LPV approach to continuous and discrete nonlinear observer design

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Abstract—In this paper we propose a new systematic design of nonlinear observers for Lipschitz nonlinear systems subject to nonlinear outputs. The new design method is dedicated to both continuous-time and discrete-time nonlinear systems having Lipschitz nonlinear outputs. By the use of the global Lipschitz property, the nonlinear system is rewritten as a linear parameter varying system subject to a linear parameter varying output. Based upon this new representation a Luenberger-like observer is designed without any linearization of the dynamics of the system or the observer. Robustness with respect to noisy measurements is also considered in a LMI setting. The present contribution is an extension of the results given in a previous work of the author [1]. We show that the existence of the observer gain is related to the solvability of a Linear Parameter Varying optimization problem and therefore, the requirement of linearization of the observation error is not needed even if the measured output is not linear. The novelty and the efficacy of the proposed design is approved by illustrative case studies.

Key-words: Nonlinear observers; Nonlinear Output Systems; Linear Matrix Inequalities (LMIs); LPV systems; Robustness; Electrical systems.

I. INTRODUCTION

Nonlinear observer design has been the subject of extensive research since the appearance of the pioneer works of Kalman [2] and Luenberger [3]. For nonlinear systems, a standard approach to solve the state observation problem, is to use a copy of the observed system and to add some correction terms attenuating the difference of the outputs, see e.g., [3], [4], [5], [6], [7], [8], [9]. Similar results have been developed in the discrete-time case, see e.g. [10], [11], [12], [13], [14], [15]. Referring to the aforementioned references, the conditions under which the observer gain exists are related to conservative conditions that require the Lipschitz constants of nonlinearities to be very small in order set up a converging observer, see [8] for more details. In order to relax the conservatism of these conditions, a new formulation of the Lipschitz property, which is less conservative than the usual one is given in [16] where the problem of discrete-time observer design along with the problem of observer-based control are addressed in LMI setting. In [17], [18] the authors propose a novel observation strategy for continuous-time nonlinear systems whose nonlinearities have positive growths. An extension of this strategy to discrete-time nonlinear systems is given in [19]. Other challenging results using convex optimization techniques are proposed in the references [17], [19], [20], [21], [22], [23], [24]. Other challenging results can be found in the references [25], [26].

The linearization of the observation error or the presence of a linear part in the system dynamics has been extensively exploited in order to make the analysis of the observer quite simple by the use of well-known stability concepts of linear systems, see e.g., [27], [28]. In this paper, we focus on giving a systematic observer design for Lipschitz nonlinear systems that may neither have a linear dynamics nor linear outputs. The novel LMI-based observation method consists in rewriting the system dynamics as a linear system with linear time-varying coefficients and therefore, there is no need for linearization by state transformation. Extension of the proposed method to nonlinear discrete-time systems is also given. Finally, observer design with robustness against measurement errors is also discussed and solved via the solutions of a set of LMIs. Throughout this paper, we note by $\mathbb{R}$ the set of real numbers. The notation $A > 0$ (resp. $A < 0$) means that the matrix $A$ is positive definite (resp. negative definite). $I$ is the identity matrix of appropriate dimension and $A^T$ denotes the matrix transpose of $A$. $\dot{x}$ stands for the time-derivative of the vector $x$ with respect to time. We note by $\bar{C}_{\text{ball}}$ and $\| \cdot \|$ the convex hull and the Euclidean norm, respectively.

II. OBSERVER DESIGN

In this Section, we focus on the design of nonlinear observers for globally Lipschitz systems of the following form:

$$
\begin{align*}
\dot{x} &= f(x, u) + g(y, u), \\
y &= h(x),
\end{align*}
$$

(1)

where $x = x(t) \in \mathcal{M} \subset \mathbb{R}^n$, $u = u(t) \in \mathcal{U}$ is an $m$ dimensional bounded input, $y = y(t) = h(x) \in \mathcal{Y}$ is the system output, $f : \mathcal{M} \times \mathcal{U} \rightarrow \mathbb{R}^n$ is a Lipschitz nonlinearity satisfying $f(0, 0) = 0$ and $g(y, u) \in \mathbb{R}^n$ is an input-output dependent vector. We assume that $h(x)$ is also a globally Lipschitz nonlinearity with $h(0) = 0$.

One of the motivation of this work is that state-space transformations that bring the dynamics of inherently nonlinear systems to some desired observable canonical forms are quite few. In addition, geometric conditions under which the transformations exist can hardly be met by practical nonlinear systems. Problems related to inverting diffeomorphisms encourages the treatment of systems as they appear even with nonlinear outputs.

Our first objective is to design a globally converging observer for system (1) without any approximation of the dynamics (1) and without any change of coordinates. To this end, we propose an observer of the following form:

$$
\dot{\hat{x}} = f(\hat{x}, u) + g(y, u) + L(h(\hat{x}) - y),
$$

(2)

where $L \in \mathbb{R}^{n \times p}$ stands for the observer gain. Using the mean value Theorem [29], we have for a given $n$-dimensional
smooth vector \( \varphi(s) \) and \((s_1, s_2) \in \mathbb{R}^n \times \mathbb{R}^n \)

\[
\varphi(s_1) - \varphi(s_2) = \int_0^1 \left( \frac{\partial \varphi(s)}{\partial s} \right)_{s=s_1 - \lambda(s_1 - s_2)} (s_1 - s_2) d\lambda.
\]

Then, by setting the observation error as \( e = \dot{x} - x \), we obtain the following dynamics:

\[
\dot{e} = f(\dot{x}, u) - f(x, u) + L(h(\dot{x}) - h(x)).
\]

(4)

Define \( V(e) = e^T P e \) and let \( L = P^{-1} Y \), where \( P \in \mathbb{R}^{n \times n} \) is a symmetric and positive definite matrix and \( Y \in \mathbb{R}^{n \times p} \) is an arbitrary real matrix. Let us rewrite the last dynamics as

\[
\dot{e} = \int_0^1 \left( \frac{\partial f(x, u)}{\partial s} \right)_{s=s_1 + \tau(x - \hat{x})} (\dot{x} - x) d\tau + \int_0^1 Y \left( \frac{\partial h(x)}{\partial s} \right)_{s=s_1 + \tau(x - \hat{x})} (\dot{x} - x) d\tau.
\]

(5)

Define

\[
\theta_{i,j}(x, u) = \frac{\partial f_{i,j}(x, u)}{\partial x_j}, \quad 1 \leq i, j \leq n,
\]

(6)

and

\[
\vartheta_{i,j}(x) = \frac{\partial h_{i,j}(x)}{\partial x_j}, \quad 1 \leq i, j \leq n.
\]

(7)

Then, using the fact that \( f(x, u) \) and \( h(x) \) are globally Lipschitz, we can say that each element of the sets \( \{\theta_{i,j}(x, u)\}_{1 \leq i, j \leq n} \) and \( \{\vartheta_{i,j}(x)\}_{1 \leq i, j \leq p} \) are globally bounded whatever \( x \in \mathcal{M} \) and \( u \in \mathcal{U} \) are. Based on these facts, the Jacobians of the nonlinearities \( f(x, u) \) and \( h(x) \) admit the following representations

\[
\frac{\partial f(x, u)}{\partial x} = \sum_{i=1}^n \sum_{j=1}^n \theta_{i,j}(x, u) \Gamma_{i,j},
\]

(8)

\[
\frac{\partial h(x)}{\partial x} = \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(x) \Omega_{i,j},
\]

(9)

where \( \Gamma_{i,j} \in \mathbb{R}^{n \times n} \) and \( \Omega_{i,j} \in \mathbb{R}^{p \times n} \) are real matrices. The \((k, m)\) elements of the matrices \( \Gamma_{i,j} \) and \( \Omega_{i,j} \) are defined as

\[
\Gamma_{i,j}(k, m) = \begin{cases} 1 & \text{if } k = i \text{ and } m = j, \\ 0 & \text{otherwise}, \end{cases}
\]

(10)

\[
\Omega_{i,j}(k, m) = \begin{cases} 1 & \text{if } k = i \text{ and } m = j, \\ 0 & \text{otherwise}. \end{cases}
\]

(11)

According to these notations, the dynamics of the observation error becomes

\[
\dot{e} = \int_0^1 \left[ \sum_{i=1}^n \sum_{j=1}^m \theta_{i,j}(s, u) \Gamma_{i,j} \right]_{s=s_1 + \tau(x - \hat{x})} e d\tau + \int_0^1 Y \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right]_{s=s_1 + \tau(x - \hat{x})} e d\tau.
\]

(12)

The time-derivative of the Lyapunov function \( V(e) \) along the trajectories of (12) is given by

\[
e' \left( \int_0^1 P \left[ \sum_{i=1}^n \sum_{j=1}^n \theta_{i,j}(s, u) \Gamma_{i,j} \right]_{s=s_1 + \tau(x - \hat{x})} e d\tau \right) + \left[ \sum_{i=1}^n \sum_{j=1}^n \theta_{i,j}(s, u) \Gamma_{i,j} \right]_{s=s_1 + \tau(x - \hat{x})} e d\tau.
\]

(13)

\[
e' \left( \int_0^1 Y \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right]_{s=s_1 + \tau(x - \hat{x})} e d\tau \right) + \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right]_{s=s_1 + \tau(x - \hat{x})} e d\tau.
\]

A sufficient condition to make \( V(e) \leq 0 \) for all \( s \in \mathcal{M} \) and \( u \in \mathcal{U} \) is then deduced:

\[
P \left[ \sum_{i=1}^n \sum_{j=1}^n \theta_{i,j}(s, u) \Gamma_{i,j} \right] + \left[ \sum_{i=1}^n \sum_{j=1}^n \theta_{i,j}(s, u) \Gamma_{i,j} \right] P = 0,
\]

(14)

\[
P \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right] + \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right] Y' < 0.
\]

We have proved the following statement.

**Theorem 1:** Consider system (1) under the action of a bounded input \( u \in \mathcal{U} \). Based upon (6)-(7), let us define

\[
\theta = \left\{ \theta_{i,j}(x, u), 1 \leq i, j \leq n, \ | x \in \mathcal{M}, u \in \mathcal{U} \right\},
\]

(15)

\[
\vartheta = \left\{ \vartheta_{i,j}(x), 1 \leq i, j \leq n, \ | x \in \mathcal{M} \right\},
\]

and let

\[
F(\theta) = \sum_{i=1}^n \sum_{j=1}^n \theta_{i,j}(x, u) \Gamma_{i,j},
\]

(16)

\[
C(\vartheta) = \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(x) \Omega_{i,j},
\]

where \( \Gamma_{i,j} \) and \( \Omega_{i,j} \) are defined as in (10)-(11). Let \( \hat{\theta}_{i,j} \) and \( \hat{\vartheta}_{i,j} \) be the lower and the upper bounds of the element \( \theta_{i,j}(x, u) \), respectively, and let \( \underline{\theta}_{i,j} \) and \( \underline{\vartheta}_{i,j} \) be the lower and the upper bounds of the element \( \vartheta_{i,j}(x) \) for all \( x \in \mathcal{M} \) and \( u \in \mathcal{U} \). If there exist a symmetric and positive definite matrix \( P \in \mathbb{R}^{n \times n} \), and a matrix \( Y \in \mathbb{R}^{n \times p} \) such that the following linear parameter varying inequality condition holds

\[
F' \left( \theta \right) P + PF(\theta) + YC(\vartheta) + C' \left( \vartheta \right) Y' < 0,
\]

(17)

\[
\hat{\theta}_{i,j} \leq \theta_{i,j}(x, u) \leq \underline{\theta}_{i,j}, \quad \forall i, j,
\]

(18)

then, the system

\[
\dot{x} = f(\dot{x}, u) + g(y, u) + P^{-1} Y (h(\dot{x}) - y),
\]

is a globally converging observer for system (1).
Since the parameter-dependent matrices given in (16) are expressed affinely on the parameters $\theta$, $\vartheta$, the resulting LPV matrix inequality condition given by Theorem 1 can be solved as a convex optimization problem using the Matlab software developed by Gahinet et al [30], [31]. It is also possible to transform the LPV optimization problem to a set of linear matrix inequalities where the bounded time-varying parameters are replaced by their convex-hulls.

III. ROBUSTNESS

In this Section, we study the problem of observer design with robustness against measurement errors. More explicitly, a new LMI condition, that ensures the existence of the observer gain, is given. The proposed observer offers an attenuation of the level of noise that may contain the estimates and is exponentially convergent in case of noiseless measurements. The design is summarized in the following statement.

Theorem 2: Consider system (1) with noisy measurements

\[
\begin{align*}
\dot{x} &= f(x, u) + g(y, u), \\
y &= h(x) + d(t),
\end{align*}
\]

where the state $x$, the nonlinearity $f(\cdot, \cdot)$ and the measured nonlinearity $g(\cdot, \cdot)$ are defined as in Theorem 1. We assume that $y = y(t) \in \mathbb{R}^p$ is measured and $d = d(t) \in \mathbb{R}^p$ is a bounded and unknown disturbance with $D \in \mathbb{R}^{p \times p}$ being a known real matrix. Assume that system (19) verify all the conditions of system (1) under the excitation of a bounded input $u \in \mathcal{U}$. Assume that $F(\theta)$ and $C(\theta)$ are defined as in (16). Let \((F_i)_{1 \leq i \leq \mu}\) and \((C_i)_{1 \leq j \leq \nu}\) be the convex-hull matrices such that

\[
F(\theta) \in \mathcal{G} \cap \{ F_1, F_2, \ldots, F_\mu \}, \\
C(\theta) \in \mathcal{G} \cap \{ C_1, C_2, \ldots, C_\nu \}.
\]  

Define the nonlinear observer as

\[
\dot{x} = f(\hat{x}, u) + g(\hat{y}, u) + P^{-1}Y(h(\hat{x}) - y).
\]

Then, for given $\gamma > 0$, the following $L_2$-gain inequality:

\[
\int_0^t \left( h(\hat{x}(\tau)) - h(x(\tau)) \right) \left( h(\hat{x}(\tau)) - h(x(\tau)) \right) d\tau \leq \gamma \int_0^t \|d(\tau)\|^2 d\tau + e(0)'Pe(0)
\]

is verified, whatever the initial conditions of the observer, provided that there exist two matrices $P = P' > 0$ and $Y$ of appropriate dimensions such that

\[
\begin{bmatrix}
F'(\theta)P + PF(\theta) + YC(\theta) + C'(\theta)Y' + C'(\theta)C(\theta) \\
YD \\
-\gamma I
\end{bmatrix} < 0,
\]

or equivalently,

\[
\begin{bmatrix}
F_iP + PF_i + YC_i + C_i'Y' + C_i'C_i \\
YD \\
-\gamma I
\end{bmatrix} < 0, \\
1 \leq i \leq \mu,
\]

\[
1 \leq j \leq \nu.
\]

Proof. Let $e = \hat{x} - x$ and let $V(e) = e'Pe$, where $P = P' > 0$ is some matrix to be determined. Then,

\[
\dot{e} = \hat{f}(\hat{x}, u) - f(x, u) - P^{-1}Y(h(\hat{x}) - h(x)) - P^{-1}YDd.
\]

The optimality condition (22) is verified if

\[
\int_0^t \left( h(\hat{x}(\tau)) - h(x(\tau)) \right) \left( h(\hat{x}(\tau)) - h(x(\tau)) \right) d\tau + V(e)
\]

\[
\leq \int_0^t \gamma \|d(\tau)\|^2 d\tau + e(0)'Pe(0).
\]

This implies that

\[
\int_0^t \left( h(\hat{x}(\tau)) - h(x(\tau)) \right) \left( h(\hat{x}(\tau)) - h(x(\tau)) \right) d\tau
\]

\[
+ \int_0^t V(e(\tau)) d\tau \leq \int_0^t \gamma \|d(\tau)\|^2 d\tau.
\]

Using Eqs. (19)-(11) along with the mean-value Theorem, we get

\[
\int_0^t \left( \int_0^1 \frac{\partial h(s)}{\partial s} \bigg|_{s = \hat{x} - \lambda(\hat{x} - x)} e(\tau) d\lambda \right)'
\]

\[
\times \left( \int_0^1 \frac{\partial h(s)}{\partial s} \bigg|_{s = \hat{x} - \lambda(\hat{x} - x)} e(\tau) d\lambda \right) d\tau + \int_0^t V(e(\tau)) d\tau
\]

\[
\leq \int_0^t \gamma \|d(\tau)\|^2 d\tau.
\]

From the last inequality, we can write the new sufficient condition to fulfill the optimality condition (22) as

\[
\int_0^t \left( \int_0^1 \left[ \sum_{i=1}^p \sum_{j=1}^n \theta_{i,j} \Omega_{i,j} \bigg|_{s = \hat{x} + \lambda(\hat{x} - x)} \right] e(\tau) d\lambda \right)'
\]

\[
\times \left( \int_0^1 \left[ \sum_{i=1}^p \sum_{j=1}^n \theta_{i,j} \Omega_{i,j} \bigg|_{s = \hat{x} + \lambda(\hat{x} - x)} \right] e(\tau) d\lambda \right) d\tau
\]

\[
+ \int_0^t \left( \int_0^1 P \left[ \sum_{i=1}^p \sum_{j=1}^n \theta_{i,j} \Omega_{i,j} \bigg|_{s = \hat{x} + \lambda(\hat{x} - x)} \right] e(\tau) d\lambda d\tau
\]

\[
+ \int_0^t \left( \int_0^1 Y \left[ \sum_{i=1}^p \sum_{j=1}^n \theta_{i,j} \Omega_{i,j} \bigg|_{s = \hat{x} + \lambda(\hat{x} - x)} \right] e(\tau) d\lambda d\tau
\]

\[
- \int_0^t \dot{e}(\tau)YDd(\tau)d\tau - \int_0^t d'(\tau)DY' e(\tau) d\tau
\]

\[
- \int_0^t \gamma d'(\tau)d(\tau)d\tau \leq 0.
\]
Since
\[
\int_t^0 \left( \int_0^1 \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right]_{s=\hat{x}+\lambda(x-\hat{x})} e(\tau) d\lambda \right)' \times \left( \int_0^1 \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right]_{s=\hat{x}+\lambda(x-\hat{x})} e(\tau) d\lambda \right) d\tau \leq \int_0^t \int_0^1 e(\tau) \left[ \sum_{i=1}^p \sum_{j=1}^n \vartheta_{i,j}(s) \Omega_{i,j} \right]_{s=\hat{x}+\lambda(x-\hat{x})}' e(\tau) d\tau.
\]
(29)

Then, (28) is verified if the following holds
\[
\int_0^t \left( e(\tau) \right)' \left[ F'(\theta) P + PF(\theta) + YC(\theta) + C'(\theta) Y' + C'(\theta) C(\theta) - D' Y' - \gamma D \right] \left( e(\tau) \right) d\tau < 0.
\]
(30)

Finally, we conclude, by the Schur complement that, the last inequality is verified if (23) or (24) is satisfied. This ends the proof.

**IV. THE DISCRETE-TIME CASE**

Before giving the main result of this section, let us recall the following statement.

**Theorem 3:** Consider the discrete-time nonlinear system:
\[
\begin{align*}
x_{k+1} &= A x_k + f(x_k, u_k) + g(y_k, u_k), \\
y_k &= C x_k,
\end{align*}
\]
(31)
where the pair \((A, C)\) is assumed to be detectable, \(x_k \in \mathbb{M} \subset \mathbb{R}^n\), \(u_k \in \mathbb{U}\) is an \(m\) dimensional control input, \(\mathbb{U}\) is the set of bounded inputs for which system (31) is observable, \(y_k \in \mathbb{R}^p\) is the system output, and \(f : \mathbb{M} \times \mathbb{U} \rightarrow \mathbb{R}^n\) is a Lipschitz nonlinearity verifying
\[
\frac{\partial f(s, u_k)}{\partial s} \in C_{\text{hull}} \{J_1, J_2, \ldots, J_r\},
\]
(32)
for a given bounded input \(u_k \in \mathbb{U}\) that makes system (31) observable. If there exist a positive and definite matrix \(X = X' \in \mathbb{R}^{n \times n}\) and a matrix \(Z \in \mathbb{R}^{n \times p}\) such that the following linear matrix inequalities hold
\[
\begin{bmatrix}
-X & (A + J_1)X + C'Z' \\
X(A + J_1) + ZC & -X
\end{bmatrix} < 0,
\]
(33)
then, the states of the following system:
\[
\dot{x}_{k+1} = A \dot{x}_k + f(\dot{x}_k, u_k) + g(y_k, u_k) + X^{-1} Z (C \dot{x}_k - y_k),
\]
(34)
converge exponentially to those for system (31).

**Proof.** The proof is given in [1].

In the next statement, we show how to deal with the case of nonlinear discrete-time systems subject to nonlinear outputs.

**Corollary 1:** Consider the discrete-time system:
\[
\begin{align*}
x_{k+1} &= f(x_k, u_k) + g(y_k, u_k), \\
y_k &= h(x_k),
\end{align*}
\]
(35)
where \(x_k \in \mathbb{R}^n\) is the system states, \(g(y_k, u_k) \in \mathbb{R}^n\) and \(y_k \in \mathbb{R}^p\), and \(u_k \in \mathbb{U}\) is a bounded input. Assume that the Jacobians of \(f(x_k, u_k)\) and \(h(x_k)\) verify (16) and (20), respectively. If there exist a positive definite matrix \(X = X' \in \mathbb{R}^{n \times n}\) and a matrix \(Z \in \mathbb{R}^{n \times p}\) such that the following linear matrix inequalities hold
\[
\begin{bmatrix}
-X & F(\theta)'X + C(\theta)'Z' \\
XF(\theta) + ZC(\theta) & -X
\end{bmatrix} < 0,
\]
(36)
or
\[
\begin{bmatrix}
-X & F_i X + C_j Z' \\
XF_i + ZC_j & -X
\end{bmatrix} < 0, 1 \leq i \leq \mu, 1 \leq j \leq \nu.
\]
(37)
Then, the following system:
\[
\dot{x}_{k+1} = f(\dot{x}_k, u_k) + g(y_k, u_k) + X^{-1} Z (h(\dot{x}_k) - y_k),
\]
(38)
is an exponentially converging observer of system (35).

**Proof.** The proof is similar to the proof of Theorem 3.

**V. SIMULATION**

**Example 5.1:** The following example has been served as a historical example to show hyper-chaos in electrical systems.
\[
\begin{align*}
\dot{x} &= \begin{bmatrix}
-6 & 6 & -2 & 0 \\
-60 & -60 & 0 & -20 \\
1 & 0 & 1 & 0 \\
0 & 1.5 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
2 \\
-20 \\
0 \\
0
\end{bmatrix} f(x_1, x_2), \\
y &= x_1,
\end{align*}
\]
(39)
where \(f(x_1, x_2) = -1.6 \left( |x_2 - x_1 - 1| - |x_2 - x_1 + 1| \right)\).

This fourth order system contains a nonlinear element and three-segment piecewise linear resistors. All the present elements are linear and passive, except an active resistor. We have chosen this particular system because it exhibits a nonlinearity with high Lipschitz constant. For all \(x \in \mathbb{R}^4\), the Jacobian of the nonlinearity is
\[
\frac{\partial}{\partial x} \begin{bmatrix}
2 \\
-20 \\
0 \\
0
\end{bmatrix} f(x_1, x_2) = \begin{bmatrix}
\theta_1 - \theta_1 & 0 & 0 \\
\theta_2 - \theta_2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}(x_1, x_2),
\]
(40)
where \(\theta_1 = 3.2 \text{sign}(x_2 - x_1 - 1) - 3.2 \text{sign}(x_2 - x_1 + 1)\), \(\theta_2 = -32 \text{sign}(x_2 - x_1 - 1) + 32 \text{sign}(x_2 - x_1 + 1)\) are bounded real-valued functions. Using the LMI package of...
MATLAB which gives us the following results:

\[
P = \begin{bmatrix} 1.7154 & 0.1478 & 0.5658 & -0.8385 \\ 0.1478 & 0.1658 & 0.0383 & 0.1070 \\ 0.5658 & 0.0383 & 0.3936 & -0.9189 \\ -0.8385 & 0.1070 & -0.9189 & 13.7693 \end{bmatrix},
\]

\[
Y = \begin{bmatrix} -11.2540 \\ -4.0597 \\ 2.4428 \\ -14.7078 \end{bmatrix}.
\]

In Figures 1, 2 and 3, we have represented the system states along with their estimates. The performance of the observer is clearly demonstrated as seen in the first instants of the simulations.

**Example 5.2:** Now, we shall present an observation problem of a nonlinear system with a nonlinear output. Consider the following nonlinear system

\[
\begin{align*}
\dot{x}_1 &= e^{-x_2^2} \log(1 + x_1^2) + \frac{1}{2} \sin(x_2), \\
\dot{x}_2 &= -\frac{x_2}{1 + x_2^2} - \frac{7}{4} \frac{x_2^3}{1 + x_2^2} - 0.1 y^3 + u, \\
y &= x_1 + \frac{1}{2} \sin(x_1).
\end{align*}
\]

(42)

where all the nonlinearities are globally Lipschitz including the output nonlinearity. By setting

\[
f(x) = \begin{bmatrix} e^{-x_1^2} \log(1 + x_1^2) + \frac{1}{2} \sin(x_2) \\ -\frac{x_2}{1 + x_2^2} - \frac{7}{4} \frac{x_2^3}{1 + x_2^2} \\ 0 \\ -0.1 y^3 + u \end{bmatrix},
\]

\[g(y, u) = \begin{bmatrix} 0 \\ -0.1 y^3 + u \end{bmatrix},
\]

and

\[h(x) = x_1 + \frac{1}{2} \sin(x_1).
\]

The Jacobian of the aforementioned nonlinearity can be rewritten as follows

\[
\frac{\partial f(x)}{\partial x} = F(\theta) = \theta_{1,1}(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \theta_{1,2}(x_2) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \theta_{2,2}(x_2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

(44)

where

\[
\begin{align*}
\theta_{1,1}(x_1) &= -2 x_1 e^{-x_1^2} \log(1 + x_1^2) + 2 x_1 \frac{e^{-x_1^2}}{1 + x_1^2}, \\
\theta_{1,2}(x_2) &= \frac{1}{2} \cos(x_2), \\
\theta_{2,2}(x_2) &= -\frac{17 x_2^3 + 17 x_2 + 4}{4 (1 + x_2^2)^2}.
\end{align*}
\]

(45)

On the other hand, the output gradient can be rewritten in linear parameter varying form as

\[\frac{\partial h(s)}{\partial s} = 1 + \frac{1}{2} \cos(s).
\]

(46)

We have, \(-0.5 < |\theta_{1,1}(x_1)| < 0.5, -0.5 < |\theta_{1,2}(x_2)| < 0.5, -2 < |\theta_{2,2}(x_2)| < -1\) and \(0.5 \leq |\theta_{1,1}(x_1)| \leq 1.5\). Using the LMI package of MATLAB, we get after solving the LMLs of Theorem 1

\[
P = \begin{bmatrix} 1.8928 & 0.6768 \\ 0.6768 & 4.7089 \end{bmatrix},
\]

\[Y = \begin{bmatrix} -6.87 \\ 1.3867 \end{bmatrix}.
\]

(47)

8210
The nonlinear observer is readily constructed as
\[
\dot{\hat{x}_1} = e^{-\hat{x}_2} \log(1 + \hat{x}_2) + \frac{1}{2} \sin(\hat{x}_2) - 3.9371 \ 
\]

VI. CONCLUSION

In this paper a useful and a simple design of nonlinear observers for systems having both nonlinear dynamics and nonlinear outputs is proposed. We showed that the design is free from any state transformation or linearization techniques generally employed for simplification of the observer analysis. Observer design with noise attenuation is also discussed in convex optimization setting. Examples showing the applicability and the efficiency of the design are given.

REFERENCES


