
(F'(t))^{2/p_-} \geq \frac{C_4}{2} F(t) \geq \frac{C_4}{2} F(0)
\]
or
\[
(F'(t))^{2/p_+} \geq \frac{C_4}{2} F(t) \geq \frac{C_4}{2} F(0);
\]
which gives, in turn,
\[ F'(t) \geq C_5(F(0))^{p-2} / 2 \]

or
\[ F'(t) \geq C_6(F(0))^{p+2} / 2. \]

Hence \( F'(t) \geq \alpha \), where \( \alpha = \min\{C_5(F(0))^{p-2}, C_6(F(0))^{p+2}\} \).

Since \( \frac{1}{p+} - \frac{1}{p-} \leq 0 \), (3.2) yields
\[ F'(t)^{2/p-}(1 + \alpha)^2(\frac{1}{p+} - \frac{1}{p-}) \geq C_4F(t), \quad \forall t \geq 0. \]

Consequently,
\[ F'(t) \geq \beta F^{p-2}(t), \quad \forall t \geq 0. \]

Simple integrating then leads to
\[ F(t)^{1/p-} \leq F(0)^{1/p-} - \frac{p-2}{p} \beta t \]

which implies that
\[ F(t) \geq \frac{1}{\left(F(0)^{1/p-} - \frac{p-2}{p} \beta t\right)^{p-2}}. \]

This shows that \( F \) blows up in a time
\[ t^* \leq \frac{2F(0)^{1/p-}}{(p-2)\beta}. \]

**Remark 3.1.** Clearly, the larger the \( F(0) \) is, the quicker the \( F \) blows up.

### 4. Numerical study

In this section, we present a 2D numerical study of the problem (P).

\[
\begin{align*}
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p(x,y)-2}\nabla u) &= |u|^{p(x,y)-2}u & \text{in } Q = \Omega \times (0, T), \\
u &= u = 0 & \text{on } \partial Q = \partial \Omega \times [0, T], \\
u(x,y,0) &= u_0(x,y) & \text{in } \Omega,
\end{align*}
\]

where \( u \) represents the velocity of a fluid in a domain \( \Omega \).

Our objective is to illustrate numerically the blow up result obtained above. We, first, start by considering the proposed numerical scheme, prove its convergence and then give an application by the end. The procedure is as follows:

1. Let us consider the linearized problem
\[
\begin{align*}
\frac{\partial u}{\partial t}^{n+1} - \text{div}(|\nabla u^n|^{m-2}\nabla u^n) &= |u^n|^{p-2}u^n & \text{in } \Omega, \\
u^n + 1 &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( t_0 = 0, u^n(x) = u(x,0), n \in \mathbb{N}, \tau = \frac{T}{n}, t_n = n \tau, u^n = u^n(x,y) = u(x,y,t_n) \) and \( \frac{\partial u^{n+1}}{\partial t} = \frac{u^{n+1} - u^n}{\tau} \).

2. The linearized problem, which is an elliptic problem, admits a solution \( u^{n+1} \) (see [5]).

3. Let us consider the well-known Röthe type approximation of the problem which consists in the following:

   for every \( i = 1, \ldots, n \), let \( u^i \) be the solution of the linear difference equation

   \[
   u^i - u^{i-1} - \tau \text{div}(|\nabla u^{i-1}|^{m-2}\nabla u^{i-1}) = |u^{i-1}|^{p-2}u^{i-1}.
   \]

   We define the Röthe function by

   \[
   u^{(0)}(t) = u^{i-1} + (t - t_{i-1}) \frac{u^i - u^{i-1}}{\tau}, \quad \text{for } t_{i-1} \leq t \leq t_i, i = 1, \ldots, n.
   \]
Fig. 1. Mesh of the domain.

Fig. 2. The exponent functions $m(x, y)$ (down) and $p(x, y)$ (up).

Fig. 3. Initial fluid’s velocity ($t = 0$).

Table 1

Evolution of $F(t)$ in time.

<table>
<thead>
<tr>
<th>Time</th>
<th>$F(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>17.8886</td>
</tr>
<tr>
<td>0.01</td>
<td>17.8525</td>
</tr>
<tr>
<td>0.02</td>
<td>17.8283</td>
</tr>
<tr>
<td>0.05</td>
<td>17.9835</td>
</tr>
<tr>
<td>0.1</td>
<td>18.4795</td>
</tr>
<tr>
<td>0.15</td>
<td>19.1978</td>
</tr>
<tr>
<td>0.2</td>
<td>20.1746</td>
</tr>
<tr>
<td>0.25</td>
<td>21.5211</td>
</tr>
<tr>
<td>0.3</td>
<td>23.4907</td>
</tr>
<tr>
<td>0.35</td>
<td>26.8085</td>
</tr>
<tr>
<td>0.4</td>
<td>36.4489</td>
</tr>
<tr>
<td>0.43</td>
<td>$6.68 \times 10^9$</td>
</tr>
</tbody>
</table>
4. Any solution $u^{n+1}$ of the linearized scheme coincides with the Röthe approximation $u^{(n)}(t)$ on $\{t_i = i\tau / i = 1, \ldots, n\}$ (it is easy to verify it).

5. The Röthe approximation $u^{(n)}(t)$ converges strongly to the solution of Problem (P) as $\tau$ goes to zero. Indeed, it is easy to prove that $u^{(n)}$ satisfies:

$$u^{(n)} \to u, \text{ a.e.}$$

$(u^{(n)})$ is bounded in $W_0^{1,m}()$ (since $u \in W_0^{1,m}(\Omega)$).

Fig. 4. Evolution of the fluid’s velocity in time.
Consequently by Proposition 3 of [3], we have \((u^{(n)})\) converges strongly to \(u\) in \(L^p(\Omega)\). Hence, \((u^{(n)})\) is an approximation of the solution \(u\) of the problem (P).

6. The proposed linearized problem admits a solution that converges to the solution of the problem (P).

We use the above linearized scheme with \(T = 5, \tau = 0.01\). Our scheme consists of solving the following sequence of problems:

Given \(u^0\) as an initial data, find the solution \(u^{n+1}\) of

\[
\int_{\Omega} \left[ \frac{u^{n+1} - u^n}{\tau} v + |\nabla u^n|^{p(x,y)-2} \nabla u^{n+1} \cdot \nabla v - |u^n|^{p(x,y)-2} u^n v \right] \, dx \, dy = 0.
\]

(4.1)
We present the full discretized problem. The numerical simulation is carried out using the FreeFem++ software [16]. We use a $P_1$ Lagrangian finite element type in space with the above linearized scheme in time. Let $V_h$ be the corresponding space of piecewise linear continuous functions where $h$ is the space step of the mesh representing the domain $\Omega$ (here $\Omega$ is a square $[-5, 5] \times [-5, 5]$ discretized with 94,440 triangles and 47,621 vertices (see Fig. 1)).

The full discretized problem reads as: find $u_{n+1}^\omega \in V_h$ such that for all $\omega \in V_h$

$$\int_{\Omega} \left[ \frac{u_{n+1}^\omega - u_n^\omega}{\Delta t} \omega + |\nabla u_n^\omega|^{m(x,y)-2} \nabla u_n^{n+1} \nabla \omega - |u_n^\omega|^{p(x,y)-2} u_n^\omega \omega \right] dx dy = 0. \quad (4.2)$$
We choose
\[ f = 0, \quad m(x, y) = |1/5x| + 1.5, \quad p(x, y) = |x| + 4 \quad \text{(see Fig. 2)} \]
and
\[ u_0(x, y) = \exp(-0.5(x^2 + y^2)) \quad \text{(see Fig. 3)}. \]

As required in Section 1, functions \( m \) and \( p \) must verify conditions (1.1)–(1.3) as follows:
- \( m_-(x, y) = 2.5, m_+(x, y) = 3.5, \) \( p_-(x, y) = 4, \) \( p_+(x, y) = 29 \) and \( m_+(x, y) = +\infty \) such that \( 2 \leq m_- \leq m(x, y) \leq m_+ < p_- \leq p(x, y) \leq p_+ < m_+(x, y) \)
- \( |m(x, y) - m(x_0, y_0)| = 1/25(|x|^2 - |x_0|^2) \)
- essential infimum \( m_+(x, y) - p(x, y) = +\infty > 0. \)

The condition of the blow up given in Theorem 3.2 is verified:
\[ \int_{\Omega} \left( \frac{1}{p(x, y)}|u_0|^{p(x, y)} - \frac{1}{m(x, y)}|\nabla u_0|^{m(x, y)} \right) dxdy = 2.75941 > 0. \]

In Table 1, the evolution of the norm of the fluid velocity \( F(t) = \frac{1}{2} \int_{\Omega} u^2(x, y, t) dxdy \) in time is shown: an estimation of the time blow up is given by the relation
\[ t^* \leq 2F(0)^{1/\beta}, \quad \text{where } F(0) = 17.886, \ p_- = 4 \text{ and } \beta \text{ is a constant} \]
which gives
\[ t^* \leq \frac{1}{\beta F(0)} = \frac{17.886}{\beta} \quad \text{where } \beta \text{ is a constant}. \]

We note that the blow up occurs at \( t = 0.43 \) as shown in Fig. 4.

Remark 4.1. In the last inequality, the constant \( \beta \) can be estimated since all the constants \( (C_i, i = 0, \ldots, 4) \) in the proof of Theorem 3.2 are known. So, an estimation of \( t^* \) can be obtained.

Remark 4.2. One can use the following numerical scheme: given \( U^0 \) an initial velocity of the domain \( \Omega_0 \), find the solution \( U^{n+1} \) of
\[ \int_{\Omega} \left[ \frac{U^{n+1} - U^n}{\Delta t}u + |\nabla U^n|^{m(x, y)} - 2\nabla U^{n+1}\nabla U - |U^n|^{p(x, y)} - 2U^{n+1}u \right] dxdy = 0, \quad \text{(4.3)} \]
which reaches the blowing-up time at \( t = 0.4 \).

Acknowledgment

The second author is sponsored by KFUPM under project IN131028.

References