Analytical solutions of a class of inverse coefficient problems✩

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1. Introduction

This study deals with analytical solutions of some class inverse coefficient problems using three types of additional conditions i.e. Dirichlet type, Neumann type, and nonlocal type. For this aim, a well known transformation is used. After applying the transformation to the given inverse problem, the unknown coefficient can be determined easily. The transformation has been applied to some problems in the classic literature. For example in [1–5], the parabolic partial differential equation with a control function, i.e.

\[ u_t(x, t) = \Delta u(x, t) + p(t) u(x, t) + \varphi(x, t), \quad 0 < t < T, \; x \in \Omega \]

where \( \Delta \) is the Laplace operator, \( \Omega \subseteq \mathbb{R}^d \) is the spatial domain of the problem and \( d = 1, 2, 3 \), is considered. Here, the functions \( u(x, t), p(t) \) are unknown. The initial and boundary conditions of the problem are \( u(x, 0) = f(x), \; x \in \Omega \) and \( u(x, t) = h(x, t), \; x \in \partial \Omega \) respectively where \( \partial \Omega \) is the boundary of \( \Omega \). For the determination of the unknown function \( p(t) \), many numerical, analytical and experimental methods have been developed in [1–5]. The transformed problems are solved in [3–5] with He’s variational iteration method introduced in [6–8]. Since the variational iteration method does not converge always, we use classical methods to solve the new inverse problem. A convergent analysis for the variational iteration method is given in [9]. The existence and uniqueness of the inverse coefficient problem are considered in [10]. Nevertheless, especially when a nonlocal additional condition is given, it is very difficult to find the exact solution pair \( \{ u(x, t), p(t) \} \) in the inverse coefficient problem. The nonlocal additional condition may be given in the form \( \int_{\Omega} u(x, t) dx = E(t), \; 0 < t < T \). In real applications to give a nonlocal additional condition is difficult since the integral may not be convergent always. However we overcome this difficulty with a very specific example as is seen in the test examples. Also the theoretical aspect of the problem can be found in [1,11].

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This article is organized as follows. In Section 2 the transformation is applied to the problems with different additional conditions and the unknown function is determined theoretically. Some examples are presented in Section 3 to convince the proposed method. A conclusion is drawn in Section 4.

2. Implementation of the transformation for unbounded and bounded domains

Consider the following problem

\[ \begin{cases} u_t(x, t) = k \Delta u(x, t) + p(t)u(x, t), & x \in \mathbb{R}^d, \ t > 0, \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{2} \]

with additional condition

\[ u(0, t) = E_1(t). \tag{3} \]

The inverse coefficient problem here consists of determining the unknown coefficient \( p(t) \) in the problem (2)–(3). In this context, for a given input \( k > 0 \) and \( p(t) \) the problem (2) will be defined as a direct (forward) problem. The constant \( k > 0 \) and the function \( p(t) \) is defined to be the input data and the function \( E(t) \) is defined to be the measured output data, respectively. At first, to solve the inverse problem take up the following transformation:

\[ w(x, t) = u(x, t) r(t) \tag{4} \]

where \( r(t) = e^{-\int_0^t p(s) ds} \). The most important feature of this function is

\[ -\frac{r'(t)}{r(t)} = p(t). \tag{5} \]

Taking the transformation (4) in problem (2) into consideration, we have the following problem:

\[ \begin{cases} w_t(x, t) = k \Delta w(x, t), & x \in \mathbb{R}^d, \ t > 0 \\ w(x, 0) = \varphi(x), & x \in \mathbb{R}^d \end{cases} \tag{6} \]

with the additional condition

\[ w(0, t) = E_1(t) r(t). \tag{7} \]

It is well known that the solution of parabolic problem (6) is given by the following Poisson formula

\[ w(x, t) = \int_{\mathbb{R}^d} S(x - y, t) \varphi(y) dy \tag{8} \]

where \( S(x, t) \) is the Green function defined by \( S(x, t) = \frac{1}{(4\pi kt)^{d/2}} e^{-\frac{x^2}{4kt}} \). Therefore the problem (6) can be solved with the formula (8) and \( w(x, t) \) is found. In order to determine the unknown function \( p(t) \), we will use an additional condition. It is clear from (7) that

\[ \frac{w(0, t)}{E_1(t)} = r(t). \]

Now, by using (5), \( p(t) \) can be determined. Consider the following additional condition instead of (3):

\[ u_x(0, t) = E_2(t) \quad \text{for a fixed } i \in \{1, 2, \ldots, d\}. \tag{9} \]

In similar way, we have

\[ \frac{w_x(0, t)}{E_2(t)} = r(t). \]

So the unknown function \( p(t) \) can be determined again by using (5). Now we consider a nonlocal additional condition [12] instead of (3) as follows:

\[ \int_{\Omega} u(x, t) dx = E_3(t). \tag{10} \]

In this case, we have the following formula to find \( r(t) \):

\[ r(t) = \frac{\int_{\Omega} w(x, t) dx}{E_3(t)}. \]
Now we will apply the transformation (4) to another important problem. For this aim, consider the following first order problem:

\[
\begin{align*}
    u_t(x, t) + k(x) u_x(x, t) &= p(t)u(x, t), \quad x \in \mathbb{R}, \quad t > 0 \\
    u(x, 0) &= \varphi(x), \quad x \in \mathbb{R}
\end{align*}
\]  

(11)

with additional condition

\[ u(0, t) = E_1(t). \]  

(12)

By using the transformation (4), the problem (11) is converted to the following problem related to the non-uniform transport equation:

\[
\begin{align*}
    w_t(x, t) + k(x) w_x(x, t) &= 0, \quad x \in \mathbb{R}, \quad t > 0 \\
    w(x, 0) &= \varphi(x), \quad x \in \mathbb{R}
\end{align*}
\]

(13)

with the additional condition

\[ w(0, t) = E_1(t)r(t) \]

where the wave speed \( k(x) \) is allowed to depend on the spatial position. The general solution of the problem (13) can be found by means of characteristic curves. Note that if \( k(x) \) is a constant and \( p(t) \) is a constant so that \( p(t) < 0 \), the first order equation in (11) models the transport of a radioactively decaying solute in a uniform fluid flow. In this case, the constant \( p(t) \) represents the rate of decay. In addition to these, the above problem can be studied with different additional conditions similar to problem (2).

A common property of above problems is that they are defined in an unbounded domain. However, the same problems can be defined in semi-infinite domains. In this case a condition is given at \( x = 0 \) as Dirichlet or Neumann conditions.

Consider the following inverse coefficient problem defined in the semi-infinite domain \( \Omega = [0, \infty)^d \):

\[
\begin{align*}
    u_t(x, t) &= k \Delta u(x, t) + p(t)u(x, t), \quad x \in \Omega, \quad t > 0 \\
    u(x, 0) &= \varphi(x), \quad x \in \Omega, \\
    u(0, t) &= 0, \quad x \in \partial\Omega, \quad t > 0, \\
    u_{\xi_i}(0, t) &= E_1(t)
\end{align*}
\]

(14)

where \( i \) is a fixed number such that \( 1 \leq i \leq d \).

By using the transformation (4), we have the following inverse problem:

\[
\begin{align*}
    w_t(x, t) &= k \Delta w(x, t), \quad x \in \Omega, \quad t > 0 \\
    w(x, 0) &= \varphi(x), \quad x \in \Omega, \\
    w(x, t) &= 0, \quad x \in \partial\Omega, \quad t > 0, \\
    w_{\xi_i}(0, t) &= E_1(t)r(t).
\end{align*}
\]

After solving the above problem and using the property of the transformation, the unknown function \( p(t) \) is determined. Instead of the additional condition \( u_{\xi_i}(0, t) = E_1(t) \), different conditions can be given as mentioned above.

To study the inverse coefficient problem in a bounded domain \( \Omega \) with smooth boundary \( \partial\Omega \), consider the following problem:

\[
\begin{align*}
    u_t(x, t) &= \Delta u(x, t) + p(t)u(x, t), \quad x \in \Omega, \quad t > 0 \\
    u(x, 0) &= \varphi(x), \quad x \in \Omega, \\
    \frac{\partial u}{\partial n}(x, t) &= 0, \quad x \in \partial\Omega, \quad t > 0 \\
    u(0, t) &= E_1(t), \quad t > 0.
\end{align*}
\]

(15)

By applying the transformation, we have the following inverse coefficient problem:

\[
\begin{align*}
    w_t(x, t) &= \Delta w(x, t), \quad x \in \Omega, \quad t > 0 \\
    w(x, 0) &= \varphi(x), \quad x \in \Omega, \\
    \frac{\partial w}{\partial n}(x, t) &= 0, \quad x \in \partial\Omega, \quad t > 0 \\
    w(0, t) &= E_1(t)r(t), \quad t > 0.
\end{align*}
\]

(16)

If the above problem is solved, the unknown function \( p(t) \) is determined. Notice that at \( x = 0 \) and \( x = a \) the Neumann conditions are given. In place of these conditions, both of them can be given as Dirichlet conditions or both Dirichlet and Neumann conditions.
3. Test examples

Example 1. Let \( \Omega = R^3 \) for the problem (2)–(3) with the following data:

\[
\begin{align*}
  k &= 1, \\
  \varphi(x, y, z) &= e^{-x-y-z}, \\
  E_1(t) &= e^t.
\end{align*}
\]

Then the analytical solution of the problem (6) is \( w(x, y, z, t) = e^{2t-x-y-z} \). From (7), we have \( r(t) = e^{2t} \). Therefore, \( p(t) = -\frac{r'(t)}{r(t)} = -2 \). To check this result, we may control the direct problem with function \( p(t) \). To do this, check that the function \( u(x, y, z, t) = \frac{w(x, y, z, t)}{r(t)} = e^{-x-y-z} \) is the solution of the following direct problem:

\[
\begin{align*}
  u_t &= u_{xx} + u_{yy} + u_{zz} - 2u(x, y, z, t), \quad (x, y, z) \in \Omega, \ t > 0 \\
  u(x, y, z, 0) &= e^{-x-y-z}, \quad (x, y, z) \in \Omega.
\end{align*}
\]

Example 2. In Example 1, the additional condition is given as the Dirichlet condition. However since the problem is defined in an unbounded domain and convergence is necessary for the integral \( \int_\Omega u(x, t)dx \), to give a nonlocal condition may be difficult. However, the following example overcomes these difficulties. For this aim, consider the following inverse coefficient problem:

\[
\begin{align*}
  u_t(x, t) &= u_{xx}(x, t) + p(t)u(x, t), \quad x \in R, \ t > 0 \\
  \lim_{t \to 0^+} u(x, t) &= \delta(x), \quad x \in R, \\
  \int_\Omega u(x, t)dx &= t, \quad t > 0
\end{align*}
\]

where \( \delta(x) \) is the Dirac Delta function defined by

\[
\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}
\]

By applying the transformation, we have the following transformed problem:

\[
\begin{align*}
  w_t(x, t) &= w_{xx}(x, t), \quad x \in R, \ t > 0 \\
  \lim_{t \to 0^+} w(x, t) &= \delta(x), \quad x \in R.
\end{align*}
\]

In this case, it is well known that the analytical solution of the above problem is \( S(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \). An important property of the solution (the Green function) is \( \int_{-\infty}^{\infty} S(x, t)dx = 1 \). By using the transformation we have the following equation for the determination of \( r(t) \):

\[
r(t) = \frac{\int_\Omega u(x, t)dx}{\int_\Omega u(x, t)dx}.
\]

Since \( \int_\Omega u(x, t)dx = 1 \) and \( \int_\Omega u(x, t)dx = t \) we conclude that \( r(t) = \frac{1}{t} \). From (5), \( p(t) \) is found as \( p(t) = \frac{1}{t} \). To check this solution, \( u(x, t) \) is found from (4) and written in the direct problem with the function \( p(t) = \frac{1}{t} \).

Example 3. Let \( \Omega = R \) for the problem (11)–(12) with the following data:

\[
\begin{align*}
  k(x) &= x, \\
  \varphi(x) &= 1, \\
  E_1(t) &= t^2 + 1.
\end{align*}
\]

The solution of the transformed problem (13) is \( w(x, t) = 1 \). Therefore \( r(t) \) and \( p(t) \) are found as \( \frac{1}{t^2+1} \) and \( \frac{2t}{t^2+1} \), respectively. Actually, \( u(x, t) = \frac{w(x, t)}{r(t)} = t^2 + 1 \) is the solution of the problem (11) with the above input data.

Example 4. Let \( \Omega = [0, \infty) \) for the problem (14) with the following data:

\[
\begin{align*}
  k &= 1, \\
  \varphi(x) &= 1, \\
  E_1(t) &= e^{t+\cos t}.
\end{align*}
\]
By applying the transformation, we have the following transformed problem:

\[
\begin{align*}
  u_t(x, t) &= u_{xx}(x, t), & x \in [0, \infty), & t > 0 \\
  w(x, 0) &= 1, & x \in [0, \infty), \\
  w(0, t) &= 0, & t > 0.
\end{align*}
\]

The solution of the problem is given by

\[
w(x, t) = \frac{1}{\sqrt{4kt^2}} \int_0^\infty \left[ e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \varphi(y) \, dy
\]

where \( \varphi(y) = 1 \) is the initial condition. After some basic calculations the solution of the above problem is found as \( w(x, t) = \text{erf} \left( \frac{x}{\sqrt{4kt}} \right) \) where the function \( \text{erf} \) is the error function. Therefore the functions \( r(t) \) and \( p(t) \) are found as \( r(t) = \sec(t) \) and \( p(t) = -\tan(t) \).

**Example 5.** Let \( \Omega = (0, \pi)^2 \) for the problem (15) with the following data:

\[
\begin{align*}
  &\varphi(x, y) = \cos(3x) \cos(3y), \\
  &E_1(t) = e^t.
\end{align*}
\]

Then we have the following inverse problem:

\[
\begin{align*}
  u_t &= u_{xx} + u_{yy} + p(t)u, & (x, y) \in \Omega, & t > 0 \\
  u(x, y, 0) &= \frac{\cos(3x) \cos(3y)}{\cos(3)^2}, & (x, y) \in \Omega, \\
  u_x(0, y, t) &= u_x(\pi, y, t) = 0, & x \in (0, \pi), & t > 0 \\
  u_y(x, 0, t) &= u_y(x, \pi, t) = 0, & y \in (0, \pi), & t > 0 \\
  u(1, 1, t) &= e^t, & t > 0.
\end{align*}
\]

Then the analytical solution of the problem (16) is \( w(x, y, t) = \frac{e^{-\frac{19t}{\cos(3)^2}} \cos(3x) \cos(3y)}{\cos(3)^2} \). From (7) (apply for \( x = 1 \), we have \( r(t) = e^{-19t} \). Therefore, \( p(t) = \frac{r'(t)}{r(t)} = 19 \). To check this result, we may control that the function \( u(x, y, t) = \frac{w(x, y, t)}{r(t)} = e^{t \cos(3x) \cos(3y)} \) is the solution of the above direct problem.

4. Conclusions

In this paper, some class of inverse coefficient problems defined in unbounded and bounded domains with different conditions are solved analytically by applying a transformation. The most important feature of the transformation is that the new transformed problem does not include the unknown parameter. Therefore the transformed problem can be solved easily by using classical methods. Test examples show that this transformation is very useful for finding analytic solutions of some class of inverse coefficient problems. It is worth pointing out that the presented method does not need complicated calculation.

**References**

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