INDIRECT FUZZY ADAPTIVE CONTROL OF A CLASS OF SISO NONLINEAR SYSTEMS

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ABSTRACT

This paper presents an adaptive fuzzy control scheme for a class of continuous-time single-input single-output nonlinear systems with unknown dynamics and disturbance. Within this scheme, the fuzzy systems are employed to approximate the unknown system’s dynamics. The proposed controller is composed of a well-defined adaptive fuzzy control term that uses the adaptive fuzzy approximators and a robustifying control term that is used to accommodate the approximation errors and disturbance. Based on a Lyapunov synthesis method, it is shown that the proposed adaptive control scheme guarantees the convergence of the tracking error to zero and the global boundedness of all signals in the closed-loop system. Moreover, The proposed controller allows initialization by zero of all adjusted parameters in the fuzzy approximators, and does not require the knowledge of the lower bound of the control gain and the upper bounds of approximation errors and disturbance. Simulation results performed on an inverted pendulum system are given to point out the good performances of the developed adaptive controller.

Key Words: Fuzzy control, fuzzy systems, adaptive control, SISO nonlinear systems.
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1. INTRODUCTION

Adaptive nonlinear control problems have been given a lot of attention in control community during the last decade. Detailed discussions can be found in [1–3] and references therein. In such adaptive control schemes, it is assumed that an accurate model of the plant is available, and the unknown parameters are assumed to appear linearly with respect to known nonlinear functions. If the plant is complex, ill-defined, and uncertain, the design of an adaptive controller becomes a truly formidable problem.

The past few years have witnessed a rapid growth in the use of fuzzy logic controllers for plants that are complex and ill-defined. In most applications of fuzzy logic controllers, the rule base of the fuzzy controller is constructed from expert knowledge. It is sometimes difficult to build the rule base of some systems, or the need may arise to tune the controller parameters if the plant dynamics change. In an attempt to overcome this problem, researchers have introduced adaptive control techniques. In such techniques, the functional approximation capability and on-line learning ability of fuzzy systems are exploited. The stability study in such schemes is performed by using the Lyapunov design approach.

There are two distinct approaches that have been formulated in the design of a fuzzy adaptive control system: direct and indirect schemes. In the direct method, a fuzzy system is used to describe the control action and the parameters of the fuzzy system are adjusted directly to meet the required control objective [4–7]. In contrast with the direct adaptive scheme, the indirect adaptive approach uses fuzzy systems to estimate the plant dynamics and then synthesizes a control law based on these estimates [4, 6, 8–11]. In [7], authors develop a fuzzy sliding mode controller for dominated linear systems with unmodeled dynamics and unknown disturbances. In [8] and [9], authors develop their indirect adaptive fuzzy controller for SISO uncertain nonlinear systems with known constant control gain. In [4, 6, and 10], they developed a stable indirect adaptive controller for affine SISO nonlinear systems with state dependent control gain. To keep the estimate of the control gain bounded away from zero, authors in [4] and [6] propose the use of a parameter projection algorithm. Notice that a projection algorithm needs at least the knowledge of the lower bound of the control gain and the knowledge of the region to which the desired values of the adjusted parameters belong. In practice, the design of such a region is not a trivial task since these desired values are unknown. On the other hand, in [10], to avoid the possible controller singularity problem, they estimate directly the inverse of the control gain. However, this approach requires the boundedness of the time derivative of the control gain by a known state dependent function. In [11], the possible controller singularity problem is avoided by using an approximation of the inverse of the estimated control gain. However, this scheme requires the knowledge of the lower bound of the control gain and the upper bounds of the approximation errors. Actually, such bounds are not easily known in a practical control system. An extension of the scheme of [11] to a class of multi-input multi-output nonlinear systems is given in [12].

In this paper we introduce a new indirect adaptive controller for SISO nonlinear systems with unknown dynamics and disturbance. We use standard fuzzy systems to estimate on-line the plant dynamics. The control law is synthesized based on these estimates and a robustifying control term is added to cancel out the effect of approximation errors and disturbance. Adaptive laws to adjust the fuzzy systems parameters and methods to deal with reconstruction errors and disturbance problems are derived using a Lyapunov function. The proposed control scheme avoids the possible controller singularity problem, does not need the knowledge of some bounding parameter values, guarantees the boundedness of all the states and the signals of the closed-loop system, and the convergence of the tracking error to zero.

This paper is organized as follows. A class of SISO nonlinear systems and control objectives are described in section 2. Section 3 reviews fuzzy systems and their useful properties. The proposed adaptive control scheme is introduced in Section 4 with stability analysis of the overall system. Simulation results for the proposed control scheme are shown in Section 5. Section 6 concludes this paper.

2. PROBLEM FORMULATION

This paper focuses on the design of an indirect adaptive fuzzy controller for a class of single-input single-output (SISO) nonlinear dynamic systems whose equation of motion can be expressed by the following set of differential equations:

\[ \dot{x}_i = x_{i+1}, \quad i = 1, \ldots, n-1 \]
\[ \dot{x}_n = f(x) + g(x)u + d \]
\[ y = x_1 \]  

(1)
where \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the overall state vector which is assumed available for measurement, \( u \in \mathbb{R} \) is the scalar control input, \( y \in \mathbb{R} \) is the scalar system output, \( d \) denotes the external disturbance, \( f(x) \) and \( g(x) \) are unknown smooth nonlinear functions.

The objective of this paper is to design a control law \( u(t) \) which ensures the boundedness of all the variables of the closed-loop system and output tracking of a given desired trajectory \( y_d(t) \).

Throughout this paper, the following assumptions are made.

**Assumption 1**

The order \( n \) of the system is known.

**Assumption 2**

The control gain \( g(x) \) is strictly positive and globally bounded away from zero by an unknown constant \( g \), i.e., \( g(x) \geq g > 0 \), for all \( x \).

Note that the control gain can be assumed strictly negative, and the controller can be similarly derived.

**Assumption 3**

Disturbance \( d \) is bounded, i.e. \( |d| \leq \overline{d} \), where \( \overline{d} \) is an unknown positive constant.

**Assumption 4**

The desired trajectory \( y_d(t) \) is a known bounded function of time with bounded known derivatives, and \( y_d(t) \) is assumed to be \( n \)-times differentiable.

Let us define the tracking error as

\[
e(t) = y_d(t) - y(t)
\]  

Define also an error metric as

\[
s(t) = \left( \frac{d}{dt} + \lambda \right)^{n-1} e(t), \quad \lambda > 0.
\]  

The equation \( s(t) = 0 \) represents a linear differential equation whose solution implies that \( e(t) \) decays exponentially to zero [1]. Thus, perfect tracking can be asymptotically obtained by maintaining this condition.

The time derivative of the error metric can be written as

\[
\dot{s} = s_r - f(x) - g(x)u - d
\]  

where \( s_r \) is given as follows

\[
s_r = y^{(n)}_d + \alpha_{n-1} e^{(n-1)} + \cdots + \alpha_1 \dot{e}
\]  

where the coefficients \( \alpha_j \) are given by

\[
\alpha_j = \frac{(n-1)!}{(n-j)!(j-1)!} \lambda^{n-j}, \quad j = 1, \ldots, n-1.
\]  

When the nonlinear functions \( f(x) \) and \( g(x) \) are known exactly, and the system has no disturbance \( (d = 0) \), to meet the control objectives, we propose the use of the following control law

\[
u = \frac{1}{g(x)} \left( -f(x) + s_r + \alpha g(x) s \right)
\]
where $\alpha$ is a positive design parameter.

Substituting (7) into (4) we can obtain

$$\dot{s} = -\alpha g(x)s.$$  \hspace{1cm} (8)

Let us consider the Lyapunov function candidate

$$V = \frac{1}{2}s^2.$$  \hspace{1cm} (9)

Then

$$\dot{V} = s\dot{s} = -\alpha g(x)s^2.$$  \hspace{1cm} (10)

Using the fact that $g(x) \geq g$, one has

$$\dot{V} \leq -\alpha g s^2.$$  \hspace{1cm} (11)

Thus, we can conclude [1] that $s(t) \to 0$ as $t \to \infty$, and therefore $e^{(i)}(t) \to 0$ as $t \to \infty$ for $i = 0, 1, \ldots, n-1$.

From Equation (11), it is shown that the larger the parameter $\alpha$ is, the more negative $\dot{V}$ will be. Hence, the convergence rate of the tracking error can be adjusted by tuning the design parameter $\alpha$.

In this study, we assume that the nonlinear functions $f(x)$ and $g(x)$ are unknown and the system is perturbed, i.e., $\delta \neq 0$, so obtaining control law (7) is impossible. In this situation, our purpose is to use fuzzy systems to approximate the unknown functions and, based on these fuzzy approximations, we develop a robust adaptive controller to meet control objectives.

3. FUZZY SYSTEMS

Mathematically, a fuzzy logic system is a function mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this paper we consider multi-input single-output (MISO) fuzzy logic systems mapping from an input vector $x = [x_1, \ldots, x_n]^T \in U \subseteq \mathbb{R}^n$ to an output $y \in \mathbb{R}$. Let $F_{ik}^1$, $k_i = 1, \ldots, p_i$, be the fuzzy sets defined on the $i$th input. The fuzzy logic system is characterized by a set of simplified if–then rules expressed in the following form [13]:

$$R^k : \text{If } x_1 \text{ is } F_{i1}^k \text{ and } \ldots \text{ and } x_n \text{ is } F_{in}^k \text{ Then } y \text{ is } y^k \text{ (} k = 1, \ldots, P)$$

where $P = \prod_{i=1}^n p_i$ is the total number of rules, and $y^k$ is the crisp output for the $k$th rule.

The final output of the fuzzy system is calculated as follows [13]:

$$y(x) = \frac{\sum_{k=1}^P \mu_k(x)y^k}{\sum_{k=1}^P \mu_k(x)}$$  \hspace{1cm} (13)

where

$$\mu_k(x) = \prod_{i=1}^n \mu_{k_i}^i(x_i), \quad k_i \in \{1, 2, \ldots, p_i\}$$

where $\mu_{k_i}^i(x_i)$ is the membership function of the fuzzy set $F_{ik}^i$.

The output given by (13) can be rewritten as

$$y(x) = w^T(x)\theta$$  \hspace{1cm} (14)

where $\theta = [y^1, \ldots, y^P]^T$ is a parameter vector, and $w(x) = [w_1(x), \ldots, w_P(x)]^T$ is a set of fuzzy basis functions defined as
It is assumed that the fuzzy system is constructed in such a way that (15) is well defined, i.e., \( \sum_{i=1}^{P} \mu_{i}(x) \neq 0 \) for all \( x \in U \).

It has been shown in [4, 13, 14] that fuzzy systems in the form of (14) can approximate continuous functions to an arbitrary degree of accuracy provided that a large enough number of rules are considered.

Let \( h(x) \) be a general smooth function defined from \( \mathbb{R}^{n} \) to \( \mathbb{R} \). There exists a fuzzy system in the form of (14) with some ideal parameters \( \theta^{*} \) such that

\[
\| w^T(x) \theta^{*} - h(x) \| \leq \varepsilon
\]

where \( \Omega_{y} \subseteq U \subseteq \mathbb{R}^{n} \) is a compact set and \( \varepsilon \) is a positive constant. Thus, we can express \( h(x) \) as follows

\[
h(x) = w^T(x) \theta^{*} + e(x)
\]

with \( |e(x)| \leq \varepsilon \) when \( x \in \Omega_{y} \).

The estimate of \( h(x) \) is given by

\[
\hat{h}(x) = w^T(x) \theta
\]

where \( \theta \) are estimates of the ideal fuzzy system parameters \( \theta^{*} \).

In this paper, we assume that the membership functions \( \mu_{i}(x) \) are fixed and known and have the shape which does not violate the universal approximation property.

4. ADAPTIVE CONTROL WITH FUZZY SYSTEMS

The control design method presented in Section 2 is useful only if \( f(x) \) and \( g(x) \) are known exactly and the system has no disturbance. If \( f(x) \) and \( g(x) \) are unknown and \( d \neq 0 \), then adaptive approaches must be employed. Let us now discuss a fuzzy-system based adaptive control scheme.

Assume that the fuzzy system described in Section 3 can approximate the nonlinear functions \( f(x) \) and \( g(x) \) as the following

\[
f(x) = w_{f}^T(x) \theta_{f}^{*} + e_{f}(x)
\]

\[
g(x) = w_{g}^T(x) \theta_{g}^{*} + e_{g}(x)
\]

where \( e_{f}(x) \) and \( e_{g}(x) \) are fuzzy approximation errors, \( \theta_{f}^{*} \) and \( \theta_{g}^{*} \) are ideal parameter vectors which minimize the functions \( |e_{f}(x)| \) and \( |e_{g}(x)| \), respectively, and \( w_{f}(x) \) and \( w_{g}(x) \) are fuzzy basis function vectors.

Assumption 5

The fuzzy reconstruction errors \( e_{f}(x) \) and \( e_{g}(x) \) are bounded, i.e., \( |e_{f}(x)| \leq \varepsilon_{f} \) and \( |e_{g}(x)| \leq \varepsilon_{g} \), where \( \varepsilon_{f} \) and \( \varepsilon_{g} \) are unknown constants.

Assumption 6

The ideal parameters are bounded by known positive values, i.e., \( \| \theta_{f}^{*} \| \leq M_{f} \) and \( \| \theta_{g}^{*} \| \leq M_{g} \), where \( M_{f} \) and \( M_{g} \) are given constants.
Since the ideal parameter vectors $\theta_f^*$ and $\theta_g^*$ are unknown, so they should be estimated by adaptation laws. The estimates are represented as $\hat{\theta}_f$ and $\hat{\theta}_g$, respectively. Let us define the parameter error vectors as
\begin{align*}
\hat{\theta}_f &= \theta_f^* - \theta_f \\
\hat{\theta}_g &= \theta_g^* - \theta_g
\end{align*}
(21)

Let the fuzzy system approximation of $f(x)$ and $g(x)$ of the actual system be
\begin{align*}
\hat{f}(x) &= w_f^T(x)\theta_f \\
\hat{g}(x) &= w_g^T(x)\theta_g
\end{align*}
(22)

Based on the above approximations and the control law given by (7), we propose the following adaptive control law
\[ u = u_{ad} + u_r. \] 
(25)

The control law (25) is a summation of an adaptive control term, $u_{ad}$, which attempts to approximate the control law (7) by using the estimates of the system’s nonlinearities, and a robustifying control term, $u_r$, which is introduced to compensate for reconstruction errors.

The adaptive control term $u_{ad}$ is given by
\[ u_{ad} = \frac{\hat{g}(x)}{\varepsilon_0 + \hat{g}^2(x)}(-\hat{f}(x) + s_x + \alpha \hat{g}(x)s) \] 
(26)
where $\varepsilon_0$ is a small positive constant.

**Remark 1**

To force the adaptive control term (26) to be well-defined even when $\hat{g}(x)$ goes to zero, we have replaced $\hat{g}(x)$ by $\hat{g}(x)/(\varepsilon_0 + \hat{g}^2(x))$, which can be considered as the Levenberg–Marquard regularized inverse [15] applied to a scalar function.

The robustifying control term is given as follows
\[ u_r = \frac{\psi s}{|\psi| + \delta^2 \exp(-\psi)} \] 
(27)
where
\[ \psi = \varepsilon + \hat{\varepsilon}_g|u_{ad} - \alpha s| + \hat{\varepsilon}_u|u_0| \] 
(28)
\[ u_0 = \frac{\varepsilon_0}{\varepsilon_0 + \hat{g}^2(x)}(-\hat{f}(x) + s_x + \alpha \hat{g}(x)s) \] 
(29)

and $\hat{\varepsilon}_g$, $\hat{\varepsilon}_u$, and $\hat{\varepsilon}_{\psi}$ are estimates of the unknown parameters $\varepsilon^* = (\varepsilon_f + \delta)/g$, $\varepsilon_g^* = \varepsilon_g/g$, and $\varepsilon_{\psi}^* = 1/g$, respectively, and $\delta^*$ is a time varying design parameter.

The parameter adaptation laws are chosen as
\begin{align*}
\dot{\hat{\theta}}_f &= -\eta_f w_f(x)s - \Phi_f \\
\dot{\hat{\theta}}_g &= -\eta_g w_g(x)(u_{ad} - \alpha s) - \Phi_g \\
\dot{\varepsilon} &= \eta_\varepsilon |s| 
\end{align*}
(30)

(31)

(32)
\[ \dot{\theta}_f = \eta_f \| u_{ad} - \alpha s \| \]  
\[ \dot{\theta}_e = \eta_e \| u_e \| \]  
\[ \delta = -\eta \delta \]  
\[ \Phi_f = \begin{cases} 
0 & \text{if } \| \theta_f \| < M_f \\
\eta_f \rho \frac{w_f'(x) \theta_f s}{\theta_f^T \theta_f} & \text{otherwise} 
\end{cases} \]  
\[ \Phi_e = \begin{cases} 
0 & \text{if } \| \theta_e \| < M_e \\
\eta_e \rho \frac{w_e'(x) \theta_e (u_{ad} - \alpha s)}{\theta_e^T \theta_e} & \text{otherwise} 
\end{cases} \]

where \( \eta_f > 0, \eta_e > 0, \eta_\delta > 0, \delta(0) > 0, \Phi_f \) and \( \Phi_e \) are defined as follows.

Then we can prove the following theorem.

**Theorem 1**

Consider the system (1). If the Assumptions 1-6 hold, and the control law is defined by (25)–(29) with adaptation laws given by (30)–(37), then the following properties are guaranteed:

1. The estimates of the parameters are bounded and satisfy: \( \| \theta_f \| \leq M_f \) and \( \| \theta_e \| \leq M_e \).
2. The state variables and the control input are bounded, i.e., \( x, u \in L_\infty \).
3. The tracking error and its derivatives decrease at least asymptotically to zero, i.e., \( e^{(i)}(t) \to 0 \) as \( t \to \infty \) for \( i = 0,1,\ldots,n-1 \).

**Proof**

To prove that \( \| \theta_f \| \leq M_f \), let us consider the following function

\[ V_f = \frac{1}{2} \theta_f^T \theta_f \]  
\[ \dot{V}_f = \theta_f^T \dot{\theta}_f \]  

Employing the adaptation law (30) we obtain

\[ \dot{V}_f = -\eta_f \theta_f^T w_f(x) s - \theta_f^T \Phi_f \]  

For the case \( \| \theta_f \| \geq M_f \) and using (36) one can obtain

\[ \dot{V}_f = -\eta_f \theta_f^T w_f(x) s - \eta_e \rho \theta_e^T w_f(x) s \]

which can be simplified to

\[ \dot{V}_f \leq -\eta_f (\rho_0 - 1) \theta_f^T w_f(x) s \]
Because $\rho_b \geq 1$ by definition, thus, $\dot{V}_f \leq 0$ and one concludes that $\|\theta_f\| \leq M_f$ must be satisfied for all time if we select $\|\theta_f(0)\| < M_f$. In the same way, one can prove that $\|\theta_g\| \leq M_g$.

Using the control law (25), Equation (4) can be written as

$$\dot{s} = s_r - f(x) - g(x)u_{wd} - g(x)u_r - d$$

which can be rewritten as

$$\dot{s} = s_r - f(x) - (g(x) - \dot{g}(x))u_{wd} - \dot{g}(x)u_r - g(x)u_r - d.$$

Using (26), we obtain

$$\dot{s} = -\left(f(x) - \dot{f}(x)\right) - (g(x) - \dot{g}(x))u_{wd} + u_0 - g(x)u_r - \alpha \dot{g}(x)s - d.$$

Adding and subtracting $\alpha g(x)s$ in (45), we can write

$$\dot{s} = -\alpha g(x)s - \left(f(x) - \dot{f}(x)\right) - (g(x) - \dot{g}(x))u_{wd} + \alpha (g(x) - \dot{g}(x))s + u_0 - g(x)u_r - d.$$

With (19)–(24), (46) becomes

$$\dot{s} = -\alpha g(x)s - w_f^T(x)\tilde{\theta}_f - w_g^T(x)\tilde{\theta}_g (u_{wd} - \alpha s) + u_0 - \varepsilon_f(x) - \varepsilon_g(x)(u_{wd} - \alpha s) - d - g(x)u_r.$$

Now, to prove the rest of Theorem 1, consider the Lyapunov function candidate

$$V = \frac{1}{2}s^2 + \frac{1}{2\eta_f} \tilde{\theta}_f^T \tilde{\theta}_f + \frac{1}{2\eta_g} \tilde{\theta}_g^T \tilde{\theta}_g + \frac{g}{2\eta_0} \tilde{e}^T \tilde{e} + \frac{g}{2\eta_0} \tilde{e}_g^T \tilde{e}_g + \frac{g}{2\eta_0} \tilde{e}_u^T \tilde{e}_u + \frac{g}{2\eta_0} \tilde{e}^T \tilde{e}.$$

where $\tilde{e} = e^* - \hat{e}$, $\tilde{e}_g = e_g^* - \hat{e}_g$, and $\tilde{e}_u = e_u^* - \hat{e}_u$.

The time derivative of (48) is

$$\dot{V} = s \dot{s} - \frac{1}{\eta_f} \tilde{\theta}_f^T \tilde{\theta}_f - \frac{1}{\eta_g} \tilde{\theta}_g^T \tilde{\theta}_g - \frac{g}{\eta_0} \tilde{e} \ddot{e} - \frac{g}{\eta_0} \tilde{e}_g \ddot{e}_g - \frac{g}{\eta_0} \tilde{e}_u \ddot{e}_u + \frac{g}{\eta_0} \ddot{e} \delta \dot{\delta}.$$

With (47), (49) becomes

$$\dot{V} = -\alpha g(x)s^2 + \dot{V}_1 + \dot{V}_2$$

where

$$\dot{V}_1 = -w_f^T(x)\tilde{\theta}_f s - w_g^T(x)\tilde{\theta}_g (u_{wd} - \alpha s) - \frac{1}{\eta_f} \tilde{\theta}_f^T \tilde{\theta}_f - \frac{1}{\eta_g} \tilde{\theta}_g^T \tilde{\theta}_g$$

$$\dot{V}_2 = -g(x)u_r + su_0 - se_f(x) - se_g(x)(u_{wd} - \alpha s) - sd - \frac{g}{\eta_0} \tilde{e} \ddot{e} - \frac{g}{\eta_0} \tilde{e}_g \ddot{e}_g - \frac{g}{\eta_0} \tilde{e}_u \ddot{e}_u + \frac{g}{\eta_0} \ddot{e} \delta \dot{\delta}.$$

Using (30) and (31), (51) becomes

$$\dot{V}_1 = \tilde{\theta}_f^T \Phi_f + \tilde{\theta}_g^T \Phi_g$$

Let us now prove that $\tilde{\theta}_f^T \Phi_f \leq 0$. If $\|\theta_f\| < M_f$, $\|\Phi_f\| = 0$, the conclusion is trivial. For $\|\theta_f\| \geq M_f$, since $\|\theta_f\| \leq M_f$, one has

$$2\tilde{\theta}_f^T \Phi_f = \|\theta_f\|^2 - \|\theta_f\|^2 - \|\theta_f - \theta_f\|^2 \leq 0.$$
\[ \hat{\theta}_g^T \Phi_g = \hat{\theta}_g^T \frac{w^\top(x) \theta_g s}{\theta_g^T \theta_g} \leq 0. \]

In the same way, we can prove that
\[ \hat{\theta}_g^T \Phi_g \leq 0. \]

The above results imply that
\[ V_2 \leq 0. \] (54)

Equation (52) can be bounded as
\[ \dot{V}_1 \leq -sg(x)u_s + \frac{\theta_g^T}{\eta_0} \left[ \left| \varepsilon^* \right| u_s + \varepsilon^* \left| u_{sd} - \alpha s \right| \right] - \frac{g}{\eta_0} \dot{\hat{e}}_g - \frac{g}{\eta_0} \dot{\hat{e}}_g - \frac{g}{\eta_0} \dot{\hat{e}}_u + \frac{g}{\eta_0} \delta \dot{\delta}. \] (55)

From (28), (29), (32), (33), and (34), (55) can be bounded as
\[ \dot{V}_2 \leq \frac{g}{\eta_0} \delta \psi \exp(-\psi) + \frac{g}{\eta_0} \delta \dot{\delta}. \] (56)

From (35) and using the fact that \( \psi \exp(-\psi) \leq 1 \), (56) can be reduced to
\[ \dot{V}_2 \leq 0. \] (57)

From (54), (57), and Assumption 2, (50) becomes
\[ \dot{V} \leq -\alpha g s^2. \] (58)

Hence, \( V \in L_\infty \), which implies that the signals \( s(t) \), \( \hat{\theta}_g(t) \), \( \hat{\theta}_e(t) \), \( \hat{e}_g(t) \), \( \hat{e}_u(t) \), and \( \delta(t) \) are bounded. This, in turn, implies the boundedness of \( x(t) \), \( u(t) \), and \( \dot{s}(t) \). Since \( V(t) \) is a non-increasing function of time and bounded from below, the limit \( \lim_{t \to \infty} V(t) = V(\infty) \) exists. By integrating (58) from 0 to \( \infty \), we have
\[ \int_0^\infty s^2(t)dt \leq \frac{V(0) - V(\infty)}{\alpha g} < \infty. \] (59)

which implies that \( s(t) \in L_2 \). Since \( s(t) \in L_2 \cap L_\infty \) and \( \dot{s}(t) \in L_\infty \), by using Barbalat’s lemma [1], we conclude that \( s(t) \to 0 \) as \( t \to \infty \). Therefore, the tracking errors and its derivatives converge asymptotically to zero [1], i.e., \( \varepsilon^{(i)}(t) \to 0 \) as \( t \to \infty \) for \( i = 0, 1, \ldots, n-1 \).

**Remark 2**

The control law and the adaptation laws used within this paper are different from those presented in the indirect schemes in [4] and [6], particularly by the presence of the term \( \alpha g(x)s \) in the control law (26) and the term \( \alpha s^2 \) in the adaptation law (31) of the parameters of \( g(x) \). In some sense, the adaptive schemes presented in [4] and [6] can be considered as a particular case of our approach when we set \( \alpha = 0 \). The additional term \( \alpha g(x)s \) is used to allow initialization by zero of the adjusted parameters \( \theta_g \) and, therefore, the designer requires no knowledge about the desired values of \( \theta_g \). In fact, if we select \( \alpha = 0 \) and \( \theta_g(0) = 0 \), we will have \( u_{sd}(t) = 0 \) and \( \theta_g(t) = 0 \) for all time. Note that initialization by zero of \( \theta_g \) is impossible within the indirect schemes of [4] and [6]. Furthermore, our approach does not impose any restrictive conditions about the time derivative of the control input as in [10].
Remark 3

Compared to the robust adaptive control scheme developed in [11], in addition to the difference of control laws and parameter adaptation laws, our approach does not require the following restrictive assumptions as in [11]:

1. The knowledge of the upper bounds of approximation errors, i.e. \( \mathcal{E}_f \) and \( \mathcal{E}_g \);
2. The knowledge of the lower bound of the control gain, i.e. \( g \);
3. The bounds \( g \) and \( \mathcal{E}_g \) verify the inequality: \( g > 3\mathcal{E}_g \).

Moreover, our approach can ensure the convergence of the tracking error to zero, while the approach of [11] can only ensure the convergence to a neighborhood of zero.

5. SIMULATION RESULTS

In this section, we test the proposed adaptive fuzzy control scheme on the tracking control of the benchmark control problem of the inverted pendulum system shown in Figure 1. The dynamic equations of such system are given by [4]

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = f(x) + g(x)u + d \\
y = x_1
\]  

with

\[
f(x) = \frac{g \sin x_1 - (m_t x_1^2 \cos x_1 \sin x_1) / (m_p + m_c)}{l(4/3 - m_p \cos^2 x_1 / (m_p + m_c))} \\
g(x) = \frac{\cos x_1 / (m_p + m_c)}{l(4/3 - m_p \cos^2 x_1 / (m_p + m_c))}
\]

where \( x_1 \) is the rotational movement, \( x_2 \) is the rotational velocity, \( g = 9.8 \text{m/s}^2 \) is the acceleration due to gravity, \( m_c \) is the mass of the cart, \( m_p \) is the mass of the pole, \( l \) is the half-length of the pole, and \( u \) is the applied force. Values \( m_c = 1 \text{kg}, \ m_p = 0.1 \text{kg} \), and \( l = 0.5 \text{m} \) are chosen in this example, and the disturbance is assumed to be \( d = 0.5 \sin(2t) \exp(-0.1t) \). The control objective is to force the output of the system to follow the desired trajectory \( y_d(t) = 0.5 \sin(t) \). Note that the desired trajectory is allowed a maximum swing of 0.5 rad, while it is limited to 0.1 rad in [4].

Two fuzzy systems in the form of (14) are used to approximate the functions \( f(x) \) and \( g(x) \). Both fuzzy systems have \( x_1(t) \) and \( x_2(t) \) as inputs. Each input \( x_i \) has five Gaussian membership functions defined as

\[
\mu_{\ell_{1}}^{(i)}(x_i) = \exp\left(-\frac{1}{2} \left( \frac{x_i - 0.8}{0.2} \right)^2 \right), \quad \mu_{\ell_{2}}^{(i)}(x_i) = \exp\left(-\frac{1}{2} \left( \frac{x_i + 0.4}{0.2} \right)^2 \right), \quad \mu_{\ell_{3}}^{(i)}(x_i) = \exp\left(-\frac{1}{2} \left( \frac{x_i}{0.2} \right)^2 \right), \\
\mu_{\ell_{4}}^{(i)}(x_i) = \exp\left(-\frac{1}{2} \left( \frac{x_i - 0.4}{0.2} \right)^2 \right), \quad \mu_{\ell_{5}}^{(i)}(x_i) = \exp\left(-\frac{1}{2} \left( \frac{x_i - 0.8}{0.2} \right)^2 \right).
\]

The initial values of the parameters \( \theta_f(t) \) and \( \theta_g(t) \) are set to zero. The design parameters used in this simulation are chosen as follows: \( \lambda = 10, \ \alpha = 5, \ \eta_f = 10, \ \eta_g = 1, \ \eta_0 = 0.001, \ \varepsilon_0 = 0.01, \ \rho_0 = 1.1, \ M_f = 20, \ M_g = 20, \ \hat{e}_f(0) = 0, \ \hat{e}_g(0) = 0, \ \hat{e}_u(0) = 0, \) and \( \delta(0) = 1. \)
The simulation results of the position $y = x_1$ and the velocity $\dot{y} = x_2$ are shown in Figure 2 and Figure 3, respectively. The control input signal $u(t)$ and the robustifying control term $u_r(t)$ are shown in Figure 4. We can find that actual trajectories converge to the desired trajectories. This simulation demonstrates the tracking capability of the proposed controller.

6. CONCLUSION

In this paper, an indirect adaptive fuzzy control scheme for a class of unknown and perturbed SISO nonlinear systems is developed. In the design, fuzzy logic systems are used for estimating the unknown dynamics of the system. The control scheme consists of an adaptive fuzzy control term with its adaptive laws, and a robustifying control term to compensate for approximation errors and disturbance. The developed adaptive controller does not require the mathematical model of the plant, guarantees the boundedness of all signals in the closed-loop system and ensures the convergence of the tracking error to zero. The main contribution of this paper is the development of a well-defined adaptive controller which permits the initialization to zero of the parameters of the fuzzy systems and does not impose any restrictive conditions about the rate of change of the control gain. In addition, this approach relaxes the a priori knowledge of the lower bound of the control gain and the upper bounds of the approximation errors and the disturbance. The simulation results show the tracking performances of the proposed method. In future works, we propose to extend the results of the paper to multi-input multi-output (MIMO) nonlinear systems.
Figure 3. $\dot{y}(t)$ (solid) and $\dot{y}_d(t)$ (dashed).

Figure 4. Control input $u(t)$ (solid), robustifying control term $u_r(t)$ (dashed).

REFERENCES


